Fundamental Graph Theory Homework (I)

1. Prove that if two graphs are isomorphic, then they have the same degree sequence.
2. Find three examples to show the converse statement of Problem 1 is not true.
3. Prove that there exist exactly two 4-regular graphs G of order 7. (Up to isomorphism.)
4. Find all non-isomorphic graphs of order 5 and size 7.
5. Prove that there exists a graph G such that $G \cong \bar{G}$ if and only if $|V(G)| \equiv 0$ or $1 \pmod 4$.
6. For positive integers $n, n_1, n_2, \ldots, n_k$ where $n = \sum_{i=1}^{k} n_i$, prove that $\binom{n}{2} \geq \sum_{i=1}^{k} \binom{n_i}{2}$.
7. Let A be the adjacency matrix of a graph G. Show that if A is indexed by $V(G) = \{v_1, v_2, \ldots, v_p\}$ as rows and columns, then the (i, j)-entry of $A^k, k \geq 1$, is the number of walks of length k from v_i to v_j.
8. Let $A = [a_{ij}]$ be an adjacency matrix of a graph G. Use the entries of A to find the number of C_3's and C_4's in G respectively.
9. Let B be the incidence matrix of a connected graph G. Prove that $rank(B) = |V(G)| - 1$. Moreover, if G has k components, then $rank(B) = |V(G)| - k$.
10. The eigenvalues of a graph G are the eigenvalues of its adjacency matrix. Find the eigenvalues of C_5 and C_6 respectively.

11. Prove that two isomorphic graphs have the same spectrum, but the converse statement may not be true.

12. Prove that $f(3,6) = 14$ and the graph which attains this bound is unique. ((3, 6)-cage)

13. Prove that if G does not contain a 3-cycle(or K_3), then G has at most $\lfloor \frac{|V(G)|^2}{4} \rfloor$ edges.

14. Prove that in a set of n points in the plane with no pair more that distance 1 apart, then the maximum number of pairs separated by distance more than $\frac{1}{\sqrt{2}}$ is $\lfloor \frac{n^2}{3} \rfloor$.

15. Prove that a graph G is bipartite if and only if G contains no odd cycles.

16. Let $k \geq 3$ and G be the graph where $V(G)$ is the collection of all k-subsets of $\mathbb{Z}_{2k+1}$ and two vertices are adjacent if and only if the corresponding two sets are mutually disjoint. Prove that the girth of G is 6.

17. Prove that if G is a tree of order p, then G has exactly $p-1$ edges.

18. Let $< d_1, d_2, \ldots, d_p >$ be a sequence of p positive integers such that $\sum_{i=1}^{p} d_i = 2(p-1)$. Prove that there exists a tree G of order p such that $< d_1, d_2, \ldots, d_p >$ is a degree sequence of G.

19. Let $A =< a_1, a_2, \ldots, a_p >$ be a non-decreasing sequence of positive integers such that $a_1 \leq \sum_{i=2}^{p} a_i$ and $\sum_{i=1}^{p} a_i$ is even. Prove that there exists a multigraph with degree sequence A.

20. Prove that $rad(G) \leq diam(G) \leq 2rad(G)$, moreover, for any two positive integers m and n satisfying $m \leq n \leq 2m$, there exists a graph G with $rad(G) = m$ and $diam(G) = n$.

21. If G is a simple graph, then $diam(G) \geq 3$ implies that $diam(\bar{G}) \leq 3$.

22. Prove that there exactly p^{p-2} distinct (labeled) trees defined on $\mathbb{Z}_p$.

23. In a digraph G define a *king* of D is a vertex x from which every vertex of $D \setminus \{x\}$ is reachable by a directed path of length at most 2. Prove that every tournament has a *king*.

24. Let T and T' be two spanning trees of a connected graph G. For $e \in E(T) \setminus E(T')$, prove that there is an edge $e' \in E(T') \setminus E(T)$ such that both $T' + e - e'$ and $T - e + e'$ are spanning trees of G.

25. For $p \geq 4$, let G be a simple graph of order p with size $2p - 3$. Prove that G has two cycles of equal length.

26. Prove or disprove that for each graph H, there exists a supergraph G of H such that G is $\triangle(G)$-regular, $|V(G)| \leq 2|V(H)|$ and H is an induced subgraph of G.

27. An edge e is a *cut-edge* of G if $G - e$ has more components than that of G. For $k \geq 2$, prove that a k-regular bipartite graph has no *cut-edge*. Use an example to show the statement is not true for general simple graph.

28. Prove that if $\delta(G) \geq 2$, then G contains a cycle.

29. Let G be a subgraph of $K_{7,7}$ such that G contains no C_4's. Prove that G has at most 21 edges.

30. Let $p = ta + s$. Prove that if G is a graph of order p which does not contain K_{t+1} as a subgraph, then G has at most $\binom{p-a}{2} + (t-1)\binom{a+1}{2}$ edges and this bound can be attained.