

Lecture 4 and 5

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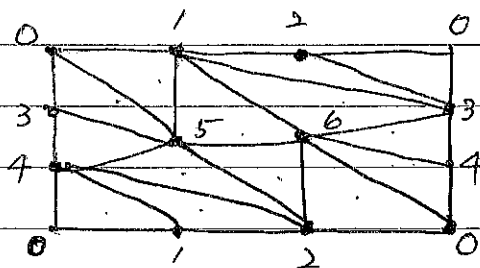
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An imbedding of K_7 in the torus

(*) K_8 can not be embedded in the torus?

(*) The ^{max} number of faces: $\lfloor \frac{56}{3} \rfloor = 18$.

$$8 - 28 + 18 = 2 - 2n$$

$$2n = 4, \quad \underline{n = 2}$$

(*) How about K_p

$$p - \binom{p}{2} + \left\lfloor \frac{p(p-1)}{3} \right\rfloor = 2 - 2n$$

$$p - \frac{p(p-1)}{2} + \left\lfloor \frac{p(p-1)}{3} \right\rfloor = 2 - 2n$$

$$\left\lfloor p - \frac{p(p-1)}{2} + \frac{p(p-1)}{3} \right\rfloor = 2 - 2n$$

$$\left\lfloor \frac{6p - 3p(p-1) + 2p(p-1)}{6} \right\rfloor = 2 - 2n$$

$$\Rightarrow n = \left\lfloor \frac{(p-3)(p-4)}{12} \right\rfloor. \quad (\text{Known!})$$

(a) The following theorem considers pseudographs, i.e., loops and multiedges are allowed.

Theorem (Euler-Poincaré Theorem) (Ex. 1-4)

Let G be a (p, q) -pseudograph which has a 2-cell embedding on S_n . Then, $p - q + f = 2 - 2n$ where f is the number of faces in the embedding.

Proof. By induction on n and it's true when $n=0$ (by

Euler's planar graph formula). Assume that the assertion is

true when $n=k \geq 0$ and G is a (p, q) -pseudograph which

has a 2-cell embedding on S_{k+1} . Since $k+1 \geq 1$, there exists

a handle in the embedding, see Figure 34(a). It suffices to

consider the embedding such that there exists at least one

edge which passes through the handle (on the surface). Note that

if we can pull back an edge without passing the handle, then

pull it back, see Figure (b). Now, we apply the idea of

cut and past to obtain a 2-cell embedding on S_k .

By using a circle around the handle, we can slit the handle through the circle and obtain $\tilde{G}$, see Figure 3.46(c). As a consequence, if there are t edges passing through the handle, the graph $\tilde{G}$ is embedded in S_p . Then $|\tilde{G}| = p + 2t$, $\|\tilde{G}\| = q + 3t$,

and the embedding in S_p has $f + t + 2$ faces. Hence,

$$(p + 2t) - (q + 3t) + (f + t + 2) = 2 - 2k.$$

This implies that $p - q + f = 2 - 2(k+1)$.

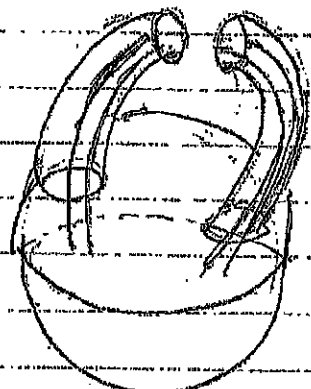
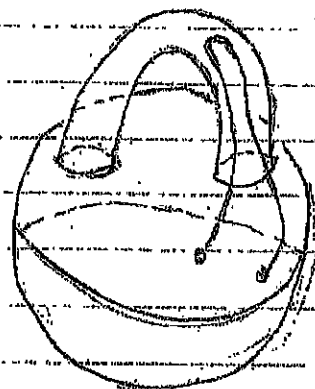
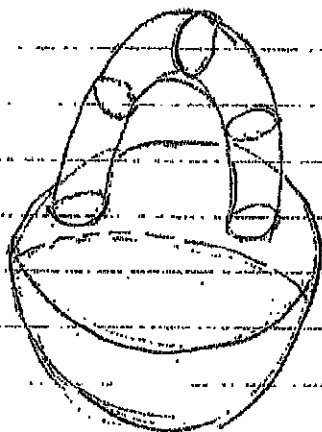


Figure (a) Figure (b), pull edges back Figure (c)

(*) Non-orientable embedding $\tilde{\gamma}(G)$ crosscaps.

$$p - q + f = 2 - \tilde{\gamma}(G).$$

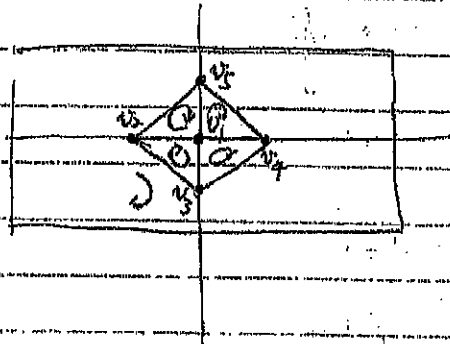
Embedding Schemes

Lecture 4,

Date

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(*) How to find a 2-cell embedding on S_g ?



The above figure provides an embedding of K_5 on S_1 , their regions are (v_1, v_2, v_3) , (v_1, v_3, v_4) , (v_1, v_4, v_5) , (v_1, v_5, v_2) , and $(v_2, v_3, v_4, v_5, v_1)$.

(*) Each arc of D_5 occurs exactly once in a region (face).

(**) By considering each vertex, we observe that there are five permutations to determine this embedding.

$$\pi_1 = (2, 3, 4, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix} \text{ (Defined on } v_1 \text{)}$$

$$\pi_2 = (3, 1, 5, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{pmatrix} \text{ (Defined on } v_2 \text{)}$$

$$\pi_3 = (4, 1, 2, 5)$$

$$\pi_4 = (3, 2, 5, 1)$$

$$\pi_5 = (1, 4, 3, 2)$$

On the other hand, if we are given five permutations one from each vertex, then we have a 2-cell embedding.

For example, $\pi_1 = (3, 2, 4, 5)$, $\pi_2 = (3, 1, 5, 4)$, $\pi_3 = (4, 1, 2, 5)$, $\pi_4 = (3, 2, 5, 1)$, and $\pi_5 = (1, 4, 3, 2)$.

Now, if we start from the arc (v_1, v_2) , then the permutation of π_2 will give the next vertex of the oriented 2-cell containing

$$(v_1, v_2), \quad v_1 - v_2 - v_{\pi_2(1)} \Rightarrow v_1 - v_2 - v_5 - v_{\pi_5(2)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_{\pi_1(5)}$$

$$\Rightarrow v_1 - v_5 - v_5 - v_1 - v_3 \Rightarrow \dots \Rightarrow (v_1, v_2, v_5, v_1, v_5, v_2, v_4, v_3).$$

$$(v_2, v_5), \pi_5(2) = 1 \quad (v_{\pi_5(4)} = v_1 \text{ and } v_{\pi_1(3)} = v_2)$$

(p, q)

(*) Therefore, given a graph G , we can define p permutations for p vertices v_i such that each permutation is a cycle using the ^{indices of} vertices in $N_G(v_i)$. Then, a 2-cell embedding will be obtained.

(Heffter)

Theorem (The Rotational Embedding Scheme)

Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \dots, v_p\}$.

For each 2-cell embedding of G on a surface, there exists a

unique p -tuple $(\pi_1, \pi_2, \dots, \pi_p)$ where for $i = 1, 2, \dots, p$, π_i ^{where}

is a cyclic permutation of $V(i)$ that describes the subscripts of the vertices in $N_G(v_i)$ in counterclockwise order about v_i .

Conversely, for each p -tuple $(\pi_1, \pi_2, \dots, \pi_p)$, there exists a

2-cell embedding of G on some surface such that for $i =$

$1, 2, \dots, p$, the subscripts of the vertices adjacent to v_i are in

counterclockwise order about v_i as given by π_i . Moreover,

the set $\{\pi_1, \pi_2, \dots, \pi_p\}$ induces a mapping π such that

$\pi(v_i, v_j) = \pi(v_j, v_i) = (v_j, v_{\pi_j(i)})$ for each pair of adjacent vertices v_i and

v_j , $1 \leq i \neq j \leq p$.

Proof. The scheme was first observed and used by Dyck (1888).

and Heffter (1891). A formalized version was obtained later in 1960

by Edmonds.

The main idea has been mentioned before the statement above of Theorem. The genus of surface can be obtained after the number of faces (regions) has been determined. □

Theorem $\gamma(K_{m,n}) = (m-1)(n-1)$, $m \leq n$.

Proof. For convenience, let $K_{m,n} = (A, B)$ where

$A = \{a_1, a_2, \dots, a_{m-1}\}$ and $B = \{a_2, a_3, \dots, a_n\}$. Note that m may

not be equal to n . See Figure 35 for the case $m=6$ and $n=8$.

Now, let

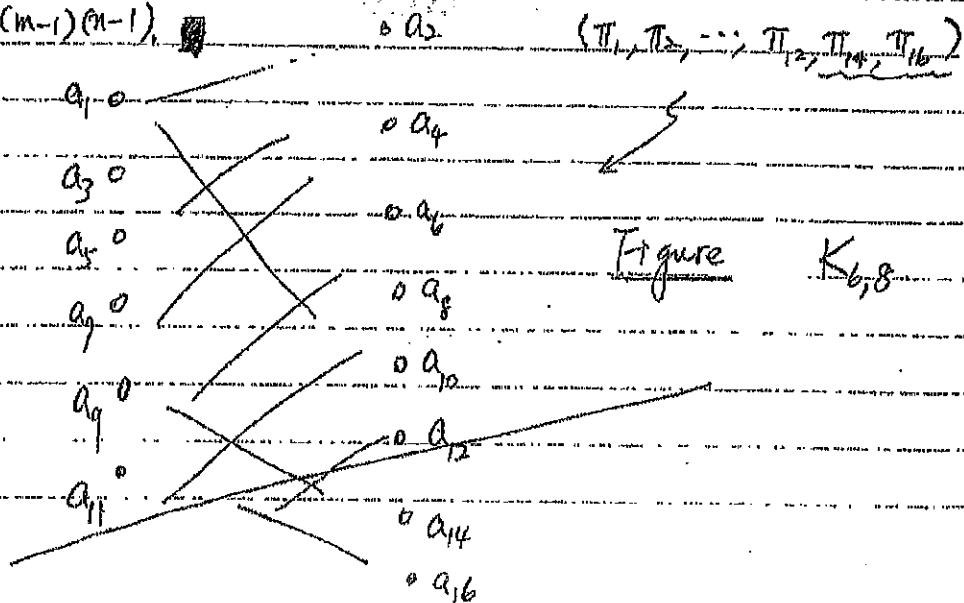
$$\left\{ \begin{array}{l} \pi_1 = \pi_5 = \dots = \pi_{4m-3} = (2 \ 4 \ 6 \ \dots \ 4n); \\ \pi_3 = \pi_7 = \dots = \pi_{4m-1} = (4n \ 4n-2 \ \dots \ 6 \ 4 \ 2); \\ \pi_2 = \pi_6 = \dots = \pi_{4m-2} = (1 \ 3 \ 5 \ \dots \ 4m-1); \text{ and} \\ \pi_4 = \pi_8 = \dots = \pi_{4m} = (4m-1, 4m-3 \ \dots \ 5 \ 3 \ 1). \end{array} \right.$$

We may check that this embedding does have $2mn$ regions

each of them is bounded by a 4-cycle. Hence, by Theorem E-P, (See (4) and (5))

$$2\gamma(G) = 2 - p + q - r = 2 - (2m + 2n) + 4mn - 2mn = 2mn - 2m - 2n + 2.$$

Hence, $\gamma(G) = (m-1)(n-1)$.



- (oo) There are $\prod_{i=1}^p (\deg(v_i) - 1)!$ p -tuples of $(\pi_1, \pi_2, \dots, \pi_p)$.
- (ooo) The 2-cell embedding with the largest number of faces gives the genus of G .
- (*) The 2-cell embedding with the smallest number of faces gives the "maximum" genus of G , denoted by $\gamma_M(G)$. See 5', 5".
- (**) Finding $\gamma(G)$ is a very difficult problem in general.
- (***) Finding $\gamma_M(G)$ is comparatively easier.

Crossing Number of G

Theorem , Let $cr(G)$ denote the crossing number of G .

Then, $cr(K_3) = cr(K_{3,3}) = 1$ and $cr(K_6) = 3$.

Proof. Since $\gamma(K_3) = \gamma(K_{3,3}) = 1$, the proof follows by a drawing with "1" crossing number. Now, we consider K_6 . By Figure 35,

$cr(K_6) \leq 3$. Let $cr(K_6) = k$. Then, we may convert the

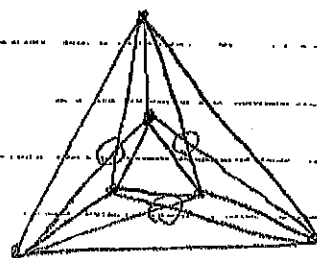
crossings into new vertices. Hence, we have

a planar graph G (obtained from above): $|G| = 6 + k$

and $\|G\| = 15 + 2k$. By the fact

$15 + 2k \leq 3(p+k) - 6$, we have $k \geq 3$. ■

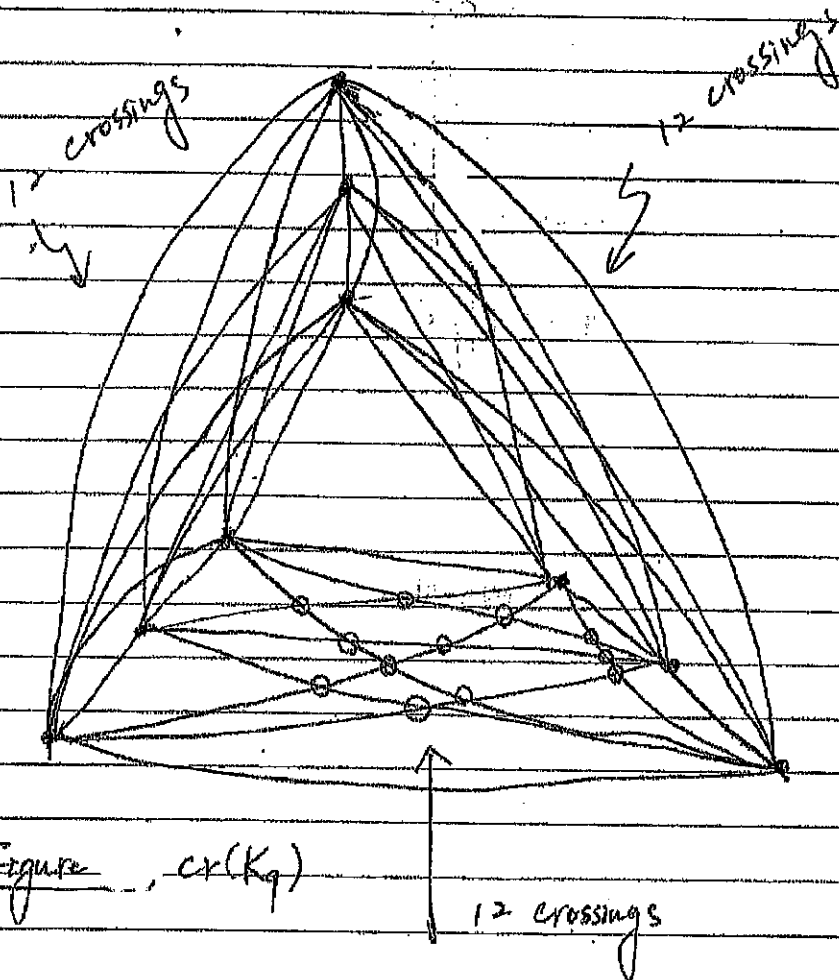
Figure



Theorem: $cr(K_9) = 36$.

Proof: For the upper bound, it suffices to give a drawing which has exactly 36 crossings. But, it is very technical to show the lower bound. (Can you prove it?) Here, we provide a drawing for $cr(K_9) \leq 36$, see

Figure.



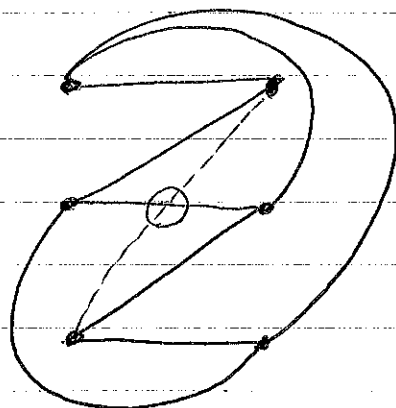
Figure, $cr(K_9)$

Conjecture $cr(K_p) = \frac{1}{4} \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{p-3}{2} \right\rfloor$.

(True for $1 \leq p \leq 10$). (By Saaty)

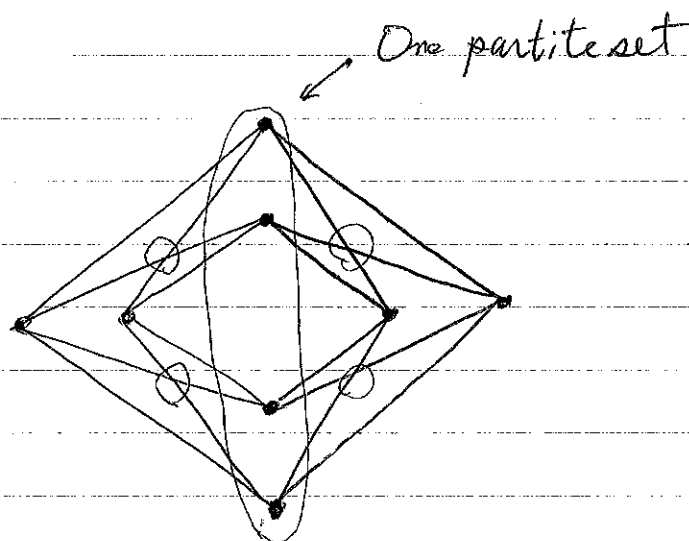
Now, we consider $cr(K_{m,n})$. Since $K_{3,3}$ is not planar, $cr(K_{3,3}) \geq 1$.

By the following figure, it concludes that $cr(K_{3,3}) = 1$.



$$cr(K_{3,3}) \leq 1.$$

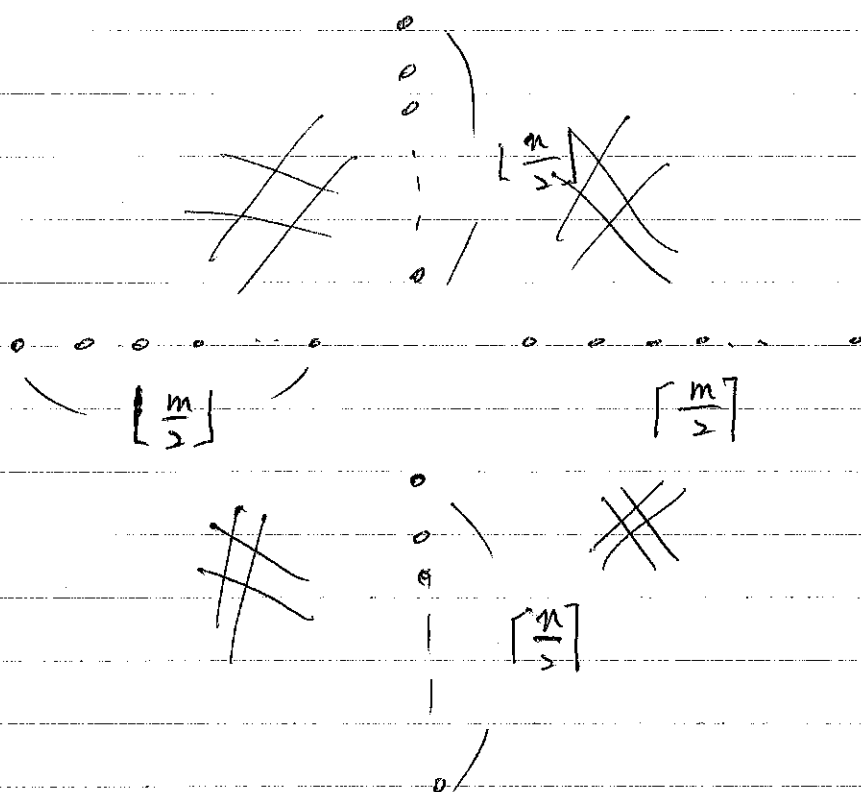
Example, $cr(K_{4,4}) \leq 4$.



Conjecture For $m, n \in \mathbb{N}$, $cr(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

Known results : $m \leq n$, $m \leq 6$, the conjecture holds.
(D. Kleitman)

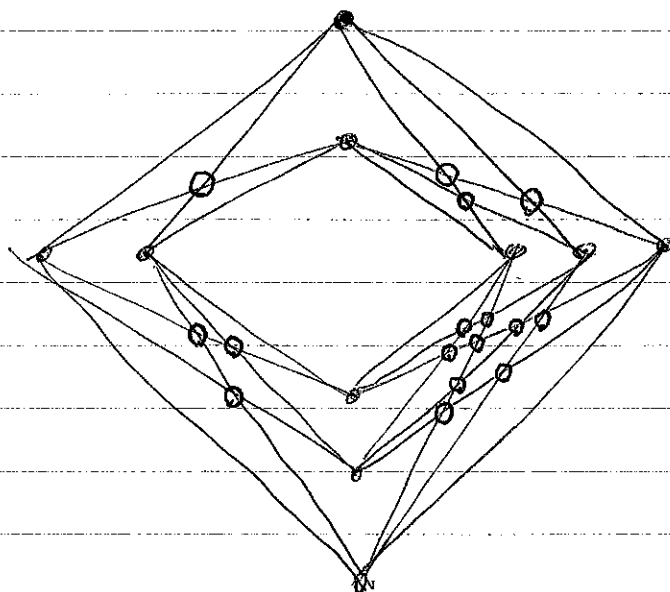
(*) The upper bound of $cr(K_{m,n})$ can be obtained by the drawing in next page.



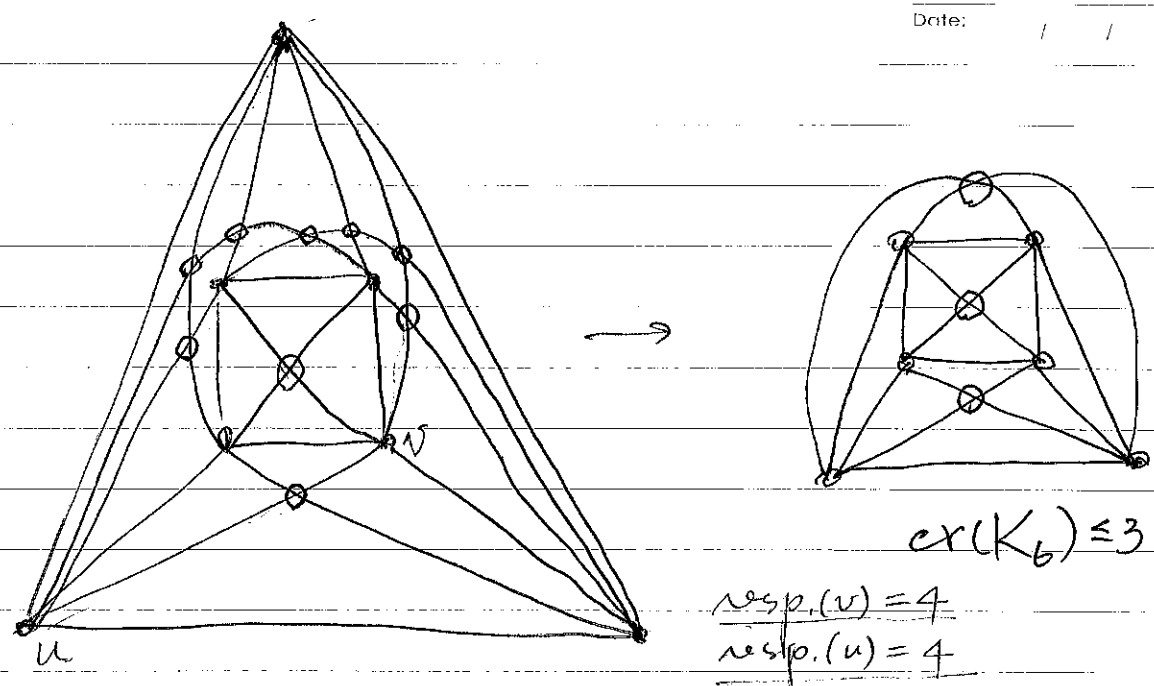
$$cr(K_{m,n}) \leq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor.$$

(Verify!)

Example $cr(K_{5,5}) \leq 16.$



Ex. 1-5. Find $cr(K_8)$ and $cr(K_{5,5})$.

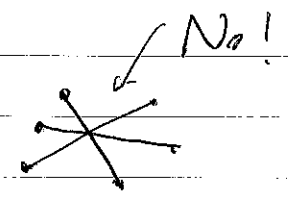


$$cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor$$

$$cr(K_7) = \frac{1}{4} \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 9$$

(*) The drawing above gives $cr(K_7) \leq 9$.

(*) The idea of "responsibility".



Fact 1. Deleting a vertex of a good drawing of K_n produces a drawing of K_{n-1} . (一共有 n 个这种 subdrawing, 其中每一个 crossing (只通 4 条), 令 x 保留在其中 $n-4$ 个 $\rightarrow$ 中.)

$resp.(v)$

Fact 2. The responsibility of a vertex v in a drawing is the total # of edge crossing of edges incident with v .

(*) The sum $\sum_{v \in V(G)} \text{resp.}(v)$ counts every crossing "four" times. 10

Fact 3. Let G be a graph with n vertices and let D be a good drawing of G with $\text{cr}(D)$ crossings. Then,
 $\exists v \in V(G)$ with $\text{resp.}(v) \geq \lceil 4 \text{cr}(D)/n \rceil$.

根据鸽巢原理

(*) For $n \geq 5$, $\text{cr}(K_n) \geq \lceil \frac{n}{n-4} \cdot \text{cr}(K_{n-1}) \rceil$. (*)

From Fact 1, $n \cdot \text{cr}(K_{n-1}) \leq n \cdot \text{cr}(K_n) - 4 \text{cr}(K_n)$

$$\Rightarrow \text{cr}(K_n) \geq \frac{n}{n-4} \text{cr}(K_{n-1})$$

$$\Rightarrow \text{cr}(K_n) \geq \lceil \frac{n}{n-4} \text{cr}(K_{n-1}) \rceil$$

$$\Rightarrow \text{cr}(K_7) \geq \lceil \frac{7}{3} \cdot \text{cr}(K_6) \rceil = 7.$$

$$Z(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

$$Z(n-1) = \frac{1}{4} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor$$

$$\lceil 4 \text{cr}(K_n)/n \rceil = Z(n) - Z(n-1) = \frac{1}{4} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-4}{2} \right\rfloor \right)$$

去掉一个顶点的 crossings!

$$n=7, \quad Z(7) - Z(6) = \frac{1}{4} \cdot 3 \cdot 2 \cdot 2 \cdot (3-1) = 6$$

$$\lceil 4 \text{cr}(K_n)/n \rceil = 6 \Rightarrow \text{cr}(K_7) \geq 6.$$

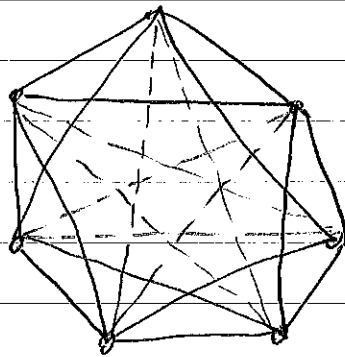
(*) Graph Decomposition

$\{H_1, H_2, \dots, H_k\}$ is a graph decomposition of G if

(a) $H_i \leq G$, $i=1, 2, \dots, k$, and (b) $\bigcup_{i=1}^k E(H_i) = E(G)$.

(Decomposing the edge set of G into k subgraphs.)
(induced)

Example K_7 can be decomposed into three Hamilton cycles.



(*) Definition If all H_i 's are isomorphic to H , then we say G has an H -decomposition, denoted by $H|G$.

(*) Definition If all H_i 's are planar, then G has a planar subgraphs decomposition. The minimum k among all planar subgraphs decomposition is known as the thickness of G , denoted by $\theta(G)$.

$$\theta(K_p) \geq \frac{\binom{p}{2}}{3p-6}.$$

(*) Can you prove $\theta(K_9) > 2$?

(*) $\theta(K_{22}) = 4$ and $\theta(K_{28}) = 5$.

$$\theta(K_p) \approx \left\lfloor \frac{p+1}{6} \right\rfloor ?$$

$$\theta(K_{m,n}) \approx \left\lceil \frac{mn}{2(m+n-2)} \right\rceil ?$$

$\theta(G) \sim cr(G)$ 有關係嗎?

mappingignorance

MATHEMATICS • SCIENCE

The history of the crossing number of the complete graph

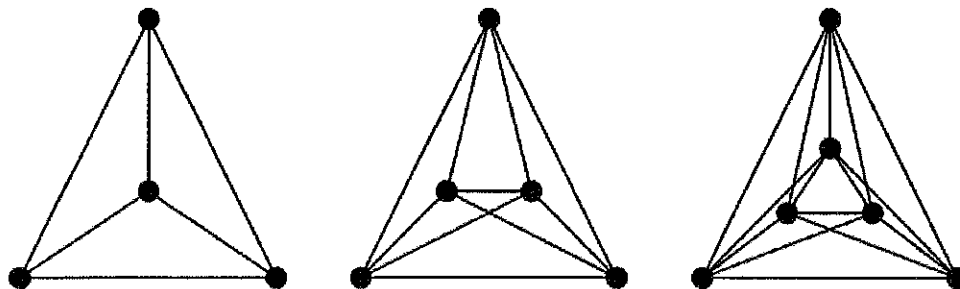
By David Orden October 24, 2013 2 comments

Easy text

We have already introduced the first half of a paper by Lowell Beineke and Robin Wilson¹, which traces the origins of the crossing number of a graph. After focusing on [the brick factory problem](#), proposed by the Hungarian number-theoretist Paul Turán, the second half of the paper is devoted to the crossing number of the complete graph and the geometrical explorations of the British artist Anthony Hill.

Lacking of any formal training in higher mathematics, Anthony Hill described himself as a *constructivist working as a geometric formalist* and made his way to deserve an Honorary Research Fellow in the Department of Mathematics at University College, London.

Unaware of the brick factory problem (which wonders about the smallest possible number of rail crossings when connecting m kilns and n storage yards), Hill considered given a set of n points and wondered about the smallest possible number of crossings when joining by curves every pair of those points. In mathematical terms, the brick factory problem deals with the bipartite graph $K_{m,n}$, while this crossing-number problem deals with the complete graph K_n .



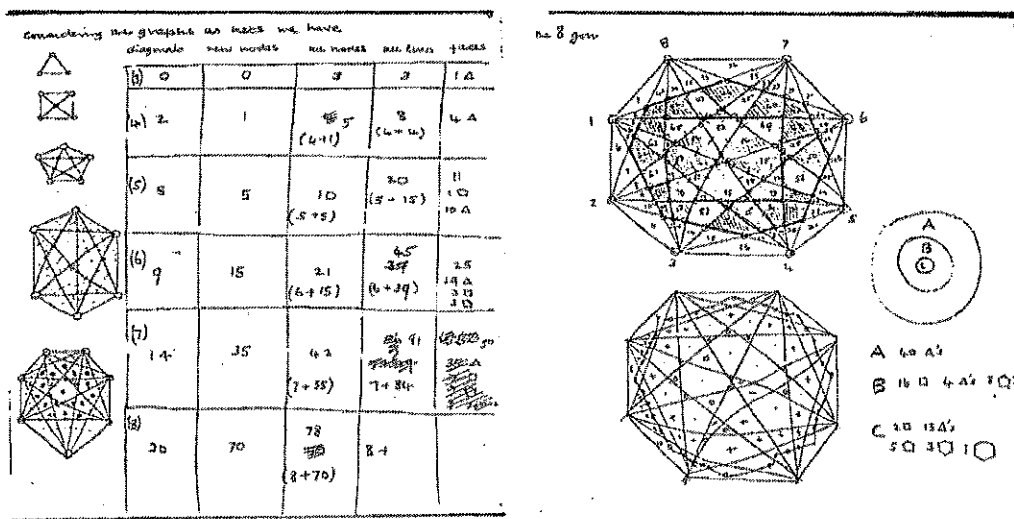
Credit: David Orden

Figure 1 shows drawings of K_n for $n \in \{4, 5, 6\}$ having, respectively, 0, 1, and 3 crossings. Indeed, it can be proved that there is no drawing with fewer crossings, so for those graphs the crossing number is $cr(K_n) = 0, 1, 3$ respectively.

ing, is actually $cr(K_4) = 0$, $cr(K_5) = 1$, and $cr(K_6) = 3$. Furthermore, observe that in those drawings the curves which join the vertices are actually straight-line segments. Such an extra condition leads to a different problem, interesting on its own.

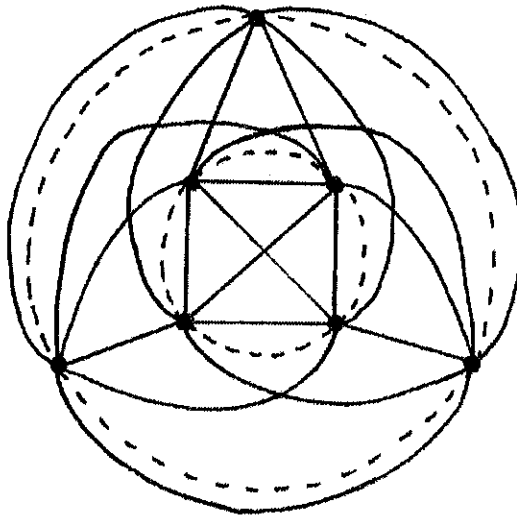
The problem of finding the crossing number of the complete graph has a more fuzzy history than its bipartite counterpart. It is such a natural question that several people might have attacked it before giving up, discouraged by its difficulty. In particular, Paul Erdős claimed in 1960 that he had already been thinking on the problem for at least 20 years (which might sound a bit strange, since he was well known for spreading open problems).

According to several unpublished notes and correspondence, it seems that the first serious investigations on the problem are due to Hill, around 1958. After consulting some colleagues about whether the problem was known, Hill performed a great deal of experimentation which allowed him to find drawings of K_6 with 3 crossings (like the one in Figure 1), of K_7 with 9 crossings, of K_8 with 18 crossings, and of K_9 with 36 crossings. Figure 2 shows part of Hill's notes.



Credit: Beineke & Wilson (2010)

His idea was *looking at diagonals of polygons, some inside and some outside*. This can be checked in Figure 3, which shows Hill's construction for K_7 with the vertices placed on two dashed circles (note that the edges in this case are actually curves).



Credit: Beineke & Wilson (2010)

Such an idea led Hill to state, in the late 1950s, which is probably the first conjecture for the crossing number of the complete graph, with two cases according to the parity of n . Those two cases were later combined into a single formula by J. Blažek and M. Koman², who in addition proved Hill's conjectured value to be actually an upper bound for the crossing number of the complete graph:

$$cr(K_n) \leq 1/4 \cdot \lfloor n/2 \rfloor \cdot \lfloor (n-1)/2 \rfloor \cdot \lfloor (n-2)/2 \rfloor \cdot \lfloor (n-3)/2 \rfloor.$$

By the spring of 1959, Hill and his friend the American artist John Ernest had arrived at the above conjecture and approached some professional mathematicians about the problem. One of them suggested contacting Andrew Booth at Birkbeck College, University of London, who was a computer pioneer in the 1940s. In May 1961, H. P. Goodman, a student of Booth's, wrote a letter to Nature³ describing their attempts to solve this seemingly intractable problem using a computer:

This problem does not appear tractable analytically, so it was programmed for the University of London Mercury computer. The programme was written on the assumption that a minimum for $n+1$ points can be obtained by adding an extra point in a suitable place on a minimum solution [with m crossings] for n points. However, the computations have proved that this apparently natural assumption is false: Two different minimum configurations for $n=7$, $m=9$ were taken, and one led to the true minimum $n=8$, $m=18$, while the other led to $n=8$, $m=19$.

By December 1960, their computer had yielded the values of 60 for $n=10$ and 100 for $n=11$, results that were later been proved correct.

Hill had already communicated the problem to Professor Ambrose Rogers, from University College, when Richard Guy gave a seminar there in November 1959, attended by Hill. Hearing about the complete graph problem, probably via Rogers, Guy published the first paper on it⁴ in 1960. Around this time, Hill communicated his results to the graph-theorist Frank Harary, with whom he wrote a paper⁵ summarizing the progress on the problem.

But it was also Guy who, in a later paper⁶, proved that the conjecture does provide the right crossing numbers for $n \leq 10$. Actually, the conjecture is known both as the Harary-Hill conjecture or as Guy's conjecture. Since then, 47 years were needed until Pan and Richter⁷ in 2007 extended the

n	5	6	7	8	9	10	11	12
$cr(K_n)$	1	3	9	18	36	60	100	150

Credit: Beineke & Wilson (2010)

which is the current status of the crossing-number problem for the complete graph. Nevertheless, very recent results³ seem to have settled some particular cases and might have opened a door towards the resolution of the problem.

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written by

David Orden

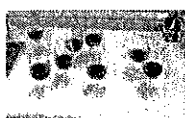


PhD in Mathematics (Universidad de Cantabria, 2003), associate professor of applied mathematics at the Universidad de Alcalá. His main research interests are Discrete and Computational Geometry, Graph Theory, and their applications to other fields.

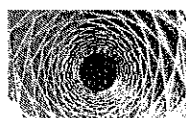
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The Crossing Number of K_{11} is 100

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Abstract

The crossing number of K_n is known for $n \leq 10$. We develop several simple counting properties that we shall exploit in showing by computer that $cr(K_{11}) = 100$, which implies that $cr(K_{12}) = 150$. We also determine the numbers of non-isomorphic optimal drawings of K_9 and K_{10} .

1 Introduction

Guy [4] conjectured that the crossing number $cr(K_n)$ of the complete graph K_n is equal to

$$Z(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

He proved this for $n \leq 10$ and also determined that, for $n = 4, 5, 6, 7, 8$, the number of optimal drawings of K_n is 1, 1, 1, 1, 5, 3, respectively.

In general, we know that $.8594 Z(n) \leq cr(K_n) \leq Z(n)$. The latter inequality follows from the existence of a drawing (see [8] for one example) and the former is proved in [2].

In this paper we use some simple counting properties to provide the basis of an algorithm which we programmed to show that $cr(K_{11}) = Z(11)$. In particular, we determine that K_9 and K_{10} have 3080 and 5679 optimal drawings, respectively. Along the way, we answer affirmatively an open question of Brodsky, Durocher and Gethner [1] by showing that every good drawing (to be defined in the next section) of K_n induces a 3-connected planar graph.

2 Theory

In this section, we provide the simple theoretical background required for our algorithm.

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A *drawing* of a graph G consists of a set of distinct points of the sphere, one for each vertex, and a simple curve for each edge, joining the points representing the ends of the edge, without any vertex-point in its interior. Two drawings are *isomorphic* if there is a homeomorphism of the sphere to itself mapping the image of one drawing to the image of the other such that vertex-points are mapped to vertex-points.

A *crossing* in a drawing D of a graph G is an ordered pair $(x, \{e_1, e_2\})$ consisting of a non-vertex-point x of the sphere and distinct edges e_1, e_2 of G whose representing curves both contain x . The *crossing number* $cr(D)$ is the number of crossings of D . Choosing the vertex-points as the corners of a convex polygon and using line-segments for the edges shows that G has a drawing with finitely many crossings. We shall only be concerned with drawings having finitely many crossings. The *crossing number* of a graph G , denoted by $cr(G)$, is the minimum $cr(D)$, taken over all drawings D of G . A drawing is *optimal* if its number of crossings achieves the minimum.

A drawing is *satisfactory* if two edges share at most one common point, including endpoints, and each non-vertex intersection between two edges is a transverse crossing. It is an easily proved folklore fact that every optimal drawing is satisfactory. A satisfactory drawing is *good* if no non-vertex point of the plane is in three edge-representing arcs. Likewise, it is an easy and well-known fact that if D is a satisfactory drawing of a graph G , then there is a good drawing D' of G having the same number of crossings. In particular, some optimal drawing of G is good.

The main theoretical result we need is the following.

Theorem 1.

- 1) For $n \leq 8$, every optimal drawing of K_n contains an optimal drawing of K_{n-1} .
- 2) A good optimal drawing of K_9 contains a good drawing of K_8 with at most 20 crossings. Any good drawing of K_8 with at most 20 crossings contains an optimal drawing of K_7 .
- 3) A good drawing of K_{11} with fewer than 100 crossings contains a good drawing of K_{10} with at most 62 crossings. Any good drawing of K_{10} with at most 62 crossings contains an optimal drawing of K_9 .

We need the following facts to prove Theorem 1; the first is due to Kleitman [5], while the other two are standard counting results.

Lemma 2. *If n is odd, then the number of crossings in any good drawing of K_n has the same parity as $Z(n)$.*

Deleting a vertex of a good drawing of K_n produces a drawing of K_{n-1} . Any crossing occurs in $n - 4$ of these n subdrawings. That is,

Lemma 3. For $n \geq 5$,

$$cr(K_n) \geq \left\lceil \frac{n}{n-4} \cdot cr(K_{n-1}) \right\rceil.$$

The *responsibility* of a vertex v in a drawing is the total number of edge crossings of edges incident with v . Notice that the sum of all the responsibilities counts every crossing four times. That is,

Lemma 4. Let G be a graph with n vertices and let D be a good drawing of G with $cr(D)$ crossings. Then there is a vertex v of G with responsibility at least $\lceil 4cr(D)/n \rceil$.

Proof of Theorem 1. In every case, we compute $\lceil 4cr(D)/n \rceil$. For (1), it is easy to verify that, for $n \leq 8$, $\lceil 4cr(K_n)/n \rceil = Z(n) - Z(n-1)$, while $\lceil 4cr(K_9)/9 \rceil = 12$ yields the first part of (2). For the second part of (2), $\lceil 4 \cdot 20/8 \rceil = 10$ shows that every good drawing of K_8 with at most 20 crossings has a drawing of K_7 with at most 10 crossings. Now Lemma 2 implies any such drawing has in fact at most 9 crossings.

For (3), Lemma 3 implies that any good drawing D of K_{11} has at least 95 crossings. Lemma 2 implies $cr(D)$ is even. So $cr(D) < 100$ implies $cr(D) \in \{96, 98\}$. If $cr(D) = 96$, then $\lceil 4 \cdot 96/11 \rceil = 35$, so some K_{10} is drawn in D with at most $96 - 35 = 61$ crossings. If $cr(D) = 98$, then $\lceil 4 \cdot 98/11 \rceil = 36$, so some K_{10} is drawn in D with at most $98 - 36 = 62$ crossings. Finally, if D is a drawing of K_{10} with $cr(D) \in \{60, 61, 62\}$, then $cr(D) - \lceil 4cr(D)/10 \rceil$ is either $60 - 24 = 36$ or $61 - 25 = 36$ or $62 - 25 = 37$. Thus, some subdrawing of K_9 has at most 37 crossings; by Lemma 2 it has at most 36. $\square$

3 Algorithm

In this section, we describe our algorithm for showing $cr(K_{11}) \geq 100$. It is based on extending good drawings of K_{n-1} to good drawing of K_n .

A *face* of a drawing D of a graph G in the sphere $\mathbb{S}$ is a component of $\mathbb{S} \setminus D$. If $cr(D) < \infty$ and G is connected, then each face is homeomorphic to an open disc. The *induced planar graph* of D , denoted as G_D , is the graph with vertices $V(G_D) = V(G) \cup \{\text{crossings}\}$, where the edges are the components of $D \setminus V(G_D)$ and v is incident to e if and only if v is in the closure of e . Any vertex in $V(G)$ is a *non-crossing (vertex)* of G_D , and any crossing of D is a *crossing (vertex)* of G_D . We define the *dual graph* of a good drawing D to be the dual graph of the induced planar graph G_D .

Let $\mathcal{D}$ be a set of good drawings. In this section we shall use the following notation:

- $cr(\mathcal{D})$: the minimum number of crossings over all the drawings in $\mathcal{D}$;
- D_F^+ : the set of all good drawings obtained by inserting a new vertex v in a face F of a good drawing D , and drawing new edges from v to all vertices of D ;
- $\mathcal{D}^+$: the set of all good drawings so that deleting some vertex and its incident edges leaves a drawing in $\mathcal{D}$;
- $\mathcal{K}_n^c$: the set of all good drawings of K_n with c crossings;
- $\mathcal{K}_n^{\leq c}$: the set of all good drawings of K_n with at most c crossings.

Obviously, $\mathcal{D}^+ = \bigcup_{D \in \mathcal{D}} \left(\bigcup_{F \in \mathcal{F}(D)} D_F^+ \right)$, where $\mathcal{F}(D)$ denotes the set of faces of a drawing D .

3.1 Idea

Given a good drawing D of K_n with vertices v_i , $i = 1, 2, \dots, n$, and a face $F \in \mathcal{F}(D)$, let $d(F, v_i)$ be the minimum distance in the dual graph from F to the faces incident to v_i . Then $cr(D) + \sum_{i=1}^n d(F, v_i)$ is a lower bound $Lb(D, F)$ for $cr(D_F^+)$. Therefore, letting $Lb[\mathcal{D}] = \min\{Lb(D, F) \mid D \in \mathcal{D}, F \in \mathcal{F}(D)\}$,

$$cr(\mathcal{D}^+) \geq Lb[\mathcal{D}]. \quad (1)$$

Here is how we will show $cr(K_{11}) \geq 100$. Theorem 1(3) says $\mathcal{K}_{11}^{\leq 99} \subseteq \left(\mathcal{K}_{10}^{\leq 62}\right)^+$, so if $\mathcal{K}_{11}^{\leq 99} \neq \emptyset$, then Inequality (1) implies that

$$99 \geq cr(\mathcal{K}_{11}^{\leq 99}) \geq Lb\left[\mathcal{K}_{10}^{\leq 62}\right].$$

The algorithm will show that $Lb\left[\mathcal{K}_{10}^{\leq 62}\right] \geq 100$, giving a contradiction. Thus $\mathcal{K}_{11}^{\leq 99} = \emptyset$, i.e., $cr(K_{11}) \geq 100$.

We give an algorithm for generating all the drawings in $\mathcal{K}_{10}^{\leq 62}$ as follows.

Input: a set $\mathcal{D}$ of good drawings of K_n ($n \geq 4$), and an integer $\delta \in \{0, 1, 2\}$.

Output: all the drawings in $\mathcal{D}^+$ having at most $cr(K_{n+1}) + \delta$ crossings.

Applying the following inputs $(\mathcal{D}, \delta)$, one after the other, to the algorithm:

$$(\mathcal{K}_4^0, 0), (\mathcal{K}_5^1, 0), (\mathcal{K}_6^3, 1), (\mathcal{K}_7^9, 2), (\mathcal{K}_8^{\leq 20}, 0), (\mathcal{K}_9^{36}, 2),$$

by Theorem 1, we shall get the sequence of sets of drawings:

$$\mathcal{K}_5^1, \mathcal{K}_6^3, \mathcal{K}_7^9, \mathcal{K}_8^{\leq 20}, \mathcal{K}_9^{36}, \mathcal{K}_{10}^{\leq 62}.$$

3.2 Generating Drawings

For each D in the input $(\mathcal{D}, \delta)$ and each face F in $\mathcal{F}(D)$, a good drawing in D_F^+ consists of D plus the new vertex and curves joining the new vertex to the vertices in D . In general, these curves correspond to walks in the dual graph. Claim 1 below shows that, in our context, it suffices to consider paths.

The algorithm first adds a new vertex v in F , then searches for all the paths in the dual graph from v to a face incident to v_i with length at most $d(F, v_i) + 2$. Denote such a set of paths by S_i . The algorithm then checks each combination $(P_1, P_2, \dots, P_n)$, where $P_i \in S_i$. For each combination $(P_1, P_2, \dots, P_n)$, if

- 1) any two new edges can be drawn without crossing each other,
- 2) any new edge can be drawn by crossing each edge in D at most once, and
- 3) the total length is no more than $cr(K_{n+1}) + \delta - cr(D)$,

then $(P_1, P_2, \dots, P_n)$ determines a way to add new edges to D so that the new drawing of K_{n+1} is valid for output.

The following claim shows that it is sufficient to search for paths in the dual graph.

Claim 1. *Let $\mathcal{D}$ be a set of good drawings of K_n , let $\delta \in \{0, 1, 2\}$, let $D \in (\mathcal{D}, \delta)$, and let $F \in \mathcal{F}(D)$. Let $D' \in D_F^+$, let v be the new vertex and let $(W_1, W_2, \dots, W_n)$ be the dual walks in D corresponding to the curves incident with v . Then each W_i is a path.*

We first prove a lemma, which answers affirmatively an open question in [1].

Lemma 5. *For $n \geq 4$, the induced planar graph G_D of a good drawing D of K_n is 3-connected.*

Proof. For $n = 4$, there are only two good drawings of K_4 ; one is optimal and the other has a unique crossing. The corresponding induced planar graphs are K_4 and a 4-wheel, which are both 3-connected.

For $n \geq 5$ suppose otherwise there is a separating set $S \subseteq V(G_D)$ with size at most 2. Then there is a partition C_1, C_2 of $V(G_D) \setminus S$ into nonempty sets so that there is no edge of G_D between C_1 and C_2 .

Let m_i be the number of non-crossings in C_i , $i = 1, 2$, and m_0 be the number of non-crossings in S . Then $m_1 + m_2 + m_0 = n$. First we prove that $m_1 > 0$ and $m_2 > 0$. Suppose $m_1 = 0$. Let v be a crossing in C_1 . Then in G_D there are four internally disjoint paths from v to four non-crossings $u_j \in S \cup C_2$, $j = 1, 2, 3, 4$. Hence each of these paths goes through a vertex in S . However, $|S| \leq 2$, a contradiction. So $m_1 > 0$. Similarly $m_2 > 0$.

Let $u_1, u_2, \dots, u_{m_1}$ be non-crossings in C_1 , and $v_1, v_2, \dots, v_{m_2}$ be non-crossings in C_2 . Then, for each $j \in \{1, 2, \dots, m_1\}$ and $k \in \{1, 2, \dots, m_2\}$, there is $\{u_j, v_k\}$ -path P_{jk} in G_D going through only crossings. Since D is a good drawing, $P_{j1}, P_{j2}, \dots, P_{jm_2}$ are internally disjoint for any fixed j . So

$m_2 \leq |S| - m_0 \leq 2 - m_0$. Similarly $m_1 \leq 2 - m_0$. Thus $n = m_1 + m_2 + m_0 \leq 4 - m_0 \leq 4$, which contradicts the assumption that $n \geq 5$. $\square$

Proof of Claim 1. For the drawing D of K_n in the input $(\mathcal{D}, \delta)$, let D' be the good drawing of K_{n+1} determined by the combination of walks $(W_1, \dots, W_n)$. Since D is a good drawing, G_D is a simple graph and, since D' is good, no W_i can have one face F both immediately preceded and succeeded by the same face F' . By Lemma 5, G_D is 3-connected. It is well known that the dual graph of any simple and 3-connected graph is simple and 3-connected (for example, see Theorem 2.6.7 in [7]), so the dual graph of G_D , i.e., the dual graph of D , is also simple and 3-connected.

Suppose some walk W_i is not a path. Then there is a subsequence $(F_j, F_{j+1}, \dots, F_{j+k})$ in W_i such that $F_j = F_{j+k}$. Removing the subsequence $(F_{j+1}, \dots, F_{j+k})$ from W_i gives a new walk W'_i . Replacing W_i with W'_i gives a different new drawing D'' of K_{n+1} . By our earlier remark, $k \geq 3$. If D'' remains a good drawing, $D'' \in \mathcal{D}^+$. Then

$$cr(D'') \leq cr(D) - k \leq cr(\mathcal{D}_n^+) + \delta - 3 < cr(\mathcal{D}_n^+),$$

which contradicts that $D'' \in \mathcal{D}^+$. In D'' there may be crossings between new edges. These are easily eliminated to give a good drawing $D''' \in \mathcal{D}^+$, which also leads to a contradiction by a similar argument. $\square$

3.3 Checking Isomorphism

To determine the number of non-isomorphic drawings, we need to tell a computer how to check drawing isomorphism. For any two good drawings D^1, D^2 of K_n , $n \geq 5$, by Lemma 5, their induced planar graphs G_{D^1}, G_{D^2} are 3-connected, and obviously simple. According to Whitney's Theorem (e.g., see Theorem 4.3.2 in [3], page 96), a planar graph has a unique drawing, up to isomorphism, if it is simple and 3-connected. Then the problem is reduced to checking graph isomorphism.

In our code we use **nauty** (no automorphisms, yes?), created by McKay [6], to determine graph isomorphism. **nauty** is a set of very efficient programming procedures for calculating the automorphism group of a vertex-colored graph. It can be used to test graph isomorphism.

3.4 Results

The results from our code show that, for $n = 5, 6, 7, 8$, the number of non-isomorphic drawings of K_n is indeed 1, 1, 5, 3 respectively. Our results also show that there are 3080 optimal drawings of K_9 and 5679 optimal drawings of K_{10} , up to isomorphism.

With all the drawings in $\mathcal{D}_{10}^{\leq 62}$, $Lb[\mathcal{D}_{10}^{\leq 62}]$ can be calculated straightforwardly by applying any shortest path algorithm. The value is 100 from our results. Therefore, $cr(K_{11}) \geq 100$, and so $cr(K_{11}) = 100$.

Moreover, by Lemma 3, it is easy to show that, for any odd n , $cr(K_n) = Z(n)$ implies $cr(K_{n+1}) = Z(n+1)$. Therefore $cr(K_{12}) = Z(12) = 150$.

We also found that there are many optimal drawings of K_9 which generate no optimal drawing of K_{10} . It was known that not every optimal drawing of K_n contains some optimal drawing of K_{n-1} (see [4] for an example). Now we also know that not every optimal drawing of K_n is contained in some optimal drawing of K_{n+1} .

Our code can be downloaded at

www.math.uwaterloo.ca/~brichter/pubs/publications.html.

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