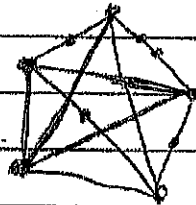


Lecture 3.

Characterization of planar graphs

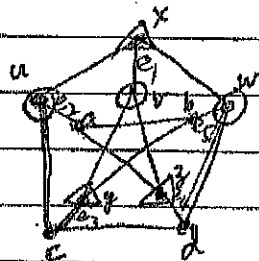
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- (*) K_5 and $K_{3,3}$ are not planar graphs, nor are their homeomorphic graphs.



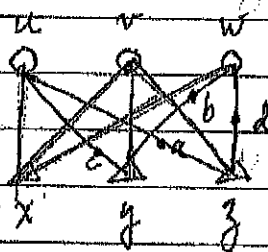
Subdivision of edges

- (*) A graph which contains K_5 , $K_{3,3}$ as minors.



$$(((G/e_1)/e_2)/e_3)/e_4/e_5$$

$$G/e_1, e_2, e_3, e_4, e_5 \rightarrow K_5$$



delete ab, cd

Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proof. (Outline)

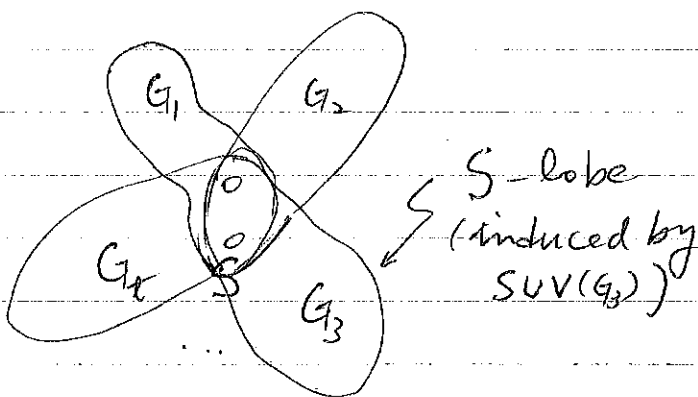
Definition (Minimal non-planar graph)

A non-planar graph such that every proper subgraph is planar.

(*) Every minimal non-planar graph is 2-connected.

Proof. Straight forward?

(*)₁ If G is non-planar and $\{x, y\}$ is a separating 2-set of G , then $G_i + xy$ is non-planar for some S -lobe of G .



Proof. If $G_i + xy$ is planar for all $i = 1, 2, \dots, t$, then

G is also planar. (Put them together with care.)

(*)₂ If G is a graph with fewest edges among all non-planar graphs without Kuratowski sub-graphs, then G is 3-connected!

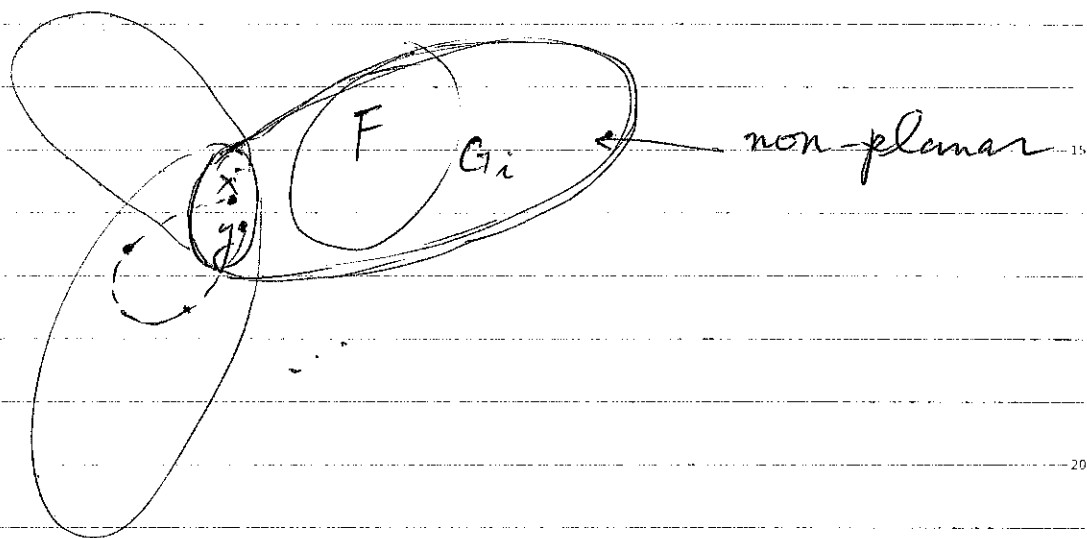
(Subdivisions of K_5 and $K_{3,3}$ are Kuratowski subgraphs.)

Proof. We claim no separating 2-sets exist in G .

Suppose not. Let $S = \{x, y\}$ be a separating 2-set and G be minimal non-planar graph. Since G is non-planar, $H =$

$xy + G_i$. (G_i is an S -lobe.) is non-planar. Since $H < G$,
(By (i),)

H contains a Kuratowski subgraph F . All of F appears in G except possibly the edge xy .



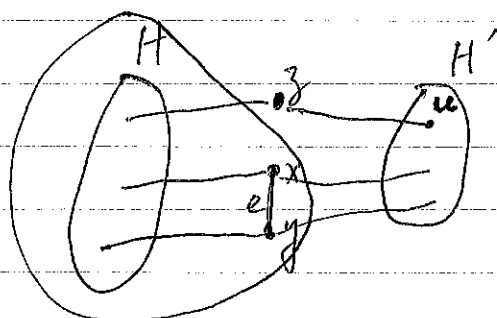
Now, if $xy \in E(G)$, then G contains a Kuratowski subgraph, a contradiction. On the other hand, if xy is not an edge of G and the Kuratowski subgraph does include xy , then we can find a path $x-y$ from the other S -lobe, see figure above. Thus, G contains a K. subgraph, a contradiction. This concludes the proof. ▀

(*)₃ [Thomassen, 1980] Every 3-connected graph G with at least five vertices has an edge e such that G/e is (contraction) 3-connected.

Proof. Suppose not. $\forall e \in E(G)$, such that G/e has a separating 2-set S . Since G is 3-connected, S must include the vertex z' obtained from contracting x and y . Let $S = \{z', z\}$.

(Here, we call z the mate of e .) Now, $\{x, y, z\}$ is a separating 3-set of G .

By our assumption, there does not exist an edge e such that G/e is 3-connected, each e has a mate. Among all edges of G , choose $e = xy$ with mate z such that $G - \{x, y, z\}$ has a component H with largest order. Clearly, $G - \{x, y, z\}$ has another component H' in $G - \{x, y, z\}$, see the following figure



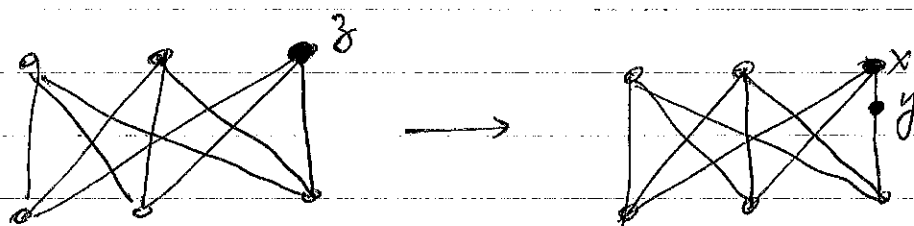
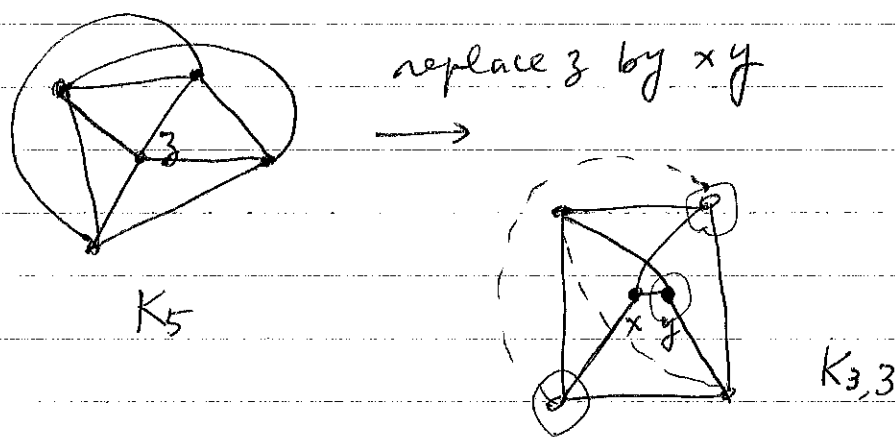
Since $\{x, y, z\}$ is a minimal separating set of G , each of $\{x, y, z\}$ has a neighbor in H , so ^{does} H' . Let u be a neighbor of z in H' and v be the mate of uz .

Again, we conclude that $G - \{u, z, v\}$ is disconnected. However, $\tilde{H} = \langle V(H) \cup \{x, y\} \rangle_G$ is connected. Note that v is not a cut-vertex of $\tilde{H}$, for otherwise $G - \{z, v\}$ is disconnected. Thus, $\tilde{H} - v$ is connected, which ^{is} a component of $G - \{u, z, v\}$. Since $\tilde{H} - v$ is of order larger than H , a contradiction. This concludes the proof, i.e. there exists an edge e such that G/e is 3-connected.

(*) If G has no Kuratowski subgraph, then G/e has no Kuratowski subgraph for any edge e in G .

Proof. We claim that if G/e contains a K_5 subgraph, so does G . Let H be a K_5 subgraph contained in G/e , let z be the vertex obtained from contracting $e = xy$. Clearly, if z is not in H , then H is a K_5 subgraph of G . Now, consider $z \in V(H)$.

On the other hand, z is a vertex in $V(H)$. First, if $\deg_H(z) = 2$, then replace z with x or y or the edge xy , we obtain a K subgraph G . It's left to consider $\deg_H(z) \geq 3$ in a subdivision of H , where H is either a subdivision of K_5 or $K_{3,3}$.



By the figures, we conclude that G does contain a K subgraph.

(*) We can extend the above "subdivisions of H ", by using minors.

Theorem (Tutte, Thomassen)
1960 1980

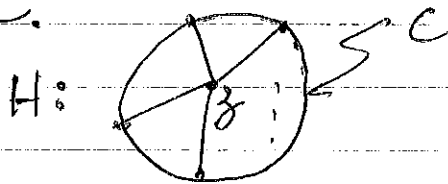
If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then G has a convex embedding in the plane with no three vertices on a line.

Corollary ($\Rightarrow$) G is planar.

Proof. By induction on $n = |G|$. Clearly, $n=4$ is true.

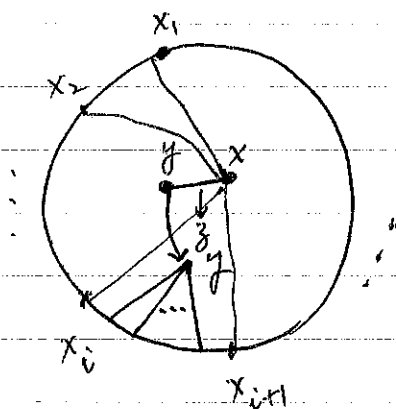
($G \cong K_4$ for $n=4$.) Now, assume that the assertion is true for $n=p \geq 5$. Consider G , $|G|=p+1$.

Let $e \in E(G)$ such that $H = G/e$ is 3-connected. By induction, H does have an embedding required. (Let z be the vertex from e .) As a consequence, H does not contain a K_5 subgraph. Since $H-z$ is 2-connected, the neighbors of z forms a cycle C .



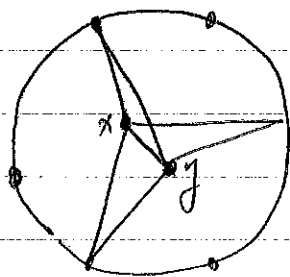
Note that all vertices on C is incident to x or y in G .

Let $x_1, x_2, \dots, x_k$ be the neighbors of x in cyclic order on C .



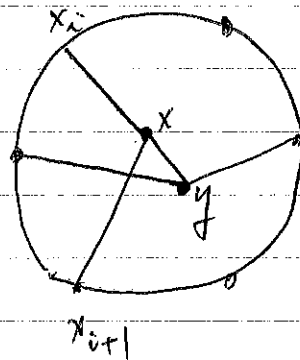
(.) y 的 neighbors 如果落在 x_i 与 x_{i+1} 之间, 可以移动 y 来得到一个适合的 embedding.

(.) 不然就会出现交叉的情况, (不可能) 而造成 G 包含 K 子图。 (不可能) 发生



x, y 有不同的 neighbors:
 u, v, w (3-connected)

↓
Subdivision of K_5



另一种情况

↓
Subdivision of $K_{3,3}$

This concludes the proof. ■

Local Structure of Planar Graphs

G : A simple graph, $|G| = p$, $\|G\| = q$, the # of faces f .

$d(v)$: The degree of v (in G). ($d_G(v)$)

$\bar{d}(G)$: The average of vertex degrees in G .

j -vertex: A vertex with degree j ,

j^+ -vertex: A vertex with degree at least j .

(j^-) -vertex: ————— at most j .

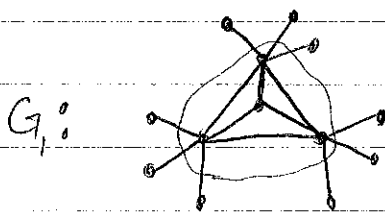
$\delta(G)$: minimum degree of G .

$\Delta(G)$: maximum degree of G .

$wt(H)$: The weight of a subgraph H of G

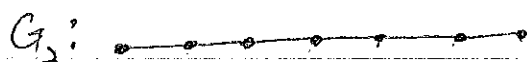
$$\| \sum_{v \in V(H)} d_G(v) \|$$

$mad(G)$: The maximum average degree over all subgraphs of G .



$$\bar{d}(G_1) = \frac{9+3+18}{13} = \frac{30}{13}$$

$$mad(G_1) = \frac{12}{4} = 3$$



$$\bar{d}(G_2) = \frac{12}{7}$$

$$mad(G_2) = \frac{12}{7}$$

Discharging Method

Proposition G is a (p, q) -graph (planar) with f faces.

1. $\sum_{v \in V(G)} \underbrace{(d(v) - 6)}_{\text{vertex charging}} + \sum_{f \in F(G)} \underbrace{(2l(f) - 6)}_{\text{face charging}} = -12.$ ($l(f)$: the # of edges in f)
2. $\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (l(f) - 6) = -12.$
3. $\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (l(f) - 4) = -8.$ (average charging)

Proof. $p - q + f = 2$

$$(1.) \quad -6p + 6q - 6f = -12 \quad (\text{multiply } -6)$$

$$2q - 6p + 4q - 6f = -12$$

$$\left(\sum_{v \in V(G)} d(v) - 6p \right) + 2 \sum_{f \in F(G)} l(f) - 6f = -12.$$

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2l(f) - 6) = -12.$$

$$(2.) \quad \text{Use } 4q - 6p + 2q - 6f = -12.$$

$$(3.) \quad -4p + 4q - 4f = -8$$

Use $2q - 4p + 2q - 4f = -8$ to obtain the proof. ■

Idea (a) Use charging rule (initial charges) first to make sure the total charge is negative (-12) .

*(b) Discharge the charges following discharging rule to ensure the total charge remains unchanged.

(*) 假如在具有特殊圖的結構下, 總電量或為非負, 就表示這樣的結構不可能存在。

Example 1 Every planar graph with $\delta(G) \geq 5$ contains a 5-vertex adjacent to a 7-vertex.

Proof. Since $\delta(G) \leq 5$ for a general planar graph, there exist 5-vertices in G . Assume that all 5-vertices do not have neighbors of degree ≤ 7 , i.e. all neighbors have degree at least 8. We use discharging method to prove that it's impossible.
Let the initial charges $\overset{ch_0(v) \text{ and } ch_0(f)}{\checkmark}$ be

(1) on vertex v , $ch_0(v) = d(v) - 6$; and

(2) on face f , $ch_0(f) = 2l(f) - 6$.

Then, the total charge

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2l(f) - 6) = -12.$$

Now, apply discharging rule (on vertices only):

Every 8^+ -vertex sends $\frac{1}{4}$ to each of its neighbor.

Since each f has $l(f) \geq 3$, $2l(f) - 6 \geq 0$.

For vertex v with degree 5, after discharging, the final charge is $(-1) + 5 \cdot \frac{1}{4} = \frac{1}{4} \geq 0$.

For vertex v , $d(v) \geq 8$; after discharging rule, we have

the charge of v is equal to $(d(v) - 6) - \frac{1}{4}(d(v)) \geq 0$. Thus,

the total charge becomes non-negative. ~~→~~ Hence,

there exists a 5-vertex adjacent to a 7-vertex. 

Example 2

If $\delta(G) \geq 3$, then a 3-vertex incident with a 5-face or a 5-vertex incident with a 3-face.

Proof. Let the initial charge be $ch_0(v) = d(v) - 4$ and $ch_0(f) = l(f) - 4$. Then, by 3. (balanced (average) charging),

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (l(f) - 4) = -12.$$

Now, for each 6^+ -face (resp. 6^+ -vertex) discharges $\frac{1}{3}$ to

its incident vertices (resp. faces). Suppose that the assertion is

not true, then a 3-vertex is incident ^{to} only 6^+ -faces and a

3-face is incident to only 6^+ -vertices.

Following the discharging rules, we conclude that

$$\begin{aligned}
 & \sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (l(f) - 4) \\
 & \geq \sum_{\substack{v \in V(G) \\ d(v)=3}} (-1) + \sum_{\substack{f \in F(G) \\ l(f)=3}} (-1) + 3 \cdot \frac{1}{3} \cdot (\# \text{ of 3-vertices}) + 3 \cdot \frac{1}{3} \cdot (\# \text{ of 3-faces}) \\
 & = 0. \quad \rightarrow \text{This concludes the proof.}
 \end{aligned}$$

Exercise 1-3

Use discharging method to prove the following properties.

(a) If G is a planar graph with $\delta(G) \geq 3$, then G contains a 3-vertex on a 5-face, a 4-vertex on a 3-face, or

a 5-vertex with four incident 3-faces.

(b) If G is a planar graph which has no edges joining 5-vertices, and $\delta(G) = 5$, then it has 60 edges whose end vertices are

of degree 5 and 6.

(c)* Let G be a planar graph with $\delta(G) = 5$ and $e_{i,j}$ denote the number of edges with end vertices whose degrees are i and j . Prove that

$$\frac{26}{11} e_{5,5} + e_{5,6} \geq 60.$$

(* : bonus).

$(c)^* \Rightarrow (b)$