

GT Lecture 12

Date / / No. 1

Direct Constructions of d-disjunct matrices

Review

Macula's Method

$M: \binom{n}{d} \times \binom{n}{k}$ matrix, indexed by $\binom{N}{d}$ and $\binom{N}{k}$ where $|N| = n$.

e.g. $N = 7, d = 2, k = 3, N = \mathbb{Z}_7$.

	012	013	014	015	...
01	1	1			
02	1	0			
03	0	1			
12	0	0	1		
13	0	1	0	1	
56	1	0	0	0	...

567

$\binom{7}{2} \times \binom{7}{3}$
21 x 35

$M(i,j) = 1$ if the i^{th} d -subset is contained in j^{th} k -subset of N
 $M(i,j) = 0$ otherwise

$\Rightarrow M$ is d -disjunct. (So, the above pooling design is 2 -disjunct)

Proof. Consider $d+1$ distinct columns, i.e., $d+1$ distinct k -subsets of N , $C_1, C_2, \dots, C_{d+1}$. Since they are distinct

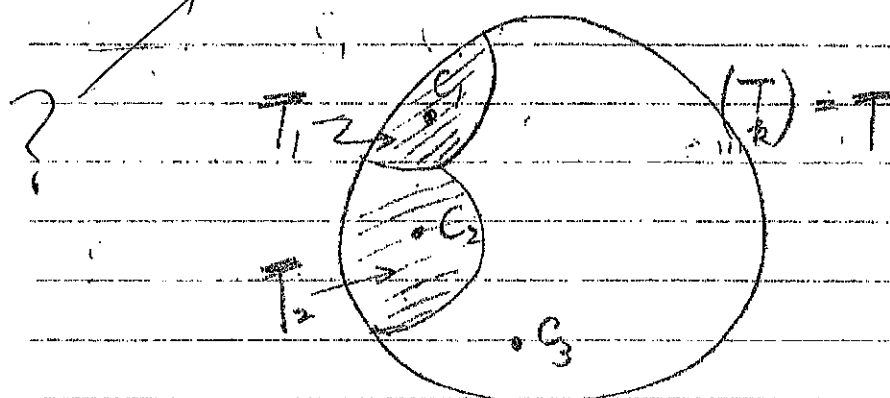
In general, this idea is called "Intersection Property" or column-intersection method.

Algorithm

Let $|T| = t$ and $\binom{T}{k} = \{k\text{-subsets of } T\}$. Define

$$r = \left\lceil \frac{t}{16d^2} \right\rceil, k = 4dr \text{ and } m = 4r.$$

距離 1 的不選!



Choose C_1

Choose $C_2 \in \bar{T}_1$

$$\bar{T}_1 = \{C \mid C \in T \text{ and } |C \cap C_1| \geq m\}$$

$$\bar{T}_2 = \{C \mid C \in T, C \notin \bar{T}_1, \text{ and } |C \cap C_2| \geq m\}$$

$$\bar{T}_m = \dots$$

$$\bar{T}_{m+1} = \emptyset$$

Choose $C_n \in \bar{T} \setminus \bigcup_{i=1}^{n-1} \bar{T}_i$

Theorem Let $C_1, C_2, \dots, C_n$ be the columns and $1, 2, \dots, t$ be the rows-indices. Then, we have a $t \times n$ d -disjunct matrix X

with $n \geq 2.3 \left(\frac{t}{16d^2} - 1 \right)$.

Proof Since any two columns intersect in at most $m-1$ rows, (elements)

to cover a column, it takes at least $\left\lceil \frac{k}{4r-1} \right\rceil$ columns, which
($k=4dr$)

is larger than d . Hence the pooling design is d -disjunct.

Now, we estimate n .

For each T_j , $j=1, 2, \dots, n$, $|T_j| \leq \sum_{i=m}^k \left(\binom{k}{i} \cdot \binom{t-k}{k-i} \right)$ // $\sum_{i=m}^k \boxed{b_i}$
Not in common

This implies that $n \geq \frac{\binom{t}{k}}{\sum_{i=m}^k b_i}$. Common elements

Observe that, for $3r+1 \leq i < k=4dr$,

$$\frac{b_i}{b_{i-1}} = \frac{\left(\binom{k}{i} \binom{t-k}{k-i} \right)}{\left(\binom{k}{i-1} \binom{t-k}{k-i+1} \right)}$$

$$= \frac{\binom{k}{i} \cdot \binom{t-k}{k-i}}{\binom{k}{i-1} \cdot \binom{t-k}{k-i+1}}$$

$$= \frac{\cancel{k!} \cdot \binom{t-k}{k-i}}{i! (k-i)! \cdot \binom{t-k}{k-i+1}} \cdot \frac{\binom{t-k}{k-i+1}}{\binom{t-k}{k-i} \cdot (k-i+1)!}$$

$$= \frac{k-i+1}{i} \cdot \frac{(k-i+1)}{(t-2k+i)}$$

$$= \frac{(k-i+1)^2}{(i)(t-2k+i)} \leq \frac{(4dr-3r)^2}{3r \cdot (16d^2r-8dr+3r)} \quad (i=3r+1)$$

$$= \frac{(4d-3)^2}{3(16d^2-8d+3)} < \frac{1}{3}$$

$$b_i < \left(\frac{1}{3}\right)^{i-3r} \cdot b_{3r} \quad \forall \quad 3r+1 \leq i \leq k = 4dr.$$

$$\sum_{i=m}^k b_i = \left(\frac{1}{3}\right)^{-3r} \cdot b_{3r} \cdot \sum_{i=m}^k \left(\frac{1}{3}\right)^i$$

$$= \left(\frac{1}{3}\right)^{-3r} \cdot b_{3r} \cdot \left(\frac{1}{3}\right)^m \cdot \left(1 + \frac{1}{3} + \dots + \left(\frac{1}{3}\right)^{k-m-1}\right) \rightarrow \frac{1}{1-\frac{1}{3}}$$

$$< b_{3r} \cdot \left(\frac{1}{3}\right)^{m-3r-1} = 2b_{3r} \cdot 3^{r-1}$$

Since $\binom{k}{r} = \sum_{i=0}^k b_i > \binom{k}{r} \left[\frac{b_{3r}}{\sum_{i=0}^k b_i} \right] > 2 \cdot 3^{r-1} \geq 2 \cdot 3^{\frac{k}{16d^2} - 1}$

Too small! (Not efficient!)

∴ Solve for k .

$$n \geq 2 \cdot 3^{\frac{k}{16d^2} - 1}$$

Can we choose a larger value? $b_{3r} = \sum_{i=3r}^k \binom{k}{i} \binom{i-k}{r-i}$

$$\log_3 n \geq \log_3 2 + \frac{k}{16d^2} - 1$$

$$k \leq 16d^2 (\log_3 n - \log_3 2 + 1).$$

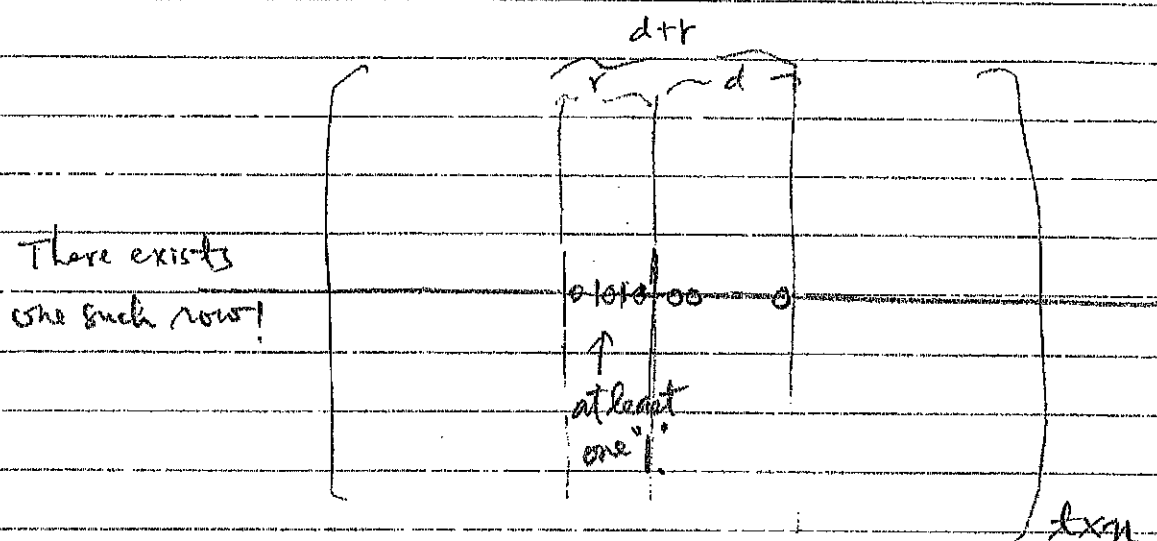
$$t(d, n) \leq 16d^2 (\log_3 n - \log_3 2 + 1).$$

$$t(d, n) \leq 16d^2 (\log_3 2) \log n (1 + o(1))$$

(Non-constructive Method!)

With the help of designs, we can do better.

★ Definition A $t \times n$ binary matrix is (d, r) -disjunct if for $d+r$ arbitrary columns, the union of any r columns is not contained in the union of the other d columns.



D'yachkov and Rykov, A survey of superimposed code theory, Problems, Control Inform. Theory, 12 (1983), 1-13.

A $t \times n$ (d, r) -disjunct matrix is equivalent to the existence of a superimposed (d, n, r) -code of length t .

Definition (Superimposed code) t, n, d (positives)

A superimposed code (binary) $STC(n, N, m)$ consists of N

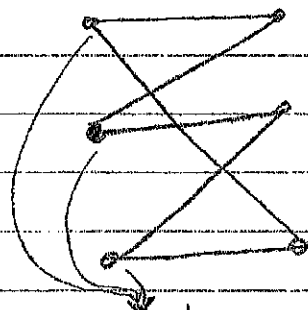
binary codewords of length n , with the property that from the Boolean sum of any m -subset we are able uniquely determine the individual codewords.

Construction of (d, ℓ) -dispenser matrices

Definition An (ℓ, d, ε) -dispenser is a bipartite graph $G = (A, B)$ satisfying the following conditions:

- (1) $\forall S \subseteq A$ with $|S| \geq \ell$ is adjacent to at least $(1-\varepsilon)|B|$ vertices of B ; and
 $(|N(S)| \geq (1-\varepsilon)|B|)$

- (2) $\forall v \in A, \deg(v) = d$.



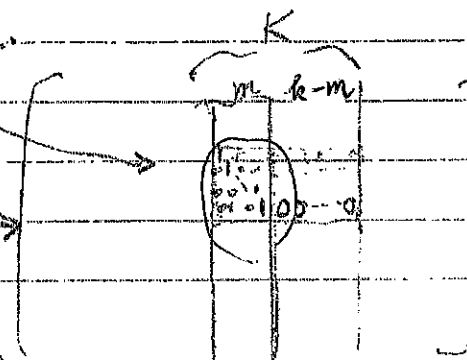
$(2, 2, \frac{1}{10})$ -dispenser

Review

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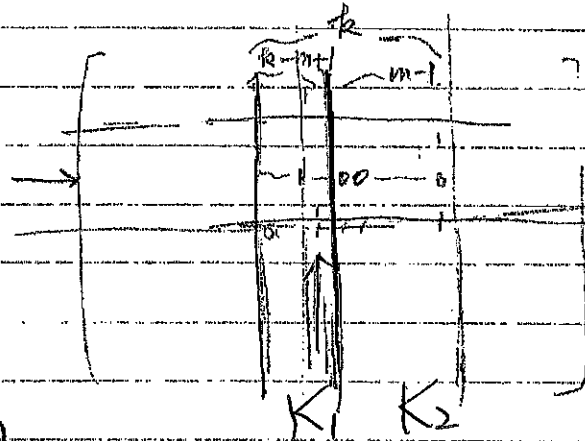
Definition A (k, m, n) -selector is a binary matrix with n columns such that for any k -set K of columns, at least m of them are isolated in K .

A column is called isolated if there exists a row incident to it but not to any other column.



Fact A (k, m, n) -selector is $(m-1, k-m+1)$ -disjunct.

Proof



The union of any $k-m+1$ columns is not contained in the union of $m-1$ other columns since there exists a column (isolated) in K_1 which is not contained in the union of $m-1$ columns in K_2 .

(In a (k, m, n) -selector, at least m out of k are isolated.)
columns

(*) Note, if $k-m+1=1$, i.e. $k=m$, then the matrix is in fact a

$(k-1)$ -disjunct matrix. Therefore, in any k columns, one of them is not contained in the union of the other $k-1$ columns.

So, consider $d=m-1$ and $r=k-m+1$. A (d, r) -disjunct matrix can be obtained by constructing a (k, m, n) -selector.

© B.S. Chlebas and D.R. Kowalski, Almost optimal explicit selector.

LNCS 3623 (Springer-Verlag, 2005) 270-280.

(*) Let M be an $s \times n$ (h, h, n) -selector $((h-1)$ -disjunct)

and $H = (A, B)$ a (k, d, ε) -dispenser.

$A = \mathbb{Z}_n$, $B = \mathbb{Z}_g' = \{0', 1', 2', \dots, g-1'\}$ and rows of M are indexed by $\mathbb{Z}_s$. Define an $sg \times n$ binary matrix M^* as follows:

$\forall v \in A$ and $i = as + b$ where $a \geq 0$ and $0 \leq b < s$, $M^*(i, v) = 1$

if and only if v is neighbor of vertex a' in B and $M(b, v) = 1$.

Then, M^* is a (k, h, n) -selector provided $(1-\varepsilon)gh \geq kd$.

(See figure in next page!)

Proof. Suppose not. Then, there exists a k -set K of columns

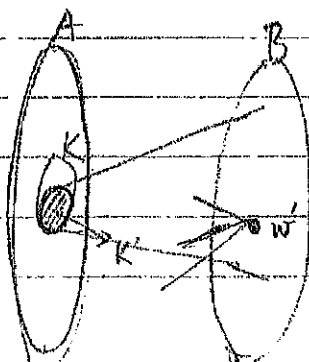
and a $(k-h+1)$ -subset $K' \subseteq K$ such that every column in K' is

not isolated in K . (Since we need h isolated columns, we have

at most $k-h$ non-isolated columns in K .)

Now, consider the (k, d, ε) -dispenser H . K' (and K) can be

"recognized" as a vertex subset of A . Let $w' \in B \cap N_H(K)$.

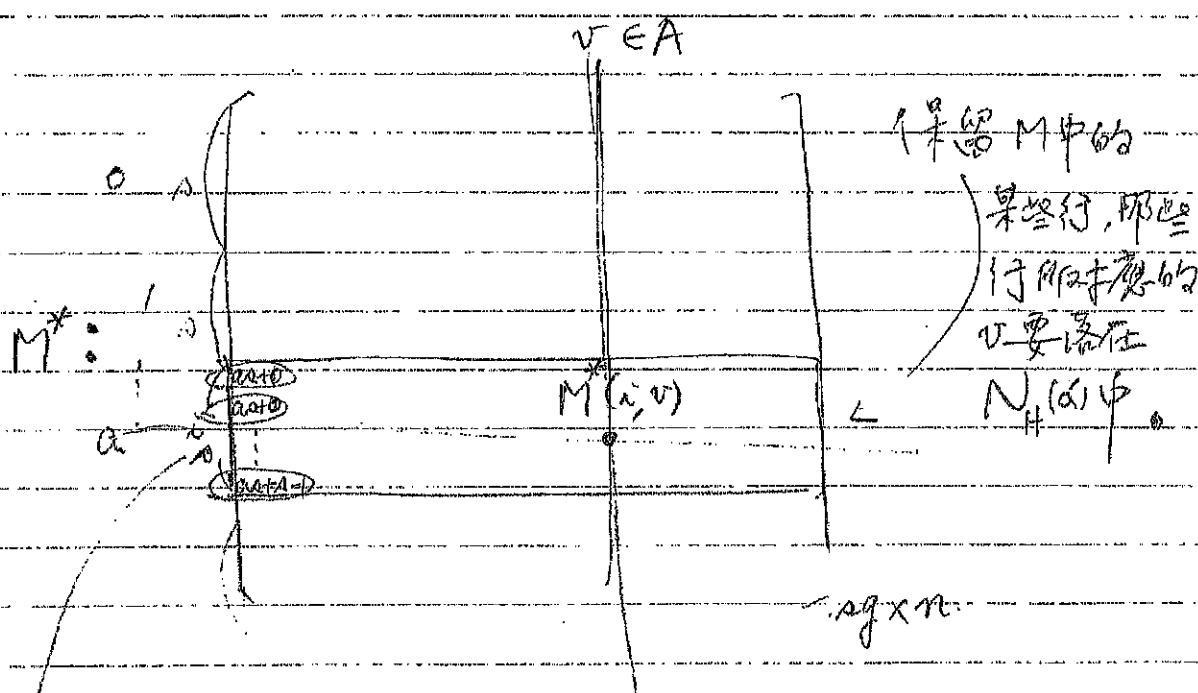
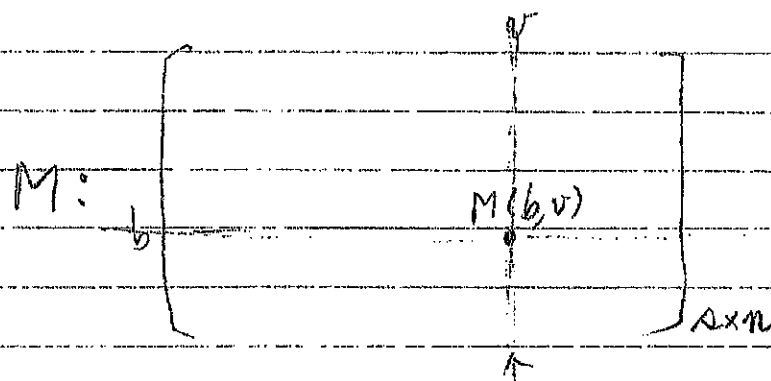


Claim:

w' has more than h neighbors in K .

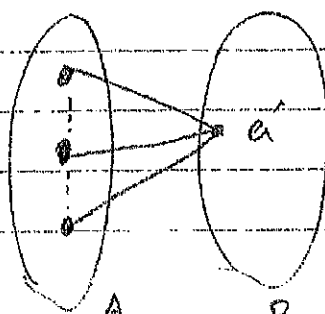
Suppose at most h .

Let $N_K(w') = \{v \in K, v \sim w'\}_H$.

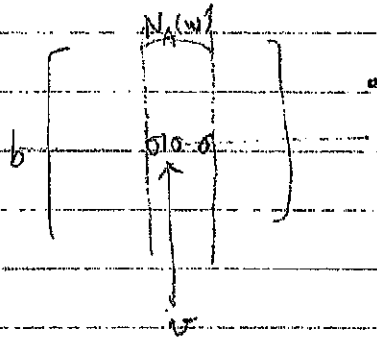


$$i = as + b, \quad a \in \mathbb{Z}_g \text{ and } 0 \leq b < s.$$

$M^*(i, v) = 1$ iff $M(b, v) = 1$ and $a' \in B$ (corresponding to a).
 $v \in N_H(a')$ where



Since M is an (h, h, n) -selector, there exists a a, b in M such that



Now, the $(wa+b)$ -th row of M^* intersects K with exactly one column (an isolated column), $\rightarrow \in K'$.

Since H is a (B, d, ε) -dispenser, $|N_H(K)| \geq (1-\varepsilon)|V| = (1-\varepsilon)g$.

Thus, $|K \cap N_H(K)| > (1-\varepsilon)gh$. However, $\forall v \in A$, $\deg_H(v) = d$,

we have at most dB edges which is less than $(1-\varepsilon)gh$ edges,

a contradiction. ■

More properties of a (d, r) -disjunct matrix.

① In a d -disjunct matrix, $t(d, 1, n) \geq \log_2 \binom{n}{d} - \log_2 \binom{d+1}{d}$.

Proof.

