

# ***Recent Progress of Grid-Block Designs***

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# ***H*-Packing**

- ★ An *H*-packing of a graph  $G$  is a collection of edge-disjoint subgraphs of  $G$  such that each of them is isomorphic to  $H$ .
- ★ If  $G$  is exactly the union of subgraphs in an  $H$ -packing of  $G$ , then we call the  $H$ -packing an *H*-design of  $G$ .
- ★ If  $G$  is the complete graph of order  $n$ , then we have an *H*-packing or an *H*-design of order  $n$ .



# ***Graph Decomposition***

- ★ In term of *graph decomposition*, an  $H$ -design of  $G$  is equivalent to a decomposition of the graph  $G$  into isomorphic copies of  $H$ .
- ★ We refer to the subgraphs isomorphic to  $H$  as the *members* of the decomposition.
- ★ Moreover, if the members are complete graphs of order  $k$ , then an  $K_k$ -design of order  $n$  is a *balanced incomplete block design* with block size  $k$  and  $\lambda = 1$ .



# ***Well-Known H-Designs***

- ★ A  $K_3$ -design of order  $n$  exists if and only if  $n \equiv 1$  or  $3 \pmod{6}$ . (Steiner triple system of order  $n$ , STS( $n$ ))
- ★  $n = 9$  (Affine plane of order 3)

|              |              |              |              |
|--------------|--------------|--------------|--------------|
| <u>1 2 3</u> | <u>1 4 7</u> | <u>1 5 9</u> | <u>1 6 8</u> |
| <u>4 5 6</u> | <u>2 5 8</u> | <u>2 6 7</u> | <u>2 4 9</u> |
| <u>7 8 9</u> | <u>3 6 9</u> | <u>3 4 8</u> | <u>3 5 7</u> |

# ***$K_4$ -Designs***

- ★ A  $K_4$ -design of order  $n$  exists if and only if  $n \equiv 1$  or  $4 \pmod{12}$ .
- ★  $n = 16$  (Affine plane of order 4)

|           |           |           |           |
|-----------|-----------|-----------|-----------|
| <u>1</u>  | <u>2</u>  | <u>3</u>  | <u>4</u>  |
| <u>5</u>  | <u>6</u>  | <u>7</u>  | <u>8</u>  |
| <u>9</u>  | <u>10</u> | <u>11</u> | <u>12</u> |
| <u>13</u> | <u>14</u> | <u>15</u> | <u>16</u> |

- (\*) Using 3 *mutually orthogonal latin squares* of order 4 to construct the Affine plane.



# ***Grid-Block***

- ★ Define  $G(r,c)$  as the *grid-block* with  $r$  rows and  $c$  columns where each grid point is a (distinct) vertex and two vertices are *collinear* if they are on the same row or column.
- ★ If we define a graph from  $G(r,c)$  by letting two vertices be adjacent if and only if they are collinear, then  $G(r,c)$  is isomorphic to the *Cartesian product* of  $K_r$  and  $K_c$  denoted by  $K_r \times K_c$ .



# ***An Example***

- ★ The green and pink grid-blocks pack  $K_9$ . Therefore, a  $G(3,3)$ -design or a **3x3 grid-block design** of order 9 exists!

|   |   |   |
|---|---|---|
| 1 | 4 | 7 |
| 2 | 5 | 8 |
| 3 | 6 | 9 |

|   |   |   |
|---|---|---|
| 1 | 6 | 8 |
| 5 | 7 | 3 |
| 9 | 2 | 4 |



# ***The Existence of $G(r,c)$ -Designs***

- ★ If a  $G(r,c)$ -design of order  $n$  exists then the following conditions hold:
  - (a)  $rc \leq n$ ,
  - (b)  $r+c-2$  divides  $n-1$ , and
  - (c)  $rc(r+c-2)$  divides  $n(n-1)$ .

# ***Lattice Rectangles***

- ★ A  $G(r,c)$ -design is called a *lattice square* provided that  $r = c = n^{1/2}$  named by Yates, 1940.
- ★ Construction of lattice squares for  $n^{1/2}$  an odd prime power was given by Raghavarao in 1971.
- ★ Lattice squares were extended to *lattice rectangles* ( $r \neq c$  and  $rc = n$ ) by Harshbarger in 1947.



# ***Real World***

- ★ In most practical uses, the grid-block has size limitation and  $n$  is large.
- ★ Thus, we have to consider the  $G(r,c)$ 's with  $r < n^{1/2}$  and  $c < n^{1/2}$ , while preserving the *unique collinearity condition*, i.e. every pair of vertices occur at most once in the same row or column. This is one of the reasons we study  $G(r,c)$ -design or  $G(r,c)$ -packing.



# ***Resolvable Packing***

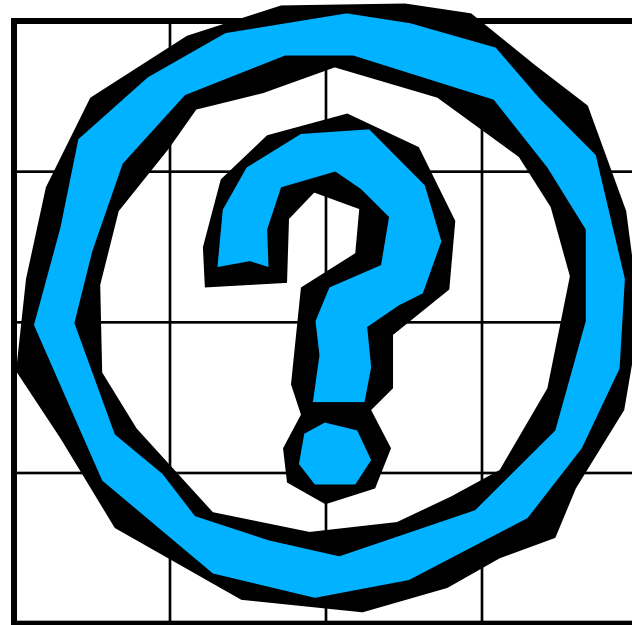
- ★ A  $G(r,c)$ -packing of order  $n$  is said to ***resolvable*** if the collection of grid-blocks can be partitioned into subclasses  $R_1, R_2, \dots, R_t$  such that every vertex of  $K_n$  is contained in precisely one grid-block of each class. Each  $R_i$  is called a ***resolution class***.
- ★ Clearly, such a packing exists only when  **$rc$  divides  $n$** .



# ***A $G(4,4)$ -Packing of order 16***

***Resolvable***

|    |    |    |    |
|----|----|----|----|
| 1  | 2  | 3  | 4  |
| 5  | 8  | 6  | 7  |
| 9  | 10 | 11 | 12 |
| 13 | 15 | 16 | 14 |



# ***A Resolvable $G(3,3)$ - Packing of order 18***

|   |   |   |
|---|---|---|
| 0 | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |

|   |    |    |
|---|----|----|
| 0 | 4  | 8  |
| 5 | 6  | 9  |
| 7 | 10 | 14 |

|    |    |    |
|----|----|----|
| 0  | 9  | 13 |
| 10 | 17 | 8  |
| 12 | 7  | 3  |

|    |    |    |
|----|----|----|
| 9  | 10 | 11 |
| 12 | 13 | 14 |
| 15 | 16 | 17 |

|    |    |    |
|----|----|----|
| 1  | 3  | 15 |
| 12 | 16 | 11 |
| 17 | 2  | 13 |

|    |    |    |
|----|----|----|
| 1  | 5  | 11 |
| 6  | 15 | 2  |
| 16 | 14 | 4  |



# ***A Cyclic $G(2,4)$ -Design***

★  $n = 33$

|    |   |    |    |
|----|---|----|----|
| 0  | 1 | 3  | 9  |
| 12 | 5 | 23 | 28 |



# ***Known Grid-Block Designs***

- ★ A  $G(2,2)$ -design of order  $n$  exists if and only if  $n \equiv 1 \pmod{8}$ . This is also known as the 4-cycle system of order  $n$ .
- ★ A  $G(2,3)$ -design of order  $n$  exists if and only if  $n \equiv 1 \pmod{9}$ . (J. E. Carter, 1989)
- ★ A  $G(3,3)$ -design of order  $n$  exists if and only if  $n \equiv 1$  or  $9 \pmod{36}$ . (Fu-Hwang-Jimbo-Mutoh-Shiue, JSPI 2004)



# $G(3,3)$ -Design

## *Outline of proof.*

1. A  $G(3,3)$ -design of order 9 exists.
2. Let  $p$  be an odd prime and  $v \equiv p \pmod{2p(p-1)}$ . If there exists a cyclic  $(v,p,1)$ -BIBD, then there exists a  $G(p,p)$ -design of order  $pv$ . (F-H-J-M-S)
3. A cyclic  $(12k+3,3,1)$ -BIBD exists.  
 $\Rightarrow$  A  $G(3,3)$ -design of order  $36k+9$  exists.



$$n \equiv 1 \pmod{36}$$

- ★ The proof can be obtained right away if there exists an  $(n,9,1)$ -BIBD of order  $n = 36k+1$ . (See it?)
- ★ The proof can also be obtained by using the existence of a  $(k,s,1)$ -BIBD, a  $G(3,3)$ -design of order 37 and a  $G(3,3)$ -design of  $K_{s(36)}$ . (One vertex in common)



# ***Cyclic Constructions***

- ★ A cyclic  $(72t+1, 3, 1)$ -BIBD and a cyclic  $(72t+37, 3, 1)$ -BIBD exist.
- ★ Use Peltesohn's result (1938), we have five classes of base blocks for  $72t+1$  case:  
(a) for  $x = 0, 1, \dots, 3t-1$ , we have  $(0, 1+2x, 33t+1+x)$ ,  $(0, 9t+1+2x, 27t+1+x)$  and  $(0, 9t+2+2x, 18t+2+x)$ , (b) for  $x = 0, 1, \dots, 3t-2$ , we have  $(0, 2+2x, 24t+2+x)$  and (c)  $(0, 6t, 24t+1)$ .
- ★ Use an imagination to put them together as grid-blocks (mainly by Mutoh).



*One of the base grid-blocks looks like this.*

|           |            |            |
|-----------|------------|------------|
| 0         | $3+2x$     | $33t+2+x$  |
| $9t+3+2x$ | $9t+4+4x$  | $42t+4+3x$ |
| $27t+2+x$ | $27t+4+3x$ | $51t+4+2x$ |

**For  $x = 0, 3, 6, \dots, 3t-6$ .**



# ***4x4 Grid-Blocks***

- ★ By using the following grid-block we obtain a 4x4 grid-block design of order 97. (?)

|    |    |    |    |
|----|----|----|----|
| 0  | 1  | 3  | 11 |
| 9  | 13 | 28 | 60 |
| 14 | 74 | 67 | 40 |
| 76 | 58 | 45 | 17 |

# A $G(4,4)$ -Design of $K_{4(4)}$

- ★ Let the 4 partite sets of  $K_{4(4)}$  be  $\{0,1,2,3\}$ ,  $\{4,5,6,7\}$ ,  $\{8,9,10,11\}$  and  $\{12,13,14,15\}$ . The following two grid-blocks form the design.

|    |    |    |    |
|----|----|----|----|
| 0  | 4  | 8  | 12 |
| 5  | 2  | 14 | 11 |
| 9  | 15 | 3  | 6  |
| 13 | 10 | 7  | 1  |

|    |    |    |    |
|----|----|----|----|
| 0  | 6  | 10 | 14 |
| 7  | 2  | 12 | 9  |
| 11 | 13 | 3  | 4  |
| 15 | 8  | 5  | 1  |

# $n+1 \rightarrow mn+1$ *Construction*

## ★ *Theorem*

If a  $G(r,c)$ -design of  $K_{n+1}$  and  $K_{m(n)}$  exists respectively, then a  $G(r,c)$ -design of  $K_{mn+1}$  exists.

- ★ So, for  $G(4,4)$ -designs, we need a design of  $K_{97}$  to start with and we have constructed earlier. It's left to find a  $G(4,4)$ -design of  $K_{m(96)}$  for proper  $m$ .

## ***Continued ...***

### ★ ***Proposition***

***For  $m \geq 4$ , a  $G(4,4)$ -design of  $K_{m(96)}$  exists.***

### ★ ***Note***

***A 4-GDD of type  $24^m$  exists for each  $m \geq 4$ .***

***Using this fact and a  $G(4,4)$ -design of  $K_{4(4)}$  we can prove the proposition. (?)***



# The Missing Cases

## ★ *Theorem*

*A  $G(4,4)$ -design of  $K_v$  exists if and only if  $v \equiv 1 \pmod{96}$  except possibly  $v = 193$  and  $v = 289$ .*

This is by the reason that the proposition works only for  $m \geq 4$ . *(Too bad!)*



# ***Joint Effort works!***

- ★ With the help from Zhang and Ge from Zhejiang Univ. the missing orders are settled now. Since it is quite complicate to put them up here, I only show you the key array we apply. (This one for  $v = 193$ .)

|    |     |     |     |
|----|-----|-----|-----|
| 0  | 1   | 3   | 7   |
| 5  | 14  | 25  | 39  |
| 35 | 72  | 131 | 62  |
| 82 | 150 | 110 | 183 |



# ***The case $v = 289$***

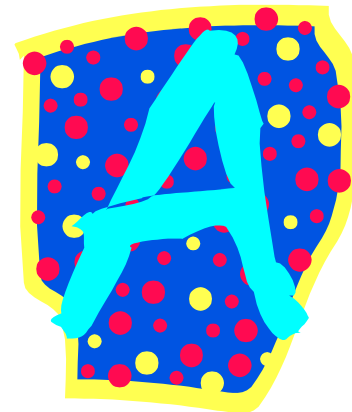
- ★ We also list a useful array here. Since  $289 = 17^2$ , we have a finite field of order 289. Let  $\omega$  be a root of a primitive polynomial  $\omega^2 + \omega + 3$ . Then we can use  $\omega^i$  as its elements. For convenience, we only list the exponents in the following array.

|          |    |     |    |
|----------|----|-----|----|
| $\infty$ | 0  | 1   | 2  |
| 3        | 4  | 5   | 6  |
| 20       | 17 | 155 | 83 |
| 46       | 70 | 221 | 7  |

# ***4x4 Grid-Block Design***

- ★ Combining the works above we have proved the following theorem.
- ★ **Theorem.** A 4x4 grid-block design of order  $v$  exists if and only if  $v$  is congruent to 1 modulo 96.

**This result reserves to have an**



# ***3x4 Grid-Block Design***

- ★ If a 3x4 grid-block design of order  $v$  exists, then  $v \equiv 1, 16, 21, 36 \pmod{60}$ .
- ★ The cases  $v \equiv 1, 21 \pmod{60}$  have been settled. (Zhang, Ge, Fu, Kuo)
- ★ The other two cases remain unsolved.
- ★  $v = 16$  is not possible.
- ★  $v = 36$  and  $76$  are O.K.
- ★ We need more input to settle the remaining cases.



# ***Results on Packings***

- ★ For practical use, we focus on resolvable  $G(r,c)$ -packings. Therefore, we consider only the cases  $rc$  divides  $n$ .
- ★ In a resolvable  $G(r,c)$ -packing the number of resolution classes  $t$  is at most  $\lfloor (v-1)/(r+c-2) \rfloor$ . If a resolvable  $G(r,c)$ -packing has this number of resolution classes, then it is *optimal*.



# ***Optimal Resolvable Packings***

- ★ There exists an optimal resolvable  $G(2,2)$ -packing of order  $n$  for each  $n \equiv 0 \pmod{4}$ . This is also a resolvable maximum 4-cycle packing of order  $n$ .
- ★ An optimal resolvable  $G(q,q)$ -packing of order  $q^m$  exists for a prime power  $q$  and an integer  $m$ . Moreover, when  $m$  is even and  $q$  is odd, we have a resolvable  $G(q,q)$ -design of order  $q^m$ . (M-J-F, 2004)



# ***Ready for Tests?***

- ★ In *DNA library screening*, we have a set of oligonucleotides (*clones*) and a *probe*  $\overline{X}$  which is a short DNA sequence. Let  $\overline{X}$  denote the dual sequence of  $X$  obtained by first reversing the order of letters and then interchanging A with T and C with G. A clone is called *positive* if it contains  $\overline{X}$  as a subsequence and *negative* if not.
- ★ *The goal is to identify all the positive clones.*



# ***Group Testing***

- ★ Economy of time and costs requires that the clones be assayed in groups.
- ★ Each group is called a *pool*.
- ★ A pool gives a negative outcome, all clones contained in it are found to be negative. On the other hand, if a pool is positive, at the second stage we test each clone individually.
- ★ ***Two-stage Test!***



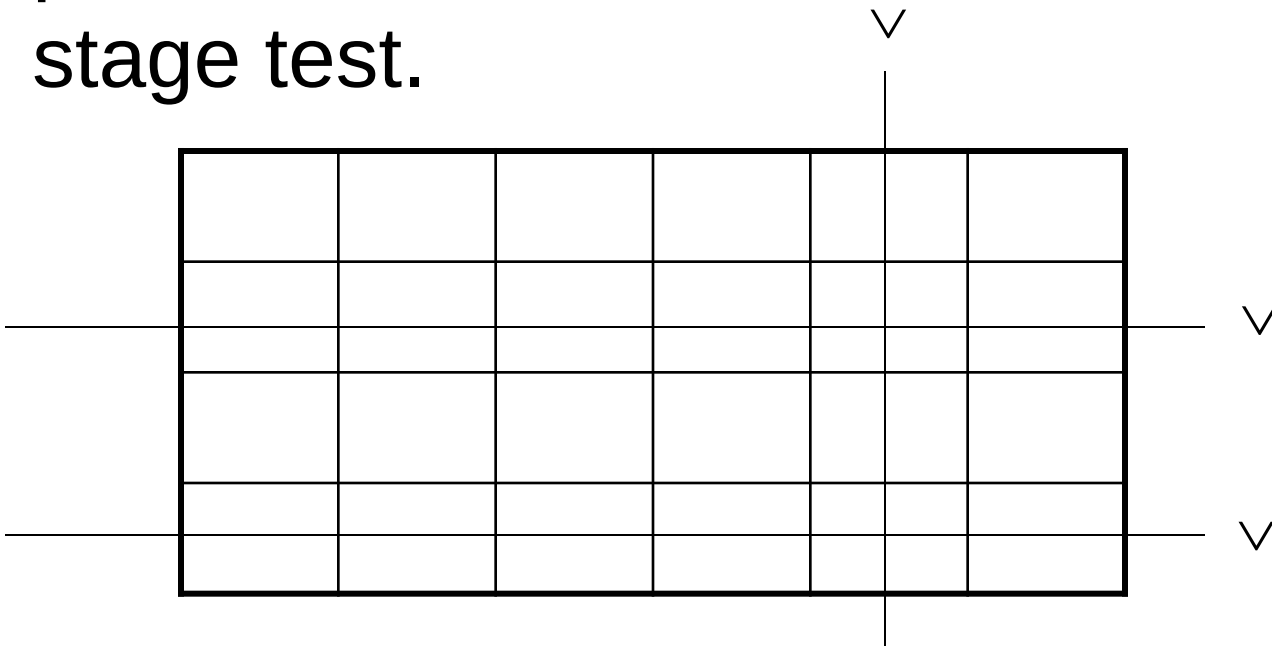
# ***Library Screening***

- ★ In such screening, a *microtiter* plate, which is an array with size 8 x 12 or 16 x 24, etc. is utilized and different clones are settled in each spot, called *well*, of the plate.
- ★ *Every row and every column in a microtiter plate is tested at the same time as a pool in the first stage. ( $r + c$  tests for a plate)*



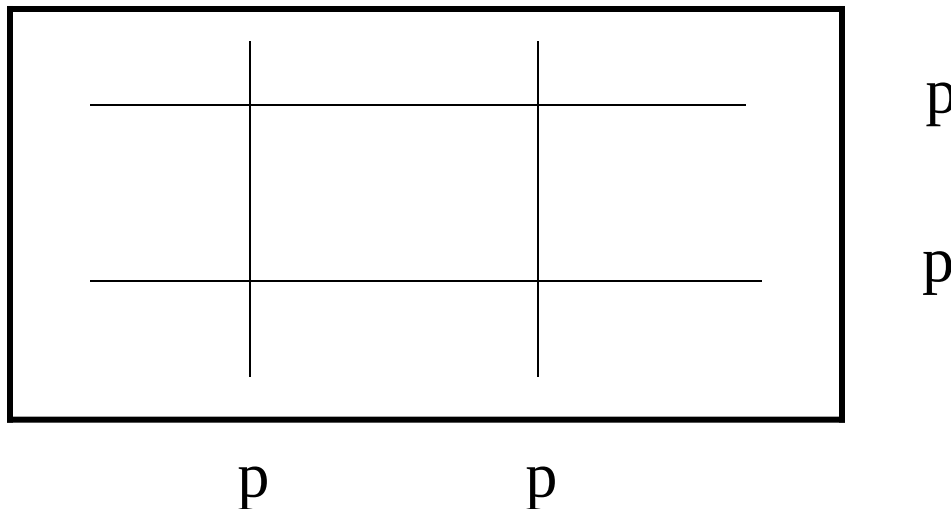
# ***Basic Matrix Method***

- ★ If there is only one *row* (or *column*) of positive then we can determine the positive clones without the second stage test.



# ***More Positive Clones***

- ★ For example if two rows and two columns are positive as follows, then we ***can not determine*** whether the four clones settled at the crossing wells of positives are really positive or not.



# ***Unique Collinearity Condition***

- ★ Thus, if it is allowed to test more than twice for each clone, then it is desired that every two clones occur at most once in the same row or the same column, which is called the *unique collinearity condition (UCC)*.
- ★ The efficiency of *UCC* was shown by Barillot et al (1991, simulation) and proved theoretically by Berger et al in 2000 at *Biometrics*.
- ★ *So, corresponds to grid-block packing.*



# Why Resolvable Packings?

- ★ The *replication number* of a vertex  $v$  in a grid-block packing is the number of grid-blocks in which  $v$  is in there.
- ★ It is a favorite property that the *number of replications of each clone should be almost the same in the first stage*.
- ★ So, take a resolution class and test (group) each grid-block at the same time guarantees the above *equal replication* property.



# ***Analysis of Replications***

- ★ A simulation result of a comparison between *constant replications* and *random replications* shows that we need less tests to find all the positives by using grid-block packings with constant replications.
- ★ As an example, if we have 1,000 clones and 0.1 is the probability of positives, then it takes around **600 tests** (Constant R.) and **750 tests** (Random R.) to find all the positives respectively. (M-J-F, 2004)



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*Thank you for your  
attention.*

謝謝!

