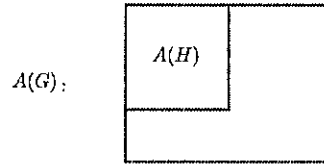


For reference (As Lecture 16)



Theorem 86

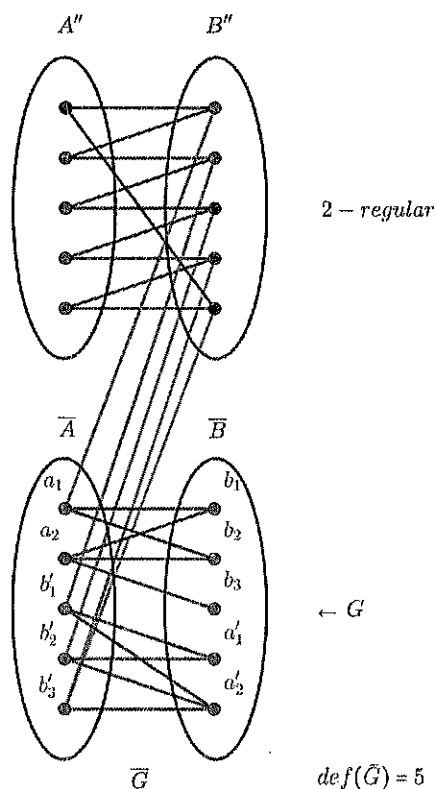
For each bipartite graph G , there exists a graph $\tilde{G}$ such that $G \leq \tilde{G}$ and $\tilde{G}$ is $\Delta(G)$ -regular.

Proof.

Let $G = (A, B)$ with $|A| \leq |B|$. Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. ($m \leq n$) First, we construct a graph $\bar{G} = (\bar{A}, \bar{B}) : (A \cup B', B \cup A')$ where $A' = \{a'_1, a'_2, \dots, a'_m\}$ and $B' = \{b'_1, b'_2, \dots, b'_n\}$ and $E(\bar{G}) = E(G) \cup \{\{b'_j, a'_i\} \mid a_i b_j \in E(G)\}$. (See Figure for example.) In fact, $\langle A \cup B \rangle_{\bar{G}} \cong \langle A' \cup B' \rangle_{\bar{G}}$. Hence, $\sum_{v \in A \cup B'} \deg_{\bar{G}}(v) = \sum_{v \in B \cup A'} \deg_{\bar{G}}(v) = 2\|G\|$, $|\bar{A}| = |\bar{B}|$, $\sum_{v \in A \cup B'} (\Delta(G) - \deg_{\bar{G}}(v)) = \sum_{v \in B \cup A'} (\Delta(G) - \deg_{\bar{G}}(v)) = \text{def}(\bar{G})$. Now, based on $\text{def}(\bar{G})$, we have two cases to consider.

Case 1. $\text{def}(\bar{G}) \geq \Delta(G)$.

Construct a bipartite graph G'' such that $G'' = (A'', B'')$, $|A''| = |B''| = \text{def}(\bar{G})$ and G'' is $(\Delta(G) - 1)$ -regular. The proof then follows by connecting A'' to $\bar{B}$ and B'' to $\bar{A}$.



Case 2. $def(\bar{G}) < \Delta(G)$.

Let $|A''| = |B''| = \Delta(G)$, and $G'' \cong K_{\Delta(G), \Delta(G)} - M$, M is a matching of size $def(\bar{G})$. The proof follows by connecting the vertices in G'' (which are incident to M) and the vertices in $\bar{G}$ whose degrees are less than $\Delta(G)$.

So, the graph $\tilde{G}$ is defined on $(A'' \cup \bar{A}, B'' \cup \bar{B})$. Moreover, $\tilde{G}$ is $\Delta(G)$ -regular with partite set size $\max\{|A| + |B| + def(G), |A| + |B| + \Delta(G)\}$.

Theorem 87

Let G be a planar graph with $\Delta(G) \geq 10$, then G is of Class 1.

Proof.

We shall apply a lemma obtained by Vizing.

Vizing's adjacency lemma

First Form: If G is of Class 2, then every vertex of G is adjacent to at least two major vertices. In particular, G contains at least three major vertices.

Second Form: Let G be a connected graph of Class 2 that is minimal with respect to edge coloring. "If $uv \in E(G)$ and $\deg_G(u) = m$, then v is adjacent to at least $\Delta(G) - m + 1$ major vertices." $\text{---}(\ast)$

We shall apply the 2nd form to prove the theorem. Suppose not. Let G be a counterexample with minimum size. Thus, G is planar, $\Delta(G) = k \geq 10$ and $\chi'(G) = k + 1$. Clearly, G is minimal with respect to chromatic index. Since G is planar, G contains vertices of degree 5 or less, let S be the set of all such vertices. Define $H = G - S$. Again, H is planar, H contains a vertex w such that $\deg_H(w) \leq 5$. By the fact $\deg_G(w) > 5$, w is adjacent to some vertices of S . Let $vw \in E(G)$ where $v \in S$. In G , $\deg_G(v) \leq 5$. By $(\ast)$, w is adjacent to at least $\Delta(G) - 5 + 1 (\geq 6)$ (major) vertices of degree $\Delta(G)$. This implies that w is adjacent to at least 6 vertices of H since all major vertices are in H . Hence, $\deg_H(w) \geq 6$. $\rightarrow\leftarrow$. This concludes the proof. ■

(•) $\sigma(G)$: Vertex cover number; $\alpha_1(G)$: matching number

Theorem 88 (König-Egeváry)

For bipartite graphs G , $\sigma(G) = \alpha_1(G)$.

Proof.

We shall apply max-flow min-cut theorem to prove the theorem.

First, we define a network as in Figure 49. Let $A = \{a_1, a_2, \dots, a_m\}$, $B = \{b_1, b_2, \dots, b_n\}$ and $G = (A, B)$. Then, the network is defined by letting u and v be source and sink respectively, $\text{cap}(u, a_i) = 1$ for $i = 1, 2, \dots, m$, $\text{cap}(b_j, v) = 1$ for $j = 1, 2, \dots, n$, and $\text{cap}(a_i, b_j) = |G| + 1$ if $a_i b_j$ is in $E(G)$. (Note that all arcs are from A to B .)

Since $\sigma(G) \geq \alpha_1(G)$ as mentioned above, it suffices to show that $\alpha_1(G) \geq \sigma(G)$.

Now, let f be a maximum flow. It is easy to see that $\text{val } f = \alpha_1(G)$. This is due to the fact that all the arcs from u and into v are of capacity 1. (No two arcs can be out of two vertices in A and ended in a vertex of B .)

So, it is left to consider the minimum cut, let it be $K = (X, \bar{X})$ where $u \in X$, $v \in \bar{X}$, $A \cap \bar{X} = A'$ and $B \cap \bar{X} = B'$, see Figure 50. Hence, K contains arcs from u to A' , $A \setminus A'$ and $B \setminus B'$, see Figure 50. Hence, K contains arcs from u to A' , $A \setminus A'$ to $B \setminus B'$ and B' to v . Notice that $\text{cap } K \leq |G|$, for example, let $X = \{u\}$. This implies that the following $(\bullet)$ is true and $A' \cup B'$ is a vertex cover and $\text{cap } K = |A'| + |B'| = \text{val } f = \alpha_1(G)$. By the fact, $|A'| + |B'| \geq \sigma(G)$, we have $\alpha_1(G) \geq \sigma(G)$. ■

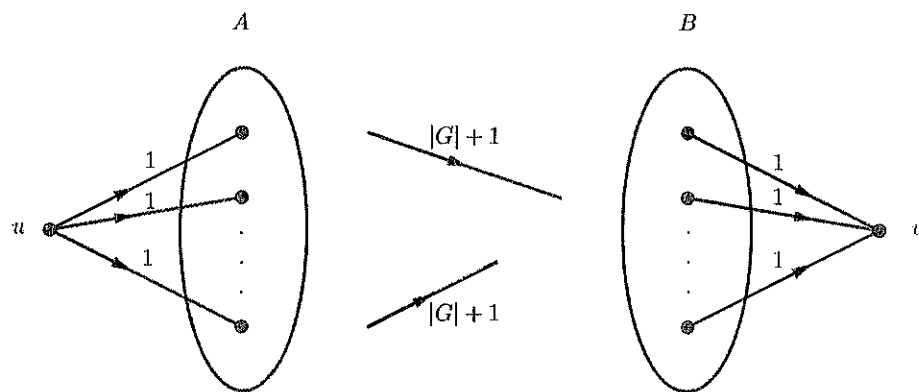


Figure 49. Net work

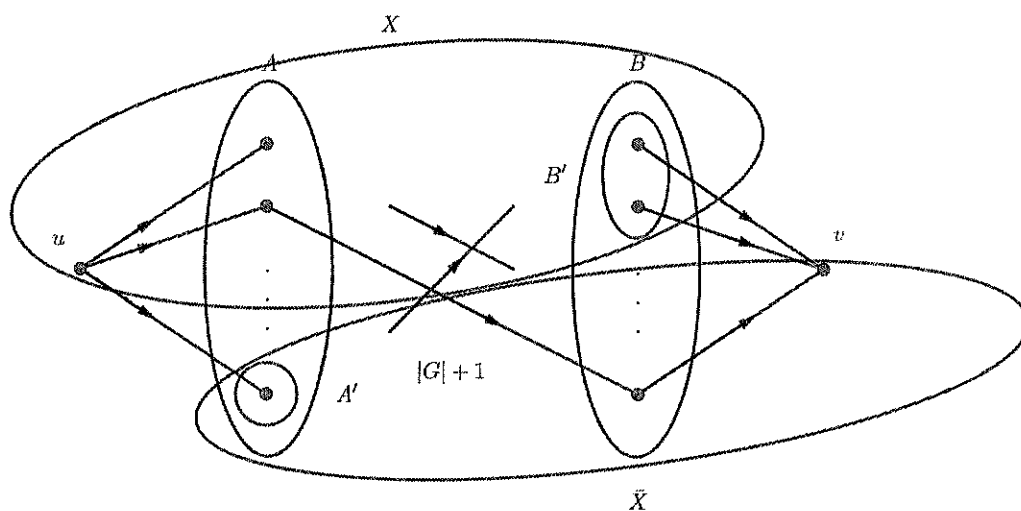


Figure 50. Cut

- (•) In $(X, \bar{X})$, there exist no edges from $A \setminus A'$ to $B \setminus B'$. For otherwise, it is not a minimum cut. This implies that all edges are incident vertices in $A' \cup B'$. $A' \cup B'$ is a vertex cover.

Theorem 89

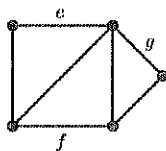
Let G be a graph of order p which has no isolated vertices. Then,
 $\alpha(G) + \sigma(G) = p$.

Proof.

Let S be a vertex cover with $\sigma(G)$ vertices of G . Then, $V(G) \setminus S$ is an independent set. Hence, $|V(G) \setminus S| \leq \alpha(G)$ and thus $p - |S| \leq \alpha(G)$. This implies that $p \leq \alpha(G) + |S| = \alpha(G) + \sigma(G)$. On the other hand, let T be an independent set of G such that $|T| = \alpha(G)$. Then, $G - T$ is vertex cover of G . By the fact $\sigma(G) \text{ (min.)} \leq |G - T| = p - \alpha(G)$, we have $p \geq \sigma(G) + \alpha(G)$. ■

Definition (Edge-cover)

An edge cover of a graph is a set of edges M such that all vertices of G are incident to M , i.e., for each $v \in V(G)$, v is incident to an edge in M . e.g.



$\{e, f, g\}$ is an edge cover.

The edge cover number of G , $\sigma_1(G) = \min \{|M| \mid M \text{ is an edge cover}\}$.

(•) $\sigma_1(G) \geq \left\lceil \frac{|G|}{2} \right\rceil$. (Each edge can cover two vertices.)

Theorem 90

$$\sigma_1(G) + \alpha_1(G) = p. \text{ (} G \text{ is a connected graph.)}$$

Proof.

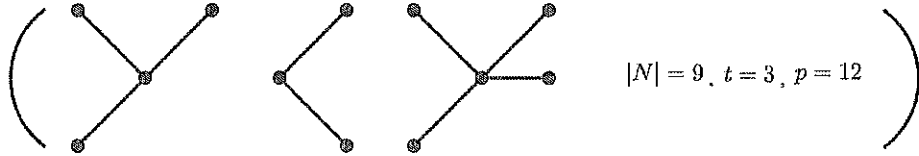
Let M be a matching in G with $\alpha_1(G)$ edges. Then, for each vertex

not in M , v is incident to a vertex in M if v is in an edge of G .

Assume that t vertices not in M , i.e., $p = 2|M| + t$. Now, by taking every edge in the matching M and the set of t edges not in M but incident to M , we have an edge cover with $|M| + t$ edges. This implies that $\sigma_1(G) \leq |M| + t$. As a consequence, we have $p = 2|M| + t = \alpha_1(G) + |M| + t \geq \alpha_1(G) + \sigma_1(G)$. On the other hand, let N be an edge cover of G with minimum number of edges, i.e., $|N| = \sigma_1(G)$. Notice that $\langle N \rangle_G$ is a disjoint union of stars. (You can not find

$$P_4 \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

in $\langle N \rangle_G$.) Assume that there are t stars. Then, $p = |N| + t$. By the fact that in $\langle N \rangle_G$ we can find a matching of size t , $p \leq |N| + \alpha_1(G)$. ($\alpha_1(G)$ (maximum matching number) $\geq t$.) ■



Theorem 90'

Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_p\}$. Then, we can use the vertices of G as variables to obtain a generating function for all dominating sets of G .

Proof.

Let f be defined as follows:

$$f(v_1, v_2, \dots, v_p) = \prod_{i=1}^p (v_i + \sum_{u \in N_G(v_i)} u).$$

Then, each summand is a product $v_1^{c_1}v_2^{c_2}\cdots v_p^{c_p}$ where $0 \leq c_j \leq p$. Now, let $S = \{v_j \mid c_j > 0, j = 1, 2, \dots, p\}$. If $u \in V(G) \setminus S$, say $u = v_k$, then in the product $v_1^{c_1}v_2^{c_2}\cdots v_p^{c_p}$, $c_k = 0$. But, one of its neighbor has been selected. This implies that v_k is incident to a vertex of S . ■

(•) For small order graphs, this is a good way to find dominating sets.

In fact, the term with maximum of "0" in powers provide a dominating set with minimum size and thus the domination number is determined.

Theorem 91

Let G be a (p, q) -graph and $A(G) = A$. Then,

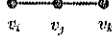
- (1) the number of triangles in G is $\frac{1}{6}tr(A^3)$;
- (2) the number of 4-cycles in G is $\frac{1}{8}[tr(A^4) - 2q - \sum_{i \neq j} a_{i,j}^{(2)}]$ where $a_{i,j}^{(2)}$ is the (i, j) -entry in A^2 ; and
- (3) the number of 5-cycles in G is $\frac{1}{10}[tr(A^5) - 5tr(A^3) - 5 \sum_{i=1}^p \sum_{j=1}^p (a_{i,j} - 2) \cdot a_{i,i}^{(3)}]$.

Proof.

It follows from the fact that the number of walks of length k from v_i to v_j is equal to $A^{(k)}(i, j)$. This can be proved by induction on k .

Hence, if triangles are concerned, then we consider $A^{(3)}(i, i)$, i.e., $tr(A^3)$. Since for each triangle (v_i, v_j, v_k) , there are 6 different ways of 3-walks: $\langle v_i, v_j, v_k, v_i \rangle$, $\langle v_i, v_k, v_j, v_i \rangle$, $\langle v_j, v_i, v_k, v_j \rangle$, $\langle v_k, v_i, v_j, v_k \rangle$ and $\langle v_k, v_j, v_i, v_k \rangle$, the result follows by using $\frac{1}{6}tr(A^3)$. For 4-cycles and

5-cycles, we have to take away those 4-walks (and 5-walks) which are not for cycles, for example



$\Rightarrow \langle v_i, v_j, v_k, v_i \rangle$. Check (2) and (3) yourself. ■

Theorem 92

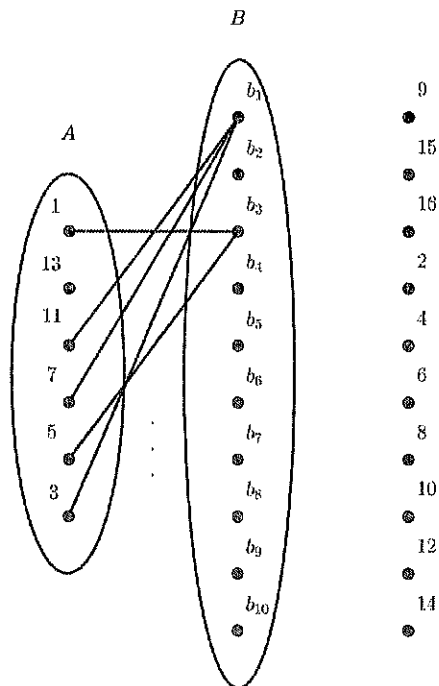
Let $G = (A, B)$ be a tree of order at most 16. Then, G has a prime labeling.

Proof.

(Sketched) We present a proof of the case $|A| = 6$ and $|B| = 10$. First, we label A by using $S = \{1, 13, 11, 7, 5, 3\}$. Then, the labeling of the vertices can be found from the set $\{1, 2, \dots, 16\} \setminus S$. Now, denote $B = \{b_1, b_2, \dots, b_{10}\}$. We claim that there exists a labeling of the vertices of B by using $[16] \setminus S$.

Let $B_i = \{x \in [16] \setminus S \mid \gcd(x, s) = 1 \text{ for each } x \in A \cap N_G(b_i)\}$. Now, consider $|S_k| = |\bigcup_{j=1}^k B_{i_j}|$. Clearly, if $k = 1$, then $|S_k| \geq 1$ since $\gcd(2, s) = 1$ for each $s \in S$. In fact, each B_i contains at least 4 elements, 2, 4, 8 and 16.

But the fact, "any two B_i 's have at most one common neighbor", we can verify that $|S_k| \geq k$ for $6 \leq k \leq 10$. Hence, by Hall's condition, the proof follows. ■



Theorem 93 (Alon, 1995)

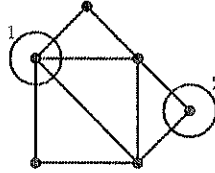
Let G be a graph of order p . Then, $\gamma(G)$ (the domination number) of G , $\gamma(G) \leq n[1 + \ln(\sigma(G) + 1)/(\sigma(G) + 1)]$.

Proof. (Probabilistic Method)

Let S be a subset of $V(G)$ with the probability of each vertex $p =_{\text{def}} \frac{\ln(\sigma(G) + 1)}{\sigma(G) + 1}$. Let $T = \{x \mid x \notin S, N_G(x) \cap S = \emptyset\}$. Since for each $y \notin S \cup T$, $N_G(y) \cap S \neq \emptyset$, $S \cup T$ is a dominating set of G . By the expectation of $\mathbb{E}(S \cup T) = \mathbb{E}(S) + \mathbb{E}(T) \leq np + n \cdot (1 - p)^{\sigma(G) + 1} \leq np + n \cdot e^{-p(\sigma(G) + 1)} = n \left(p + \frac{1}{\sigma(G) + 1} \right)$. This implies that there exists a dominating of size at most $n \cdot [1 + \ln(\sigma(G) + 1)/(\sigma(G) + 1)]$. ■

(•) A greedy algorithm for finding the dominating set:

Choose the vertices of a dominating set one by one following the idea: A vertex that covers the "maximum number" of vertices which are not covered yet is picked.



" p is not a constant."

Theorem 94 (Omit the proof)

Let $p = p(n)$. Then, we have

- (1) $p(n) = n^{-2} \longrightarrow$ No edges.
- (2) $p(n) = n^{-\frac{3}{2}} \longrightarrow G$ has a nontrivial component which grows like a tree.
- (3) $p(n) = n^{-1} \longrightarrow$ Contains a cycle.
- (4) $p(n) = \frac{\ln n}{n} \longrightarrow$ Connected.
- (5) $p(n) = (1 + \varepsilon) \longrightarrow$ Contains a Hamilton cycle.
- (•) The growth rate is getting smaller.

Theorem 95 (More about eigenvalues of $A(G)$)

Let G be a connected graph of order p and A be its adjacency matrix. Then, we have the following basic properties.

- (1) For each eigenvalue λ , $|\lambda| \leq \Delta(G)$.

- (2) $\Delta(G)$ is an eigenvalue of G if and only if G is regular. Moreover, if $\Delta(G)$ is an eigenvalue of G , then the multiplicity of $\Delta(G)$ is 1.
- (3) If $-\Delta(G)$ is an eigenvalue of G , then G is regular and bipartite.
- (4) If G is bipartite and λ is an eigenvalue then $-\lambda$ is also an eigenvalue, moreover, they have the same multiplicity.
- (5) The maximal eigenvalue, $\lambda_{\max}(G)$ satisfies $\delta(G) \leq \lambda_{\max}(G) \leq \Delta(G)$.
- (6) If $H \leq G$, then $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$.

Proof.

- (1) Let $\vec{x}$ be an eigenvector with eigenvalue λ , i.e., $A\vec{x} = \lambda\vec{x}$. Let $\vec{x} = (x_1, x_2, \dots, x_p)$ and $|x_i| \leq 1$ (by re-scaling $\vec{x}$). Suppose that $|x_j| \geq |x_i|$ for each $i = 1, 2, \dots, p$. For convenience, let $x_j = 1$. ($|x_j|$ 最大, 則令 $\vec{x} \leftarrow \vec{x} / \pm |x_j|$.) Then,

$$\begin{aligned}
 |\lambda| &= |\lambda \cdot x_j| \\
 &= \left| \sum_{i=1}^p a_{j,i} \cdot x_i \right| \\
 &\leq \sum_{i=1}^p a_{j,i} \cdot |x_i| \\
 &\leq |x_j| \cdot \deg_G(v_j) \\
 &\leq \Delta(G).
 \end{aligned}$$

■

- (2) If $\Delta(G) = \Delta$ is an eigenvalue, then as in (1), let $|x_j| = 1$, and we have $\Delta = \Delta \cdot x_j = \sum_{i=1}^p a_{j,i} \cdot x_i$. Hence $x_i = x_j = 1$ and $\deg_G(v_j) = \Delta$, whenever $v_i \sim_G v_j$. Therefore, $\deg_G(x_i) = \Delta$. Now, by the same argument, if $v_k \sim v_i$, $\deg_G(v_k) = \Delta$, then G is Δ -regular by the fact

that G is connected. This also implies that the eigenvector is

$\vec{1} = (1, 1, \dots, 1)$. The reverse statement is easy to see.

- (3) If $-\Delta(G)$ is an eigenvalue, then by (2) we have $\deg_G(v_j) = \Delta$ and $x_i = -x_j = -1$ whenever $v_i \sim_G v_j$. Since two vertices are adjacent if they have distinct weights (x_i and x_j) 1 and -1, the vertex set of G can be partitioned into two subset V_1 and V_2 such that $v_i \sim_G v_j$ iff their corresponding weights are different (1 or -1). Hence, G is bipartite.

- (4) It follows by considering $\ker(A - \lambda I_p)$ and $\ker(A + \lambda I_p)$. Let

$G = (V_1, V_2)$ and $\vec{b}(b_1, b_2, \dots, b_p)$ such that $b_i = 1$ if $v_i \in V_1$ and $b_i = -1$ if $v_i \in V_2$. Now, if $A\vec{x} = \lambda\vec{x}$ and $v_i \in V_1$, then

$$\begin{aligned} A \cdot (\vec{b} \otimes \vec{x})_i &= \sum_{j=1}^p a_{ij} \cdot b_j \cdot x_j = \sum_{v_j \in V_1} a_{ij} x_j (= 0(v_i \in V_1)) - \sum_{v_j \in V_2} a_{ij} x_j = \\ &= -\sum_{v_j \in V_1} a_{ij} x_j - \sum_{v_j \in V_2} a_{ij} x_j = -\sum_{j=1}^p a_{ij} x_j = -\lambda x_i = -\lambda (\vec{b} \otimes \vec{x})_i. \\ \vec{b} \otimes \vec{x} &=_{def} (b_1 x_1, b_2 x_2, \dots, b_p x_p) \end{aligned}$$

This implies λ and $-\lambda$ occur the same number of times in solving $A\vec{x} = \lambda\vec{x}$, i.e., $m(\lambda)$ (multiplicity of λ) = $m(-\lambda)$.

- (5) By (1), we have $\lambda_{\max}(G) \leq \Delta(G)$. Now, we claim the other inequality. Let the numerical range of A be $V(A)$, i.e.,

$V(A) = \{ \langle A\vec{x}, \vec{x} \rangle = \vec{x}^t A \vec{x} \mid |\vec{x}| = 1 \}$. Hence, let $\vec{1} = (1, 1, \dots, 1)$ and we have $\frac{1}{p} \langle A\vec{1}, \vec{1} \rangle \in V(A)$.

$$\text{Now, } \lambda_{\max} = \max.V(A) \geq \frac{1}{p} \langle A\vec{1}, \vec{1} \rangle = \frac{1}{p} \sum_{k=1}^p \deg_G(v_k) \geq \delta(G).$$

- (6) Let H be an induced subgraph of order $p-1$, i.e.,

$H = \{ \{v_1, v_2, \dots, v_{p-1}\} \}_G$. Then, $\lambda_{\max}(H) = \langle A'\vec{y}, \vec{y} \rangle$ where

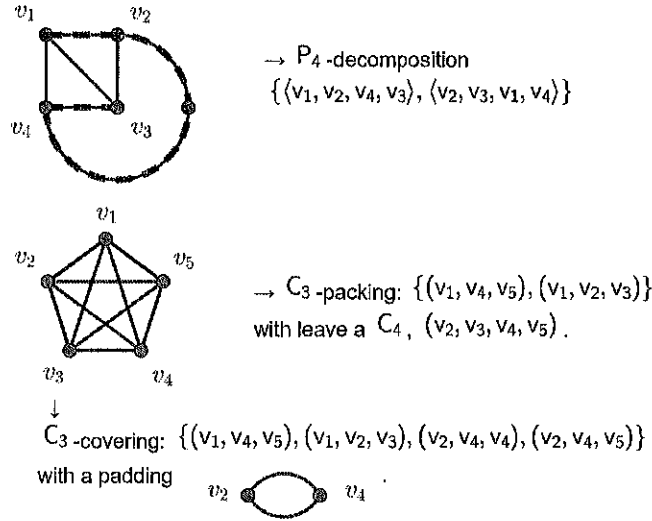
$A' = A(H)$ and $\langle \vec{y}, \vec{y} \rangle = 1$. Now, consider $\vec{x} = (y_1, y_2, \dots, y_{p-1}, 0)$

where $\vec{y} = (y_1, y_2, \dots, y_p)$. Clearly, $\langle A'\vec{x}, \vec{x} \rangle = \langle A'\vec{y}, \vec{y} \rangle = \lambda_{\max}(H)$ and

$\langle \vec{x}, \vec{x} \rangle = 1$. Since $\langle A\vec{x}, \vec{x} \rangle \in V(A)$, i.e., $\lambda_{\max}(H) \in V(H)$. This implies that $\lambda_{\max}(G) \geq \lambda_{\max}(H)$. The other inequalities can be shown similarly. $\blacksquare$

- (•) A graph G has an $\mathcal{H}$ -decomposition of $E(G)$ can be partitioned into subsets $E_1, E_2, \dots, E_k$ such that for each $i = 1, 2, \dots, k$, $\langle E_i \rangle_G \in \mathcal{H}$.
- (•) If $\mathcal{H} = \{H\}$, then an $\mathcal{H}$ -decomposition of G can be referred as an H -decomposition of G .
- (•) A graph G has an H -packing if $E(G)$ contains edge-disjoint subsets such that each of them induces a graph in $\mathcal{H}$. An H -packing of G can be defined accordingly.
- (•) A graph G has an H -covering if $E(G)$ is a subset of a edge-disjoint union of graphs in $\mathcal{H}$. An H -covering of G can be defined as well.
- (••) In packing, the edges not used induce a subgraph which is known as the "leave" of the packing. Similarly in covering, the extra edges used induce a padding of the covering.
- (•) If the graph G is the complete graph K_n , then the H -decomposition, H -packing and H -covering is also referred to as that of order n .

Examples



Theorem 96

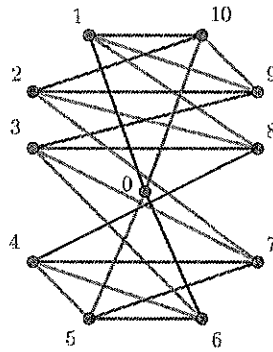
For each odd integer $n \geq 3$, K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles. For each even integer n , K_n can be decomposed into $\frac{n}{2}$ Hamilton paths.

Proof.

The following construction is known as the Waleki's method. Let $V(K_n) = \mathbb{Z}_n$ and the cycles are:

$$\left(0, 1, n-1, 2, n-2, \dots, \frac{n-1}{2}, \frac{n+1}{2}\right), \left(0, 2, 1, 3, n-1, \dots, \frac{n+1}{2}, \frac{n+3}{2}\right),$$

$$\left(0, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n+1}{2}, \frac{n-5}{2}, \dots, n-2, n-1\right). \quad \blacksquare$$



By deleting a vertex in K_{n+1} (n even), we obtain the decomposition of K_n into $\frac{n}{2}$ Hamilton paths. ■

- (*) This theorem has been extended to cycles and paths with prescribed length. We list the theorem and omit their proofs.
(The cycle case is very complicate.)

Theorem 96' (Alspach et. al, 2001)

For each odd integer larger than 3 and an integer $3 \leq m \leq n$, the complete graph K_n (n is odd) and $K_n - I$ (n is even) can be decomposed into m -cycles provided $m \mid \binom{n}{2}$ (for odd n) and $m \mid \binom{n}{2} - \frac{n}{2}$ (for even n) respectively.

- (•) The case $m = 3$ was proved in 1847 by T. P. Kirkman, and the case $m = n$ mentioned above eas obtained long time ago.

- (*) An important tool for decomposition.

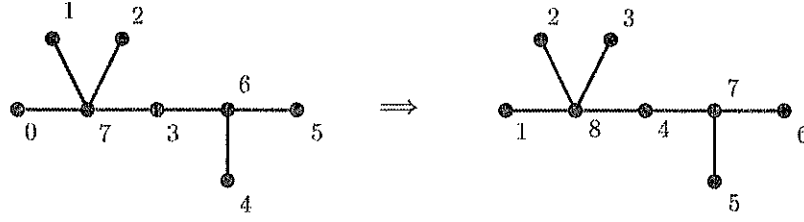
Definition (Graceful labeling, β -labeling)

A graceful labeling of a graph G is a 1-1 mapping $f : V(G) \rightarrow \{0, 1, 2, \dots, \max\{\|G\|, |G| - 1\}\}$ such that the weights of edges uv defined by $|f(u) - f(v)|$ are all distinct.

If G is connected, this *value* takes $\|G\|$.

Examples

(Shown easiler!)



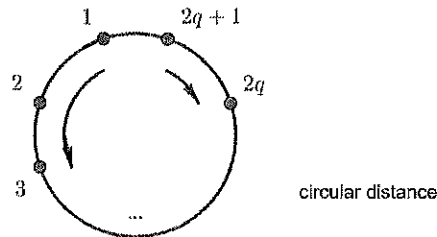
Theorem 97

Let G be a graph of size q and G has a graceful labeling. Then, K_{2q+1} can be decomposed into $2q + 1$ copies of G .

Proof.

Let $V(K_{2q+1}) = \mathbb{Z}_{2q+1}$. By arranging the vertices on a cycle, see Figure below, we notice that any two vertices have a circular distance at most q . More precisely, $\text{dist}(i, j) = \min.\{j - i, (2q + 1) - (j - i)\}$ (for $j > i$).

Now, we can add the labels of G for each one of them (taking modulo $2q + 1$) and obtain the desired decomposition.

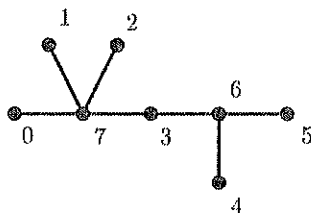


- (•) If G has a graceful labeling f and the labeling has an extra property such that $\exists c_f \in \mathbb{Z}_{2q+1}$ satisfying for each uv either $f(u) \geq c_f > f(v)$ or $f(v) \geq c_f > f(u)$, then G has an α -labeling.

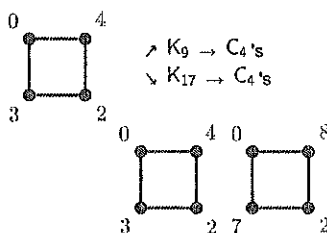
The idea of Theorem 97 is known as "difference method" and G with labeling is a "base graph".

Example

The labeling of the following graph



is an α -labeling since we can choose $c_f = 3$ or 4 or 3.5. The following labeling is also an α -labeling, $c_f = 2.5$.



Theorem 98

If G has an α -labeling and $\|G\| = q$, then $G \mid K_{2tq+1}$ where $t \in \mathbb{N}$.

($G \mid K_{2tq+1}$ denotes K_{2tq+1} has a G -decomposition.)

Proof.

By Theorem 97, $G \mid K_{2q+1}$ can be obtained by a graceful labeling of G . Now, if G has an α -labeling, we may change the labels to find t base graphs for the decomposition of K_{2tq+1} . As mentioned above on the case of C_4 's, for each label larger than c_f , we add $q, 2q, \dots, (t-1)q$ respectively. This gives a collection of t base graphs (with labels). By difference method, we have the proof. (All differences from 1 to tq have been used exactly once.) ■

- (*) If G has an α -labeling, then G must be a bipartite graph. The two partite sets of G are obtained by using the labels, larger than c_f and smaller than c_f respectively.
- (•) A graph G may have β -labeling but not α -labeling.
- (**) One of the most beautiful conjectures on labelings is "Graceful Tree Conjecture": Every tree has a graceful labeling. Of course, you may also conjecture that every tree has an α -labeling (but this is in general not true).
- (•) We have shown that for each graph G there exists a $\Delta(G)$ -regular graph H such that $G \leq H$. In fact, we can say more about this type of supergraph.

Theorem 99

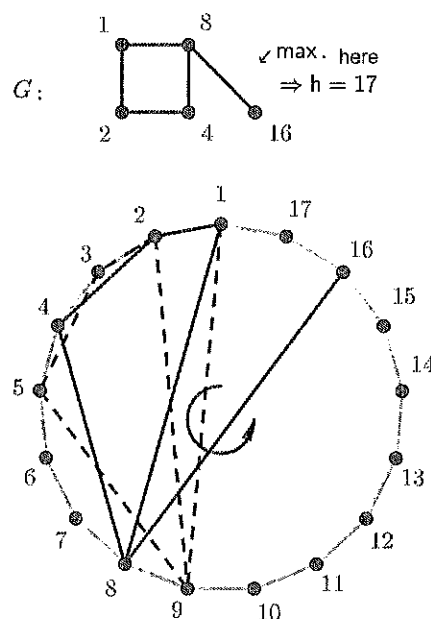
Let G be a graph of size q without isolated vertices. Then, there exists a regular graph H such that $G \mid H$. More precisely, H is a $2q$ -regular graph.

Proof.

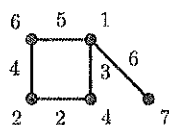
Let $G = \{v_1, v_2, \dots, v_p\}$ and f is a labeling of G such that $f(v_i) = 2^{i-1}$. Then, all edges will receive distinct weights $|f(u) - f(v)|$. We shall construct a $2q$ -regular graph of order h such that $h = 1 + 2 \max\{|f(v_i) - f(v_j)| \mid v_i v_j \in E(G)\}$.

Let $V(H) = \mathbb{Z}_h$ and G be the graph with its vertices the labels from f , see figure for an example. Then, by difference method, the base graph will generate a regular graph which use the weights in G exactly once. Since $\|G\| = q$, the graph obtained H will be a $2q$ -regular graph of order h . ■

(*) We can decrease the order of H by giving another labeling satisfying all $|f(u) - f(v)|$'s are different for $uv \in E(G)$.



Another example



H : 10-regular graph of order 13.

(•) "Graph Decomposition" is one of the most important topic in Graph Theory, many results are also related to the study of Combinatorial Designs.

(*) Problem. For which graph G , $K_3 \mid G$?

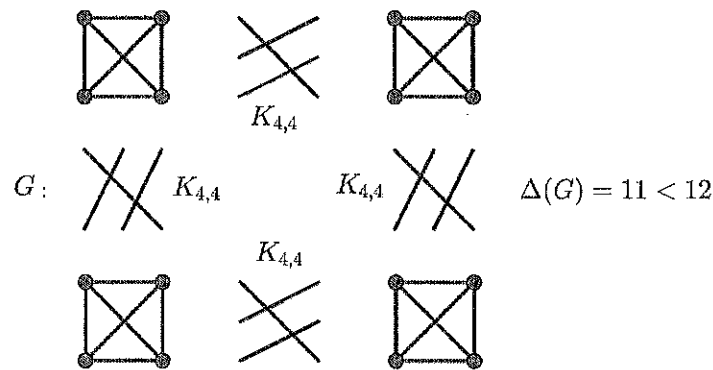
(*) Problem. For which graph G , $C_3 \mid G$?

Theorem 100

There exists a graph G with $\delta(G) < \frac{3}{4}|G|$ such that $K_3 \nmid G$.

Proof.

For general n , the construction is similar to the following graph of order 16.



Since there are four bipartite subgraphs $K_{n,n}$ in G , the K_3 -decomposition needs to use up all these edges by using one from K_n and two from $K_{n,n}$. ($K_{n,n}$ contains no odd cycles!) Hence, we need at least $\frac{1}{2}(4n^2)$ edges from four K_n 's. But, $4K_n$ has $2n(n-1)$ edges which are not enough! The K_3 -decomposition of G is not possible. ■

Nash-Williams Conjecture

For any graph G of order p , G has a K_3 -decomposition provided $\delta(G) \geq \frac{3}{4}p$ and $3 \mid \|G\|$.

How about C_4 -decomposition? Keep moving forward!