

Theorem 83

For $p \in (0, 1)$, almost all graphs obtained in Model A is of Class 1.

Proof.

Step 1.

Prove that almost all graphs have a unique major vertex.

Step 2.

By Vizing's result, every Class 2 graph has at least three major vertices, we conclude the proof. ■

Lecture 15

Some ideas in Algebraic Graph Theorem.

- ✓(•) The adjacency matrix $A(D)$ of a directed graph D is

$$A(D) = [x_{i,j}]_{n \times n} \text{ where } V(D) = \{v_1, v_2, \dots, v_n\}, x_{i,j} = 1 \text{ if and only if } (v_i, v_j) \text{ is an arc of } D.$$
- ✓(•) If D is a graph (instead of digraph), then we view each edge of D as a pair of arcs in opposite directions, and thus $A(D)$ is a symmetric $(0,1)$ -matrix.
- ✓(•) If G is a simple graph, then we can define the *Laplacian* of G , denoted by $L(G) = [l_{i,j}]_{n \times n}$ where $l_{i,i} = \deg_G(v_i)$ and $l_{i,i} = -1 = l_{j,i}$ if $\{v_i, v_j\} \in E(G)$.
- (••) We shall consider $A(G)$ where G is a simple graph in what follows.
- (•) The characteristic polynomial of $A(G)$ is defined as

$$\phi(A, x) =_{\text{def}} \phi(G, x) = \det(xI_n - A). \quad (A \approx A(G)).$$

- (•) The spectrum of A (matrix) is a list of its eigenvalues of A , the zeros of $\phi(A, x)$, together with their multiplicities.

Example

$$G \cong C_4.$$

$$\phi(C_4, x) = x^4 - 4x^2; \text{ zeros are } 2, 0, 0, -2 \text{ (0 is of multiplicity 2)}.$$

- (•) The *largest* eigenvalue of a graph G is called the index of G and the spectral radius of G is the maximum value of $\{|\lambda_i| \mid i = 1, 2, \dots, n, \text{ and } \lambda_i \text{ is an eigenvalue of } A\}$. In C_4 case, the index and spectral radius of G are equal.

$$(•) \text{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{pmatrix} \text{ and } \sum_{i=1}^t m_i = n.$$

Example

$$1. \text{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

$$2. \text{Spec}(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

Theorem 84

The diameter of a connected graph G is less than the number of distinct eigenvalues.

$$\text{diam}(K_n) = 1$$

$$\text{diam}(K_{m,n}) = 2$$

Proof.

Let r be the number of distinct eigenvalues, let them be $\lambda_1, \lambda_2, \dots, \lambda_r$. Then $\prod_{i=1}^r (x - \lambda_i)$ is the minimal polynomial of A . This implies that the linear combination of $I_n = A^0, A^1, \dots, A^r$ is the zero matrix. Now, consider $\text{diam}(G)$. Let $d(v_i, v_j) = \text{diam}(G) = k$. Then, $A^h(i, j) = 0$ for each $0 \leq h < \text{diam}(G)$. Hence, $A^0, A^1, \dots, A^k$ are linearly independent. (?) This implies that $k < r$, i.e., $\text{diam}(G)$ is less than the number of distinct eigenvalues. ■

Theorem 85

For every graph G , $\chi \leq 1 + \lambda_{\max}(G)$.

Proof.

For every graph G , let $\chi(G) = k$ and H be a vertex critical subgraph of G . That is, $\chi(H) = k$ and for each vertex $v \in V(H)$, $\chi(H - v) = k - 1$. By Theorem 56, $\delta(H) \geq k - 1$.

Now, consider $\lambda_{\max}$. If λ is an eigenvalue of $A(G)$, then there exists an eigenvector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Let

$x_j = \max_{i=1}^n \{x_i \mid \vec{x} = (x_1, x_2, \dots, x_n)\}$. Then

$$\lambda x_j = (A\vec{x})_j = \sum_{v_i \in N(v_j)} x_i \leq \deg_G(v_j) \cdot x_j \leq \Delta(G) \cdot x_j.$$

Hence, $\lambda_{\max} \leq \Delta(G)$ and Theorem 85 is an improvement of Brooks

Thm. On the other hand, $\lambda_{\max} \geq \delta(G)$. (?) This implies that

$$k \leq 1 + \delta(H) \leq 1 + \lambda_{\max}(H) \leq 1 + \lambda_{\max}(G). \quad \blacksquare$$

Cayley graph

In mathematics, a **Cayley graph**, also known as a **Cayley color graph**, **Cayley diagram**, **group diagram**, or **color group**,^[1] is a graph that encodes the abstract structure of a group. Its definition is suggested by Cayley's theorem (named after Arthur Cayley), and uses a specified set of generators for the group. It is a central tool in combinatorial and geometric group theory. The structure and symmetry of Cayley graphs make them particularly good candidates for constructing expander graphs.

Definition

Let G be a group and S be a generating set of G . The Cayley graph $\Gamma = \Gamma(G, S)$ is an edge-colored directed graph constructed as follows:^[2]

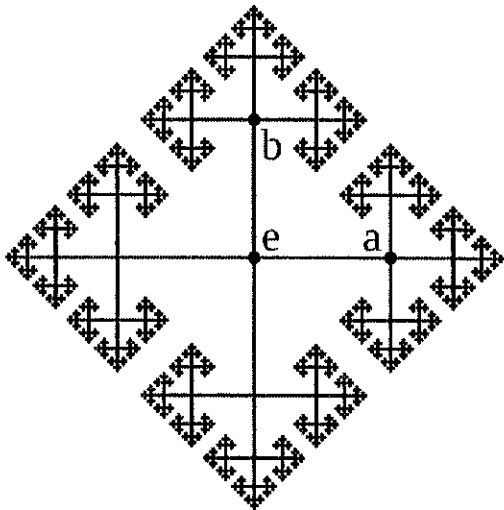
- Each element g of G is assigned a vertex: the vertex set of Γ is identified with G .
- Each element s of S is assigned a color c_s .
- For every $g \in G$ and $s \in S$, there is a directed edge of color c_s from the vertex corresponding to g to the one corresponding to gs .

Not every convention requires that S generate the group. If S is not a generating set for G , then Γ is disconnected and each connected component represents a coset of the subgroup generated by S .

If an element s of S is its own inverse, $s = s^{-1}$, then it is typically represented by an undirected edge.

The set S is often assumed to be finite, especially in geometric group theory, which corresponds to Γ being locally finite and G being finitely generated.

The set S is sometimes assumed to be symmetric ($S = S^{-1}$) and not containing the group identity element. In this case, the uncolored Cayley graph can be represented as a simple undirected graph.

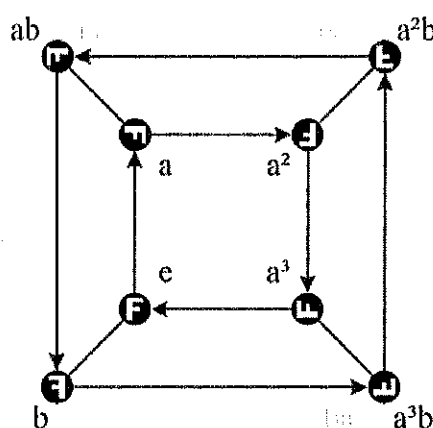


The Cayley graph of the free group on two generators a and b

Graph families defined by their automorphisms		
distance-transitive	→ distance-regular	← strongly regular
↓		
symmetric (arc-transitive)	← t -transitive, $t \geq 2$	skew-symmetric
↓		
(if connected) vertex- and edge-transitive	→ edge-transitive and regular	→ edge-transitive
↓	↓	↓
vertex-transitive	→ regular	→ (if bipartite) biregular
↑		
Cayley graph	← zero-symmetric	asymmetric

Examples

- Suppose that $G = \mathbb{Z}$ is the infinite cyclic group and the set S consists of the standard generator 1 and its inverse (-1 in the additive notation); then the Cayley graph is an infinite path.
- Similarly, if $G = \mathbb{Z}_n$ is the finite cyclic group of order n and the set S consists of two elements, the standard generator of G and its inverse, then the Cayley graph is the cycle C_n . More generally, the Cayley graphs of finite cyclic groups are exactly the circulant graphs.
- The Cayley graph of the direct product of groups (with the cartesian product of generating sets as a generating set) is the cartesian product of the corresponding Cayley graphs.^[3] Thus the Cayley graph of the abelian group $\mathbb{Z}^2$ with the set of generators consisting of four elements $(\pm 1, 0)$, $(0, \pm 1)$ is the infinite grid on the plane $\mathbb{R}^2$, while for the direct product $\mathbb{Z}_n \times \mathbb{Z}_m$ with similar generators the Cayley graph is the $n \times m$ finite grid on a torus.



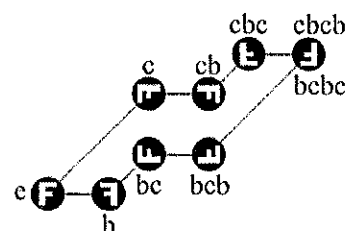
Cayley graph of the dihedral group D_4 on two generators a and b

- A Cayley graph of the dihedral group D_4 on two generators a and b is depicted to the left. Red arrows represent composition with a . Since b is self-inverse, the blue lines, which represent composition with b , are undirected. Therefore the graph is mixed: it has eight vertices, eight arrows, and four edges. The Cayley table of the group D_4 can be derived from the group presentation

$$\langle a, b \mid a^4 = b^2 = e, ab = ba^3 \rangle.$$

A different Cayley graph of D_4 is shown on the right. b is still the horizontal reflection and is represented by blue lines, and c is a diagonal reflection and is represented by pink lines. As both reflections are self-inverse the Cayley graph on the right is completely undirected. This graph corresponds to the presentation

$$\langle b, c \mid b^2 = c^2 = e, bcbc = cbcb \rangle.$$



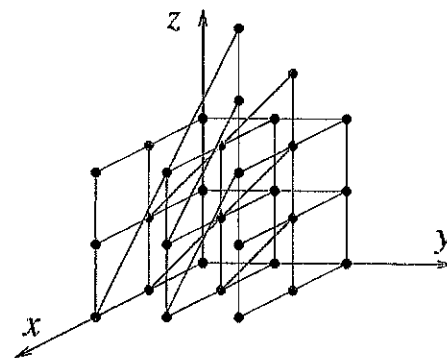
Cayley graph of D_4 , on two generators which are both self-inverse

- The Cayley graph of the free group on two generators a and b corresponding to the set $S = \{a, b, a^{-1}, b^{-1}\}$ is depicted at the top of the article, with e being the identity. Travelling along an edge to the right represents right multiplication by a , while travelling along an edge upward corresponds to the multiplication by b . Since the free group has no relations, the Cayley graph has no cycles: it is the 4-regular infinite tree. It is a key ingredient in the proof of the Banach–Tarski paradox.
- More generally, the Bethe lattice or Cayley tree is the Cayley graph of the free group on n generators. A presentation of a group G by n generators corresponds to a surjective homomorphism from the free group on n generators to the group G , defining a map from the Cayley tree to the Cayley graph of G . Interpreting graphs topologically as one-dimensional simplicial complexes, the simply connected infinite tree is the universal cover of the Cayley graph; and the kernel of the mapping is the fundamental group of the Cayley graph.
- A Cayley graph of the discrete Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{Z} \right\}$$

is depicted to the right. The generators used in the picture are the three matrices X, Y, Z given by the three permutations of 1, 0, 0 for the entries x, y, z . They satisfy the relations

$Z = XYX^{-1}Y^{-1}, XZ = ZX, YZ = ZY$, which can also be understood from the picture. This is a non-commutative infinite group, and despite being embedded in a three-dimensional space, the Cayley graph has four-dimensional volume growth.^[4]



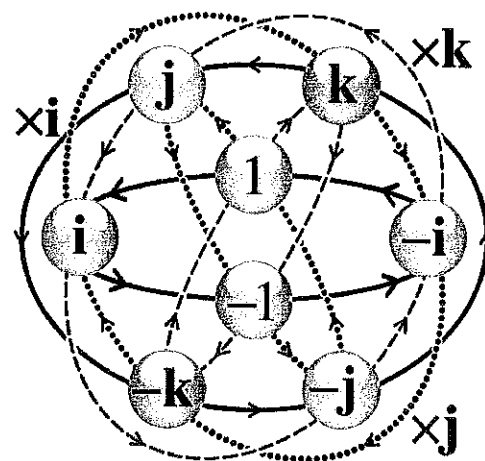
Part of a Cayley graph of the Heisenberg group. (The coloring is only for visual aid.)

Characterization

The group G acts on itself by left multiplication (see Cayley's theorem). This may be viewed as the action of G on its Cayley graph. Explicitly, an element $h \in G$ maps a vertex $g \in V(\Gamma)$ to the vertex $hg \in V(\Gamma)$. The set of edges of the Cayley graph and their color is preserved by this action: the edge (g, gs) is mapped to the edge (hg, hgs) , both having color c_s . In fact, all automorphisms of the colored directed graph Γ are of this form, so that G is isomorphic to the symmetry group of Γ .^{[note 1][note 2]}

The left multiplication action of a group on itself is simply transitive, in particular, Cayley graphs are vertex-transitive. The following is a kind of converse to this:

Sabidussi's Theorem—An (unlabeled and uncolored) directed graph Γ is a Cayley graph of a group G if and only if it admits a simply transitive action of G by graph automorphisms (i.e., preserving the set of directed edges).^[5]



Cayley Q8 graph showing cycles of multiplication by quaternions i, j and k

To recover the group G and the generating set S from the unlabeled directed graph Γ , select a vertex $v_1 \in V(\Gamma)$ and label it by the identity element of the group. Then label each vertex v of Γ by the unique element of G that maps v_1 to v . The set S of generators of G that yields Γ as the Cayley graph $\Gamma(G, S)$ is the set of labels of out-neighbors of v_1 . Since Γ is uncolored, it might have more directed graph automorphisms than the left multiplication maps, for example group automorphisms of G which permute S .

Elementary properties

- The Cayley graph $\Gamma(G, S)$ depends in an essential way on the choice of the set S of generators. For example, if the generating set S has k elements then each vertex of the Cayley graph has k incoming and k outgoing directed edges. In the case of a symmetric generating set S with r elements, the Cayley graph is a regular directed graph of degree r .
- Cycles (or closed walks) in the Cayley graph indicate relations among the elements of S . In the more elaborate construction of the Cayley complex of a group, closed paths corresponding to

relations are "filled in" by polygons. This means that the problem of constructing the Cayley graph of a given presentation $\mathcal{P}$ is equivalent to solving the Word Problem for $\mathcal{P}$.^[1]

- If $f : G' \rightarrow G$ is a surjective group homomorphism and the images of the elements of the generating set S' for G' are distinct, then it induces a covering of graphs

$$\bar{f} : \Gamma(G', S') \rightarrow \Gamma(G, S),$$

where $S = f(S')$. In particular, if a group G has k generators, all of order different from 2, and the set S consists of these generators together with their inverses, then the Cayley graph $\Gamma(G, S)$ is covered by the infinite regular tree of degree $2k$ corresponding to the free group on the same set of generators.

- For any finite Cayley graph, considered as undirected, the vertex connectivity is at least equal to $2/3$ of the degree of the graph. If the generating set is minimal (removal of any element and, if present, its inverse from the generating set leaves a set which is not generating), the vertex connectivity is equal to the degree. The edge connectivity is in all cases equal to the degree.^[6]
- If $\rho_{\text{reg}}(g)(x) = gx$ is the left-regular representation with $|G| \times |G|$ matrix form denoted $[\rho_{\text{reg}}(g)]$, the adjacency matrix of $\Gamma(G, S)$ is $A = \sum_{s \in S} [\rho_{\text{reg}}(s)]$.
- Every group character χ of the group G induces an eigenvector of the adjacency matrix of $\Gamma(G, S)$. The associated eigenvalue is

$$\lambda_{\chi} = \sum_{s \in S} \chi(s),$$

which, when G is Abelian, takes the form

$$\sum_{s \in S} e^{2\pi i js / |G|}$$

for integers $j = 0, 1, \dots, |G| - 1$. In particular, the associated eigenvalue of the trivial character (the one sending every element to 1) is the degree of $\Gamma(G, S)$, that is, the order of S . If G is an Abelian group, there are exactly $|G|$ characters, determining all eigenvalues. The corresponding orthonormal basis of eigenvectors is given by

$v_j = \frac{1}{\sqrt{|G|}} (1 \quad e^{2\pi i j / |G|} \quad e^{2 \cdot 2\pi i j / |G|} \quad e^{3 \cdot 2\pi i j / |G|} \quad \dots \quad e^{(|G|-1)2\pi i j / |G|})$. It is interesting to note that this eigenbasis is independent of the generating set S .

More generally for symmetric generating sets, take $\rho_1, \dots, \rho_k$ a complete set of irreducible representations of G , and let $\rho_i(S) = \sum_{s \in S} \rho_i(s)$ with eigenvalue set $\Lambda_i(S)$. Then the set of eigenvalues of $\Gamma(G, S)$ is exactly $\bigcup_i \Lambda_i(S)$, where eigenvalue λ appears with multiplicity $\dim(\rho_i)$ for each occurrence of λ as an eigenvalue of $\rho_i(S)$.

Schreier coset graph

If one instead takes the vertices to be right cosets of a fixed subgroup H , one obtains a related construction, the Schreier coset graph, which is at the basis of coset enumeration or the Todd–Coxeter process.

Connection to group theory

Knowledge about the structure of the group can be obtained by studying the adjacency matrix of the graph and in particular applying the theorems of spectral graph theory. Conversely, for symmetric generating sets, the spectral and representation theory of $\Gamma(G, S)$ are directly tied

together: take $\rho_1, \dots, \rho_k$ a complete set of irreducible representations of G , and let $\rho_i(S) = \sum_{s \in S} \rho_i(s)$ with eigenvalues $\Lambda_i(S)$. Then the set of eigenvalues of $\Gamma(G, S)$ is exactly $\bigcup_i \Lambda_i(S)$, where eigenvalue λ appears with multiplicity $\dim(\rho_i)$ for each occurrence of λ as an eigenvalue of $\rho_i(S)$.

The genus of a group is the minimum genus for any Cayley graph of that group.^[7]

Geometric group theory

For infinite groups, the coarse geometry of the Cayley graph is fundamental to geometric group theory. For a finitely generated group, this is independent of choice of finite set of generators, hence an intrinsic property of the group. This is only interesting for infinite groups: every finite group is coarsely equivalent to a point (or the trivial group), since one can choose as finite set of generators the entire group.

Formally, for a given choice of generators, one has the word metric (the natural distance on the Cayley graph), which determines a metric space. The coarse equivalence class of this space is an invariant of the group.

Expansion properties

When $S = S^{-1}$, the Cayley graph $\Gamma(G, S)$ is $|S|$ -regular, so spectral techniques may be used to analyze the expansion properties of the graph. In particular for abelian groups, the eigenvalues of the Cayley graph are more easily computable and given by $\lambda_\chi = \sum_{s \in S} \chi(s)$ with top eigenvalue equal to $|S|$, so we may use Cheeger's inequality to bound the edge expansion ratio using the spectral gap.

Representation theory can be used to construct such expanding Cayley graphs, in the form of Kazhdan property (T). The following statement holds:^[8]

If a discrete group G has Kazhdan's property (T), and S is a finite, symmetric generating set of G , then there exists a constant $c > 0$ depending only on G, S such that for any finite quotient Q of G the Cayley graph of Q with respect to the image of S is a c -expander.

For example the group $G = \mathrm{SL}_3(\mathbb{Z})$ has property (T) and is generated by elementary matrices and this gives relatively explicit examples of expander graphs.

Integral classification

An integral graph is one whose eigenvalues are all integers. While the complete classification of integral graphs remains an open problem, the Cayley graphs of certain groups are always integral. Using previous characterizations of the spectrum of Cayley graphs, note that $\Gamma(G, S)$ is integral iff the eigenvalues of $\rho(S)$ are integral for every representation ρ of G .

Cayley integral simple group

A group G is Cayley integral simple (CIS) if the connected Cayley graph $\Gamma(G, S)$ is integral exactly when the symmetric generating set S is the complement of a subgroup of G . A result of Ahmady, Bell, and Mohar shows that all CIS groups are isomorphic to $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z}$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$ for primes p .^[9] It is important that S actually generates the entire group G in order for the Cayley graph to be connected. (If S does not generate G , the Cayley graph may still be integral, but the complement of S is not necessarily a subgroup.)

In the example of $G = \mathbb{Z}/5\mathbb{Z}$, the symmetric generating sets (up to graph isomorphism) are

- $S = \{1, 4\}$: $\Gamma(G, S)$ is a 5-cycle with eigenvalues $2, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}$
- $S = \{1, 2, 3, 4\}$: $\Gamma(G, S)$ is K_5 with eigenvalues $4, -1, -1, -1, -1$

The only subgroups of $\mathbb{Z}/5\mathbb{Z}$ are the whole group and the trivial group, and the only symmetric generating set S that produces an integral graph is the complement of the trivial group. Therefore $\mathbb{Z}/5\mathbb{Z}$ must be a CIS group.

The proof of the complete CIS classification uses the fact that every subgroup and homomorphic image of a CIS group is also a CIS group.^[9]

Cayley integral group

A slightly different notion is that of a Cayley integral group G , in which every symmetric subset S produces an integral graph $\Gamma(G, S)$. Note that S no longer has to generate the entire group.

The complete list of Cayley integral groups is given by $\mathbb{Z}_2^n \times \mathbb{Z}_3^m, \mathbb{Z}_2^n \times \mathbb{Z}_4^n, Q_8 \times \mathbb{Z}_2^n, S_3$, and the dicyclic group of order 12, where $m, n \in \mathbb{Z}_{\geq 0}$ and Q_8 is the quaternion group.^[9] The proof relies on two important properties of Cayley integral groups:

- Subgroups and homomorphic images of Cayley integral groups are also Cayley integral groups.
- A group is Cayley integral iff every connected Cayley graph of the group is also integral.

Normal and Eulerian generating sets

Given a general group G , a subset $S \subseteq G$ is normal if S is closed under conjugation by elements of G (generalizing the notion of a normal subgroup), and S is Eulerian if for every $s \in S$, the set of elements generating the cyclic group $\langle s \rangle$ is also contained in S . A 2019 result by Guo, Lytkina, Mazurov, and Revin proves that the Cayley graph $\Gamma(G, S)$ is integral for any Eulerian normal subset $S \subseteq G$, using purely representation theoretic techniques.^[10]

The proof of this result is relatively short: given S an Eulerian normal subset, select $x_1, \dots, x_t \in G$ pairwise nonconjugate so that S is the union of the conjugacy classes $\text{Cl}(x_i)$. Then using the characterization of the spectrum of a Cayley graph, one can show the eigenvalues of $\Gamma(G, S)$ are given by $\left\{ \lambda_\chi = \sum_{i=1}^t \frac{\chi(x_i) |\text{Cl}(x_i)|}{\chi(1)} \right\}$ taken over irreducible characters χ of G . Each eigenvalue λ_χ in this set must be an element of $\mathbb{Q}(\zeta)$ for ζ a primitive m^{th} root of unity (where m must be divisible by the orders of each x_i). Because the eigenvalues are algebraic integers, to show they are integral it suffices to show that they are rational, and it suffices to show λ_χ is fixed under

any automorphism σ of $\mathbb{Q}(\zeta)$. There must be some k relatively prime to m such that $\sigma(\chi(x_i)) = \chi(x_i^k)$ for all i , and because S is both Eulerian and normal, $\sigma(\chi(x_i)) = \chi(x_j)$ for some j . Sending $x \mapsto x^k$ bijects conjugacy classes, so $\text{Cl}(x_i)$ and $\text{Cl}(x_j)$ have the same size and σ merely permutes terms in the sum for λ_χ . Therefore λ_χ is fixed for all automorphisms of $\mathbb{Q}(\zeta)$, so λ_χ is rational and thus integral.

Consequently, if $G = A_n$ is the alternating group and S is a set of permutations given by $\{(12i)^{\pm 1}\}$, then the Cayley graph $\Gamma(A_n, S)$ is integral. (This solved a previously open problem from the Kourovka Notebook.) In addition when $G = S_n$ is the symmetric group and S is either the set of all transpositions or the set of transpositions involving a particular element, the Cayley graph $\Gamma(G, S)$ is also integral.

History

Cayley graphs were first considered for finite groups by Arthur Cayley in 1878.^[2] Max Dehn in his unpublished lectures on group theory from 1909–10 reintroduced Cayley graphs under the name Gruppenbild (group diagram), which led to the geometric group theory of today. His most important application was the solution of the word problem for the fundamental group of surfaces with genus ≥ 2 , which is equivalent to the topological problem of deciding which closed curves on the surface contract to a point.^[11]

See also

- Vertex-transitive graph
- Generating set of a group
- Lovász conjecture
- Cube-connected cycles
- Algebraic graph theory
- Cycle graph (algebra)

1. Proof: Let $\sigma : V(\Gamma) \rightarrow V(\Gamma)$ be an arbitrary automorphism of the colored directed graph Γ , and let $h = \sigma(e)$ be the image of the identity. We show that $\sigma(g) = hg$ for all $g \in V(\Gamma)$, by induction on the edge-distance of g from e . Assume $\sigma(g) = hg$. The automorphism σ takes any c_g -colored edge $g \rightarrow gs$ to another c_g -colored edge $\sigma(g) \rightarrow \sigma(gs)$. Hence $\sigma(gs) = \sigma(g)s = hgs$, and the induction proceeds. Since Γ is connected, this shows $\sigma(g) = hg$ for all $g \in V(\Gamma)$.
2. It is easy to modify Γ into a simple graph (uncolored, undirected) whose symmetry group is isomorphic to G : replace colored directed edges of Γ with appropriate trees corresponding to the colors.

Notes

1. Magnus, Wilhelm; Karrass, Abraham; Solitar, Donald (2004) [1966]. *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations* (<https://books.google.com/books?id=1LW4s1RDRHQC&pg=PR2>). Courier. ISBN 978-0-486-43830-6.

2. Cayley, Arthur (1878). "Desiderata and suggestions: No. 2. The Theory of groups: graphical representation" ([https://babel.hathitrust.org/cgi/pt?id=uc1.\\$c239465;view=1up;seq=194](https://babel.hathitrust.org/cgi/pt?id=uc1.$c239465;view=1up;seq=194)). *American Journal of Mathematics*. **1** (2): 174–6. doi:10.2307/2369306 (<https://doi.org/10.2307%2F2369306>). JSTOR 2369306 (<https://www.jstor.org/stable/2369306>). In his *Collected Mathematical Papers* 10: 403–405.
3. Theron, Daniel Peter (1988). *An extension of the concept of graphically regular representations* (Ph.D. thesis). University of Wisconsin, Madison. p. 46. MR 2636729 (<https://mathscinet.ams.org/mathscinet-getitem?mr=2636729>).
4. Bartholdi, Laurent (2017). "Growth of groups and wreath products". In Ceccherini-Silberstein, Tullio; Salvatori, Maura; Sava-Huss, Ecaterina (eds.). *Groups, graphs and random walks: Selected papers from the workshop held in Cortona, June 2–6, 2014*. London Math. Soc. Lecture Note Ser. Vol. 436. Cambridge Univ. Press, Cambridge. pp. 1–76. arXiv:1512.07044 (<https://arxiv.org/abs/1512.07044>). ISBN 978-1-316-60440-3. MR 3644003 (<https://mathscinet.ams.org/mathscinet-getitem?mr=3644003>).
5. Sabidussi, Gert (October 1958). "On a class of fixed-point-free graphs" (<https://doi.org/10.1090%2Fs0002-9939-1958-0097068-7>). *Proceedings of the American Mathematical Society*. **9** (5): 800–4. doi:10.1090/s0002-9939-1958-0097068-7 (<https://doi.org/10.1090%2Fs0002-9939-1958-0097068-7>). JSTOR 2033090 (<https://www.jstor.org/stable/2033090>).
6. See Theorem 3.7 of Babai, László (1995). "27. Automorphism groups, isomorphism, reconstruction" (<http://people.cs.uchicago.edu/~laci/handbook/handbookchapter27.pdf>) (PDF). In Graham, Ronald L.; Grötschel, Martin; Lovász, László (eds.). *Handbook of Combinatorics* (<https://books.google.com/books?id=5Y9NCwIx63IC>). Vol. 1. Elsevier. pp. 1447–1540. ISBN 9780444823465.
7. White, Arthur T. (1972). "On the genus of a group" (<https://doi.org/10.1090%2FS0002-9947-1972-0317980-2>). *Transactions of the American Mathematical Society*. **173**: 203–214. doi:10.1090/S0002-9947-1972-0317980-2 (<https://doi.org/10.1090%2FS0002-9947-1972-0317980-2>). MR 0317980 (<https://mathscinet.ams.org/mathscinet-getitem?mr=0317980>).
8. Proposition 1.12 in Lubotzky, Alexander (2012). "Expander graphs in pure and applied mathematics" (<https://doi.org/10.1090%2FS0273-0979-2011-01359-3>). *Bulletin of the American Mathematical Society*. **49**: 113–162. arXiv:1105.2389 (<https://arxiv.org/abs/1105.2389>). doi:10.1090/S0273-0979-2011-01359-3 (<https://doi.org/10.1090%2FS0273-0979-2011-01359-3>).
9. Ahmady, Azhvan; Bell, Jason; Mohar, Bojan (2014). "Integral Cayley graphs and groups". *SIAM Journal on Discrete Mathematics*. **28** (2): 685–701. arXiv:1307.6155 (<https://arxiv.org/abs/1307.6155>). doi:10.1137/130925487 (<https://doi.org/10.1137%2F130925487>). S2CID 207067134 (<https://api.semanticscholar.org/CorpusID:207067134>).
10. Guo, W.; Lytkina, D.V.; Mazurov, V.D.; Revin, D.O. (2019). "Integral Cayley graphs" (<https://link.springer.com/content/pdf/10.1007/s10469-019-09550-2.pdf>) (PDF). *Algebra and Logic*. **58** (4): 297–305. arXiv:1808.01391 (<https://arxiv.org/abs/1808.01391>). doi:10.1007/s10469-019-09550-2 (<https://doi.org/10.1007%2Fs10469-019-09550-2>). S2CID 209936465 (<https://api.semanticscholar.org/CorpusID:209936465>).
11. Dehn, Max (2012) [1987]. *Papers on Group Theory and Topology*. Springer-Verlag. ISBN 978-1461291077. Translated from the German and with introductions and an appendix by John Stillwell, and with an appendix by Otto Schreier.

External links

- Cayley diagrams (<http://www.weddslist.com/groups/cayley-plat/index.html>)
- Weisstein, Eric W. "Cayley graph" (<https://mathworld.wolfram.com/CayleyGraph.html>). *MathWorld*.

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