

Lecture 14 (前面的部份以前介绍过)

111

(•••) The (original) conjecture was disproved by using Theorem 75.

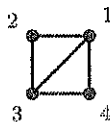
(•) The graph $G(1, 2n - 3)$ is known as "Chen and Fu graph".

(•) Prime Labeling

A prime labeling of a graph G is a mapping

$f: V(G) \xrightarrow{1-1} \{1, 2, \dots, n\}$ such that if $u, v \in E(G)$, then $f(u)$ and $f(v)$ are relatively prime, i.e., $\gcd(f(u), f(v)) = 1$.

Example



(•) $P(a, b)$: The of primes between integers a and b . $P(a, b]$, $P[a, b]$ and $P[a, b)$ can be defined accordingly.

(*) Bertrand's Postulate: (Bertrand-Chebyshev Theorem) For each $n \geq 2$, $P(n, 2n) \neq \emptyset$.

(•) $\alpha(G)$: Independence number of G

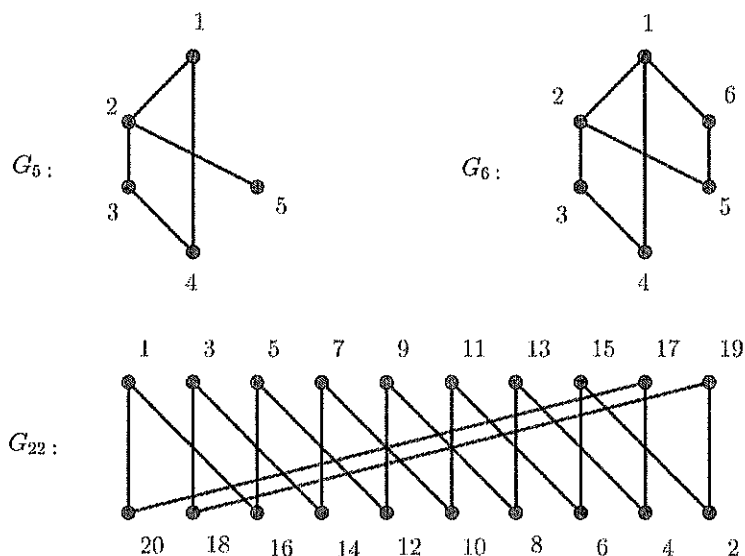
$\sigma(G)$: Vertex cover number of G

(S is a vertex cover of G if every edge e of $E(G)$ is incident at least one vertex of S .) $\Rightarrow V(G) \setminus S$ is an independent set.

Theorem 76

Let G be a graph of order n . Then the followings hold.

- (•) The prime sum graph of order n , G_n , is defined on $[n]$ and two vertices i and j in $[n]$ are adjacent if $i + j$ is a prime.



A part of edges, 23,19,41 are primes used for them.

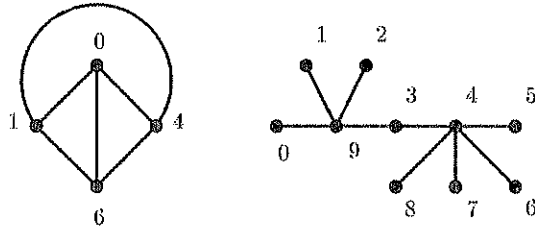
(1,22,19,4,15,8,11,12,7,16,3,20,21,2,17,6,13,10,9,14,5,18) $\leftarrow$ Hamilton cycle.

- (••) So, we are interested in determining "for which n , G_n has a Hamilton cycle?". Clearly, n must be even. (?)

- (*) It is not difficult to check, for small $n \geq 6$, G_n is indeed hamiltonian.

(•) Graceful Labelings

A graceful labeling of a graph G (with $|G| \leq \|G\| + 1$) is a mapping $f : V(G) \xrightarrow{1-1} \{0, 1, 2, \dots, |G|\}$ such that $\{|f(u) - f(v)| \mid uv \in E(G)\} = \{1, 2, \dots, \|G\|\}$.

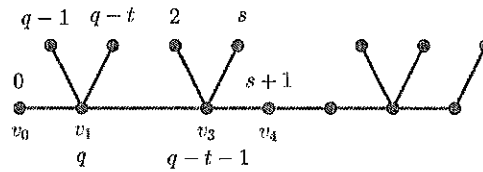


Theorem 77

Any caterpillar has a graceful labeling.

Proof.

Starting from the end vertex from one side, label the vertex v_0 with 0, then the vertex v_1 incident to v_0 is labeled with $\|G\| = q$. Now, the neighbors of v_1 are labeled by 2, 3, $\dots$ for pendant vertices and let the largest label be used in v_3 . Again, we shall start from the use of $q-1, q-2, \dots$ and let the smallest label for v_4 to use. The proof then follows. ■



(*) There are special trees which have a graceful labeling. But, to prove that all trees do have graceful labelings remains unsolved.

Graceful Tree Conjecture (Rigell-Kotzig)

All trees are graceful.

(*) We remark here, for forests, we can also find graceful labelings.

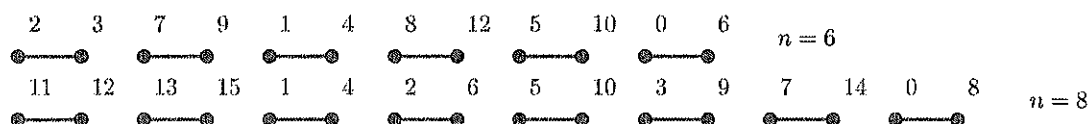
Theorem 77'

Each matching of size n , M_n , has a graceful labeling.

Proof.

This is a direct consequence of using Skolem sequences of order n .

(See a couple of examples below.) ■



(**) There are many problems in graph labelings, you may refer to the following reference for more informations.

A dynamic survey of graph labeling by J. A. Gallian.

(502 pages!)



Lecture 14

Random Graphs

(*) The notation of random graphs is different from probabilistic method. Here are two theorems which uses probabilistic method.
(Find the lower bound of $R(s)$ is another example.)

Theorem 78

There exists a tournament T_n such that T_n has at least $n!/2^{n-1}$ directed Hamilton paths.

Proof.

In K_n , there are $n!$ Hamilton paths. (Starting from one of the vertices which have been selected.) Now, if each edge is assigned with an orientation, the probability for any one Hamilton path to be a directed Hamilton path is $\frac{1}{2^{n-1}}$. (each edge has $\frac{1}{2}$ chance to be in the right direction.) So, if χ is the random variable for the number of directed Hamilton paths, $E(\chi)$ (Expectation) $= n! \cdot \frac{1}{2^{n-1}}$ and therefore, there exists a tournament satisfying this value. ■

Theorem 79

The independence number of G , $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)}$.

Proof.

Let f be a random labeling of G by using $1, 2, \dots, |G|$. For convenience, let $V(G) = \{v_1, v_2, \dots, v_p\}$ and f is a 1-1 mapping from $V(G)$ onto $\{1, 2, \dots, p\}$. Now, for each v , there exists a unique $u \in N_G[v]$ such that $f(u) = \min \{f(x) \mid x \in N_G[v]\}$. Now, let S be the set of vertices v in $V(G)$ such that $f(v) = \min \{f(x) \mid x \in N_G[v]\}$. That's is v has the smallest label, then put v in S .

Now, clearly S is an independent set. (?) Moreover, the probability of being the smallest label among all its neighbors is $\frac{1}{\deg_G(v) + 1}$, we conclude the proof since $\alpha(G) \geq |S| = \sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)}$. ■

Random Graphs

There are different models, here we consider one of the most popular one.

Model A $G(n, p), 0 \leq p \leq 1$.

The probability of the existence of an edge (independently) is p and the graph induced by using existent edges is G_p .

(•) We use G^n to denote the distribution of graphs of order n . Let q_n be the probability of the existence of "property" Q when the graphs considered are of order n .

(••) If $\lim_{n \rightarrow \infty} q_n = 1$, then we say "Q" almost always holds. In this case, we say almost all graphs have property "Q".

Theorem 80 (Gilbert, 1959)

Let p be a constant such that $0 < p \leq 1$. Then, almost all graphs are connected.

Proof.

Suppose not. Then, there are graphs G (of order n) which are not connected. Here, there exists a proper subset $S \subseteq V(G)$ such that $\langle S, V(G) \setminus S \rangle_G$ contains no edges. This implies that the probability q_n of the existence of disconnected graphs of order n satisfies

$$\begin{aligned}
0 &\leq q_n \\
&\leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (1-p)^{k(n-k)} \cdot p^c \quad (c \text{ is a constant}) \\
&\leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} n^k (1-p)^{k(n-k)} \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(n \cdot (1-p)^{n-k} \right)^k \\
&< \frac{y}{1-y} \quad (\text{where } y = (n \cdot (1-p)^{n-k})).
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} (n(1-p)^{n-k}) = 0$, $\lim_{n \rightarrow \infty} \frac{y}{1-y} = 0$. Hence, $\lim_{n \rightarrow \infty} q_n = 0$ and thus almost all graphs are connected. $\blacksquare$

- (•) It is not difficult to see the connectedness should be quite strong, not only 1-connected. We can in fact claim that any cut set of size $k-1$ is not available for a fixed k .

Following the arguments from Theorem 80, we are able to show, if n is large enough, then a random graph obtained from constant p will give a graph with high connectedness.

Theorem 81

For every constant $p \in (0, 1)$ and $k \in \mathbb{N}$, almost all graphs are k -connected.

Proof.

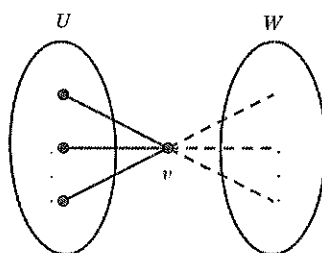
First, we claim G^p has property " $P_{i,j}$ ". ($P_{i,j}$ is the property that for any disjoint vertex sets U and W with $|U| \leq i$ and $|W| \leq j$, there exists a vertex $v \notin U \cup W$ such that $U = N_G(v)$ but $W \cap N_G(v) = \emptyset$. (See

Figure below.) Let $q = (1 - p)$. Then, the probability of such v is $p^{|U|} \cdot q^{|W|} \geq p^i \cdot q^j$. ($|U| \leq i$ and $|W| \leq j$)

Hence, the probability of no such v exists is

$$(1 - p^{|U|} \cdot q^{|W|})^{n-|U|-|W|} \leq (1 - p^i \cdot q^j)^{n-i-j} \text{ for } n \geq i + j.$$

Now, there are at most n^{i+j} $\langle U, W \rangle$ pairs, and thus the probability of " $P_{i,j}$ " (not $P_{i,j}$) is at most $n^{i+j} \cdot (1 - p^i q^j)^{n-i-j}$. By the fact i and j are constants, we have the probability "0" when $n \rightarrow \infty$.



To prove the theorem, let $i = 2$ and $j = k - 1$. Since almost all graphs G have property $P_{2,k-1}$. Let W be an arbitrary set of $k - 1$ vertices. Then, for any two vertices $x, y \in V(G) \setminus W$, then either x is adjacent to y or x and y have a common neighbor. Therefore, W is not a vertex cut of size $k - 1$. This implies that G is k -connected.

Not only the graph is with high connectivity, the graph does have very small diameter.

■ *(x and y are connected by a path.)*

Theorem 82

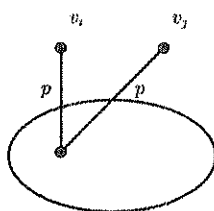
Almost all graphs are of diameter 2.

Proof.

Let $\chi_{i,j}$ be the indicator random variable such that

$$\chi_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ do not have a common neighbor; and} \\ 0 & \text{otherwise.} \end{cases}$$

So, the random variable χ of "no two vertices have a common neighbor is equal to $\sum_{j \neq i} \chi_{i,j}$. Hence,
 $\mathbb{E}(\chi) = \sum_{j \neq i} \mathbb{E}(\chi_{i,j}) = \binom{n}{2} \cdot (1 - p^2)^{n-2} \rightarrow 0$ and we conclude that almost every pair of distinct vertices have a common neighbor. ■



- (*) We may replace p by $p(n)$. Then, we have a more complicate situation to consider the random graphs.
- (**) We may also consider the probability of edges based on the vertices they are incident to. That is, the next edge will come from somewhere near a vertex with larger degree.

Problem

Show that almost all graphs there is a unique vertex with maximum degree.

- (•••) If this is true, then almost all graphs are of Class 1, since Vizing did prove that a Class 2 graph contains at least three major vertices (degree $\Delta(G)$).

For convenience in referring this result, we put it as a theorem.

Theorem 83

For $p \in (0, 1)$, almost all graphs obtained in Model A is of Class 1.

Proof.

Step 1.

Prove that almost all graphs have a unique major vertex.

Step 2.

By Vizing's result, every Class 2 graph has at least three major vertices, we conclude the proof. ■

Some ideas in Algebraic Graph Theorem.

- (•) The adjacency matrix $A(D)$ of a directed graph D is
 $A(D) = [x_{i,j}]_{n \times n}$ where $V(D) = \{v_1, v_2, \dots, v_n\}$, $x_{i,j} = 1$ if and only if (v_i, v_j) is an arc of D .
- (•) If D is a graph (instead of digraph), then we view each edge of D as a pair of arcs in opposite directions, and thus $A(D)$ is a symmetric $(0,1)$ -matrix.
- (•) If G is a simple graph, then we can define the *Laplacian* of G , denoted by $L(G) = [l_{i,j}]_{n \times n}$ where $l_{i,i} = \deg_G(v_i)$ and $l_{i,i} = -1 = l_{j,i}$ if $\{v_i, v_j\} \in E(G)$.
- (••) We shall consider $A(G)$ where G is a simple graph in what follows.
- (•) The characteristic polynomial of $A(G)$ is defined as