

- (•) This Theorem is not difficult to prove, but it is very useful in the construction of "Combinatorial Designs".
- (••) Equivalently, K_{2m} can be decomposed into $2m - 1$ 1-factors, which is also known as a 1-factorization of K_{2m} .
- (•••) If G is an r -regular graph and $\chi'(G) = r$, then G has a 1-factorization.
- (***) It was conjectured that if G is r -regular and $r \geq \frac{|G|}{2}$, then G has a 1-factorization or equivalently $\chi'(G) = r$.

Theorem 71' (D. Hoffman et al.)

A complete multipartite graph G is of Class 2 if and only if G is overfull.

Lecture 13

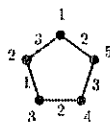
(•) "Total coloring"

A k -total coloring of a graph G is a mapping

$\varphi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that

- (i) adjacent vertices receive distinct images,
- (ii) incident edges receive distinct images, and
- (iii) each vertex has a distinct image with the images of its incident edges.

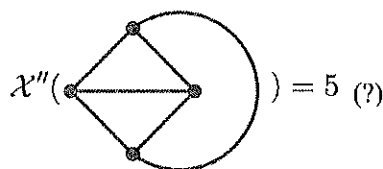
e.g.

A 4-total coloring of C_5

(•) $\chi''(G) = \min.\{k \mid G \text{ has a } k\text{-total coloring}\}$. (Total chromatic number)

Theorem 73

$$\chi''(K_{2n+1}) = \chi''(K_{2n}) = 2n + 1.$$



Proof.

$\chi''(K_{2n+1})$ can be obtained by using $\chi'(K_{2n+1}) = 2n + 1$.

TCC Conjecture $\chi''(G) \leq \Delta(G) + 2$

Note that $\chi''(G) \geq \Delta(G) + 1$. As to the total coloring of K_{2n} , we claim that $2n$ colors are not enough.

Observe that each color class has at most one vertex and $n - 1$ edges. So, $2n$ color classes will contain at most $2n$ vertices and $2n(n - 1)$ edges. Hence, there are $2n^2$ elements (vertices and edges) in total. But, K_{2n} has $2n + \frac{2n(2n - 1)}{2}$ elements to color, which is $2n^2 + n$. Clearly, $2n$ color is not enough. Since K_{2n+1} is $(2n + 1)$ -total colorable, K_{2n} is also $(2n + 1)$ -total colorable. The proof follows. ■

(•) Based on TCC Conjecture, a graph G is called Type 1 if

$$\chi''(G) = \Delta(G) + 1 \text{ and Type 2 otherwise.}$$

Theorem 74

$K_{m,n}$ is of Type 1 if and only if $m \neq n$.

Proof.

($\Rightarrow$) If $m = n$, then there are $2n + n^2$ elements to color. Since each color class contains at most n elements, $\Delta(G) + 1 = n + 1$ colors are not enough, $n(n + 1) < 2n + n^2$. Hence, $\chi''(K_{n,n}) \geq n + 2$, and thus $K_{n,n}$ is not of Type 1.

($\Leftarrow$) On the other direction, let $m = n + k$. Now, $\Delta(K_{m,n}) = n + k$. By the edge-coloring of $K_{m,n}$, we have an $n \times (n + k)$ Latin rectangle based on $\{1, 2, \dots, n + k\}$, see Figure 47. Since $k \geq 1$, we may extend this rectangle to $(n + 1) \times (n + k)$ and the last row can be used to color the vertices of A . Finally, color all vertices of B by one extra color, we have $K_{m,n} \leq n + k + 1 = \Delta(K_{m,n}) + 1$. ■

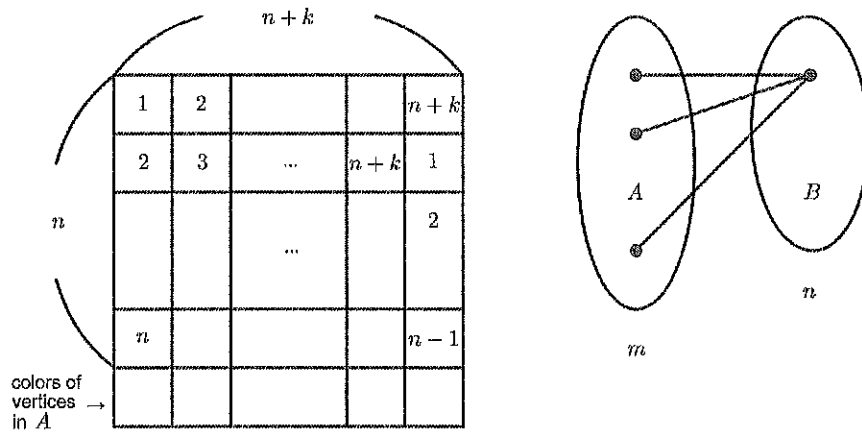
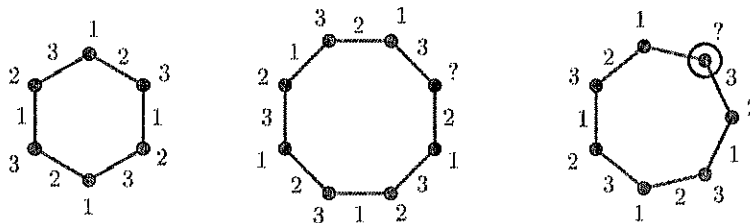


Figure 47. An edge-coloring of $K_{m,n}$

(•) A cycle C_n is of Type 1 if and only if $n \equiv 0 \pmod{6}$.



If we use three colors, then starting from one vertex and one edge, all the colors of the others are forced!

(•) The deficiency of a graph G , $def(G)$, is defined as

$$\sum_{v \in V(G)} (\Delta(G) - \deg_G(v)).$$

(••) G is *conformable* if G has a vertex coloring

$\varphi: V(G) \rightarrow \{1, 2, \dots, \Delta(G) + 1\}$ such that

$$def(G) \geq |\{i \mid |G| - |\varphi^{-1}(i)| \equiv 1 \pmod{2}\}|.$$

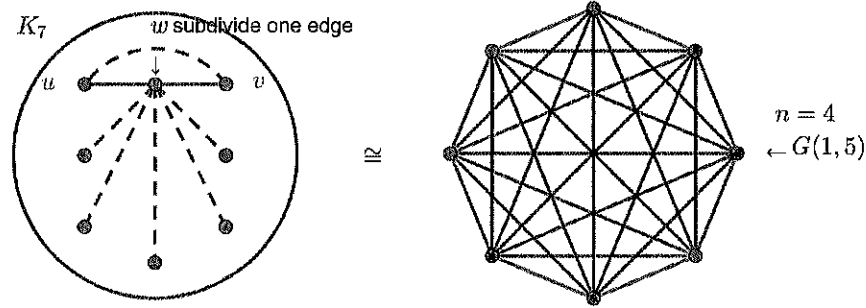
Theorem 75

Let S_i be a star with i edges. Then, $K_{2n} - S_1 - S_{2n-3} = G(1, 2n-3)$ is of Type 2.

Proof.

Assume that $G(1, 2n-3)$ is of Type 1, i.e., there exists a total coloring ψ of $G(1, 2n-4)$ using $\Delta(G) + 1 = 2n - 1$ colors. Let uv be the edge subdivided by w , see Figure 48. First, if $\psi(u) = \psi(v) = 0$, then let $\psi(w) = 1$, $\psi(uw) = 2$ and $\psi(wv) = 3$. Let r_j the number of vertices v in which j occurs in either v or an edge incident to v . Hence,

$\sum_{j=0}^{2n-2} r_j = |K_{2n}| + 2\|G(1, 2n-3)\|$. (?) Now, 0 occurs in at most $2n-2$ vertices. Hence, $\sum_{j=0}^{2n-2} r_j \leq (2n-2) + 3 \cdot (2n) + (2n-5) \cdot (2n-1) = 4n^2 - 4n + 3 < |K_{2n}| + 2\|G(1, 2n-3)\|$. $\rightarrow \leftarrow$

Figure 48. $G(1, 5)$

On the other hand, if $\psi(u) \neq \psi(v)$, a similar argument shows that $2n-1$ colors are not enough. Hence, $G(1, 2n-3)$ is of Type 2. ■

(*) If G is of Type 1, then G is conformable.

Proof.

If there exists a color i such that $|G| - |\psi_{V(G)}^{-1}(i)| \equiv 1 \pmod{2}$, then i occurs in at most $|G| - 1$ vertices. Since every color occurs around a major vertex (Δ -degree), $\Delta(G) + 1$ colors are not enough if $\text{def}(G) < |\{i \mid |G| - |\psi_{V(G)}^{-1}(i)| \equiv 1 \pmod{2}\}|$. So, if G is not conformable, then G is not of Type 1. ■

Conjecture (Chetwynd and Hilton, 1988)

Let G be a simple graph with $\Delta(G) \geq \left\lfloor \frac{|G|+1}{2} \right\rfloor$. Then, G is of Type 2 if and only if there exists a non-conformable subgraph H of G such that $\Delta(H) = \Delta(G)$ (and G is not the Chen and Fu graph). ←後來加入.