

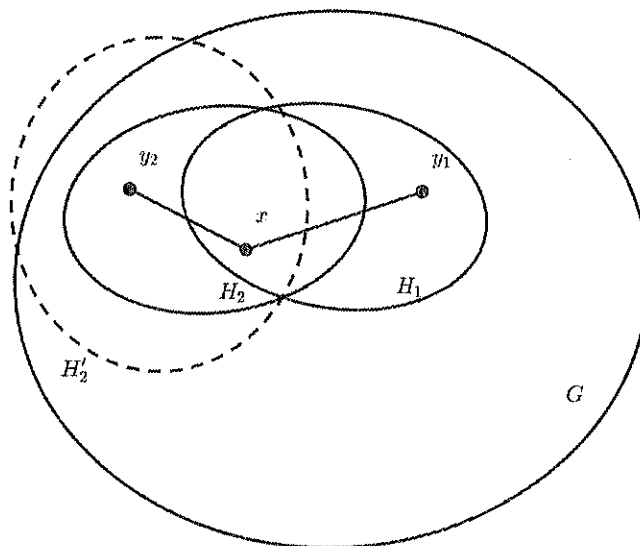


Figure 42.

Lecture 12

- (•) A k -edge-coloring is a mapping $\pi : E(G) \rightarrow \{1, 2, \dots, k\}$ such that incident edges receive distinct images (colors).
- (•) $\chi'(G) = \min \{k \mid G \text{ has a } k\text{-edge-coloring}\}$. (Chromatic index of G .) If $\chi'(G) = k$, then G is h -edge-colorable for each $h \geq k$.

Theorem 67 (Vizing, 1964)

If G is a simple graph, then $\Delta(G) \leq \chi' \leq \Delta(G) + 1$.

Proof.

The left hand inequality is easy to see, we prove the right hand inequality. By induction on $\|G\|$. We shall prove that G has a $(\Delta(G) + 1)$ -edge-coloring (coloring in short) for G and the assertion is true for smaller sizes, i.e., for each $e \in E(G)$, $G - e$ has a coloring π .

First, we observe that since each vertex v is of degree at most $\Delta(G)$, a color is missing around v . Second, if α and β be two colors

used in the coloring, then α and β induce a subgraph with components either paths or even cycles.

Finally, if " G has no colorings using $\Delta(G) + 1$ colors", then for each edge xy and any coloring of $G - xy$, there exists an $\alpha - \beta$ path from y ends in x provided α is missing at x and β is missing at y . See Figure 43 for missing colors.

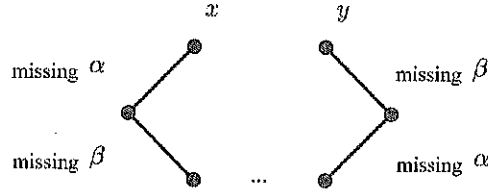


Figure 43.

(*) If $\alpha - \beta$ path does not connect x and y , then we may recolor one of the path (α, β) , to obtain a coloring of G using $\Delta(G) + 1$ colors.

Clearly, if x and y are missing the same color, then we can use that color to color xy and obtain a $\Delta(G) + 1$ coloring of G . $\rightarrow \leftarrow$

Claim: There is a way to recolor some edges in $G - xy$ such that x and y miss the same color.

Outline of proof.

Let $M(y)$ denote the colors missing at y , and $c_1 \in M(y)$. Now, consider $M(x)$. If $c_1 \in M(x)$, then color xy by c_1 results in a $\Delta(G) + 1$ coloring of G . (The claim holds.) Hence, $c_1 \notin M(x)$, let $c_0 \in M(x)$ and $\pi(xy_1) = c_1$. (See Figure 44) Then, consider $M(y_1)$ and let $c_2 \in M(y_1)$. Note that $c_2 \notin M(x)$. If $c_2 \in M(x)$, then we let $\pi(xy_1) = c_2$. Thus, c_1 becomes a missing color in $M(x)$, the coloring for xy is available,

$\pi(xy) = c_1$. This fact will continue: $c_2 \notin M(x) \Rightarrow \exists y_2, s.t. \pi(xy_2) = c_2$; and then $c_3 \in M(y_2)$, $\pi(xy_3) = c_3, \dots, c_{i+1} \in M(y_i)$, $\pi(xy_{i+1}) = c_{i+1}$. Since we only have $\Delta(G) + 1$ colors, there exists an l such that $\pi(xy_{l+1}) = c_{l+1} \in \{c_1, c_2, \dots, c_l\}$. W.L.O.G., let $c_{l+1} = c_k$, $k \in \{1, 2, \dots, l\}$.

Now, we have several cases to consider depending on whether $c_0 \in M(y_l)$ or $c_0 \notin M(y_l)$.

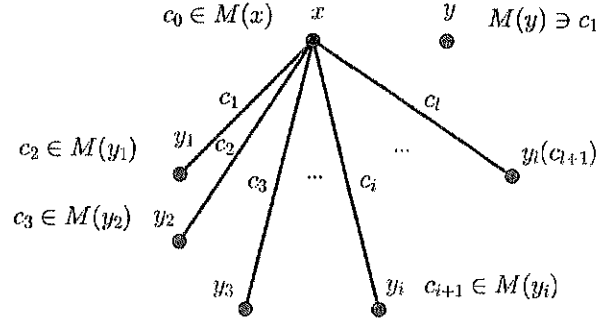


Figure 44.

(a) $c_0 \notin M(y_l)$.

Since $c_{l+1} = c_k$, $c_k \in M(y_l)$. Now, consider $c_k - c_0$ path starting from y_l .

- (i) It is a $y_l - y_k$ path. Since $\pi(xy_k) = c_k$, we may recolor them to a $c_0 - c_k$ path starting from y_k . (Note here that c_0 occurs in an edge incident to y_l . By the fact that the last color is c_k , both c_0 and c_k occur an even number of times.) Now, since $\pi(xy_k) = c_0$, the recoloring of $xy_1, xy_2, \dots, xy_{k-1}$ gives $c_1 \in M(x)$, we have the proof.
- (ii) It is a $y_l - y_{k-1}$ path. Since $c_k \in M(y_{k-1})$, this path is ended with color c_0 . That is to say c_0 is also available for xy_{k-1} (not only c_{k-1}). Hence, we color xy_{k-1} with c_0 instead of c_{k-1} , the proof follows by a similar recoloring process.

(iii) It is a $y_l - y_i$ path, $i \notin \{k-1, k\}$.

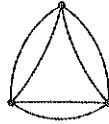
Then, either c_l or c_0 will be available for xy_i and the proof follows by recoloring process. The case (b) $c_0 \in M(y_l)$ can be done similarly. ■

(•) Based on the same proof technique, we also have a stronger result of Vizing's Theorem.

Theorem 67' (Vizing, 1964)

If G is a multigraph with multiplicity η , then $\chi'(G) \leq \Delta(G) + \eta$.

(•) The following graph has $\Delta(G) = 4$ and $\eta = 2$.



Definition (Class 1 and Class 2)

A graph (simple) is of Class 1 if $\chi'(G) = \Delta(G)$ and of Class 2 if $\chi'(G) = \Delta(G) + 1$.

Theorem 68 (König, 1916)

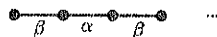
A bipartite graph is of Class 1.

Proof.

(1st)

By induction on $\|G\|$. Let $xy \in E(G)$ and $G - xy$ can be edge-colored with $\Delta(G)$ colors. Now, since $\deg_{G-xy}(x) < \Delta(G)$ and

$\deg_{G-xy}(y) < \Delta(G)$, a color is missing at x and also a color is missing at y . Let them be α and β respectively. Clearly, $\alpha \neq \beta$, and β occurs around x and α occurs around y . Now, we adapt the idea in proving Vizing's Theorem, let P be a longest $\alpha - \beta$ path from x :



First, if P is an $x - y$ path and the last edge has color α , then P is a path of even length. Hence, $P \cup \{xy\}$ is an odd cycle. G is bipartite. $\rightarrow \leftarrow$ Hence, x and y are in different components induced by the set of edges colored α and β . Now, we recolor all the edges of P by interchanging α and β . This gives a coloring in which β is missing at x and also at y . By coloring xy with β , we obtain a Δ -edge-coloring of G . ■

(2nd)

Lemma

Let G be a bipartite graph. Then, there exists a $\Delta(G)$ -regular bipartite graph $\tilde{G} \geq G$. (Exercise) By Lemma $\tilde{G}$ is a $\Delta(G)$ -regular bipartite graph and thus $\tilde{G}$ can be decomposed into $\Delta(G)$ perfect matchings by König's Theorem. This implies that $\chi'(\tilde{G}) = \Delta(G)$.

Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) \leq \Delta(G)$. Hence, we conclude the proof.

(•) A graph G is said to be overfull if $\|G\| > \left\lfloor \frac{|G|}{2} \right\rfloor \cdot \Delta(G)$.

(••) If G is overfull, then G is of Class 2.

(•) If G is overfull, then $|G|$ is odd.

Theorem 69

Petersen graph is of Class 2.

Proof.

If G is the Petersen graph and $\chi'(G) = 3$, then G can be decomposed into 3 1-factors: F_1 , F_2 and F_3 (3 color classes). Now, consider the set of 5 link-edges e_1, e_2, e_3, e_4 and e_5 , see Figure 45.

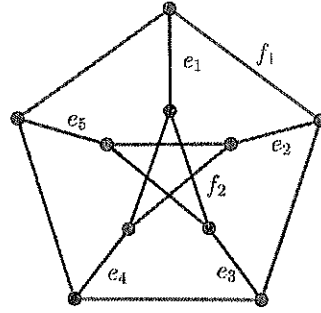


Figure 45. Petersen graph

At least one of F_1 , F_2 and F_3 will contain at least two link-edges by Pigeon-hole principle, let it be F_1 . Clearly, F_1 can not contain all the 5 link-edges. For otherwise, two C_5 's is the union of F_2 and F_3 which is impossible. So, there are three cases to consider.

$$(i) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 4$$

Let e_1 be the edge not in F_1 . But, now all the edges of $G - e_1$ not in $\{e_2, e_3, e_4, e_5\}$ are incident to an edge of $\{e_2, e_3, e_4, e_5\}$. So, no other edge can be chosen for F_1 .

$$(ii) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 3$$

Let e_1 and e_2 be the edges not in F_1 . Then, other than link-edges, we choose at most one more edge f_1 . The case e_1 and e_3 are not in F_1 has similar conclusion (only f_2 is available).

$$(iii) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 2$$

This case comes out that we can find two more edges which not link-edges. ■

(*) The proof of Theorem 69 implies that the Petersen graph contains no "Hamilton cycles".

Proof.

If G contains a Hamilton cycle C , then $\chi'(G) = 3$ by coloring the cycle with two colors and $G - C$ (1-factor) with another color. ■

Theorem 70

A 3-regular planar graph G is of Class 1.

Proof.

Let G be embedded in S_0 . Then, by 4-color Theorem, G is 4-face-colorable (or 4-map-colorable). Let the 4 colors used be obtained from the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$. Since each edge is in the boundary of two adjacent faces, let the edge be colored by $(a_1, b_1) \oplus (a_2, b_2)$ where (a_1, b_1) and (a_2, b_2) are the colors of these two adjacent faces. As a conclusion, we obtain a 3-edge-coloring of G , since $(0,0)$ will not be used. The coloring is proper since three adjacent faces will receive three different colors, see Figure 46. ■

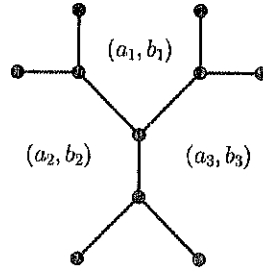


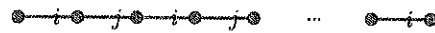
Figure 46.

Theorem 71 (Equitable edge-coloring)

If G has a k -edge-coloring f , then G has an equitable edge coloring, i.e., for any two $i, j \in \{1, 2, \dots, k\}$, $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$.

Proof.

If there exist i and j such that $|f^{-1}(i)| - |f^{-1}(j)| \geq 2$, then we consider the graph H induced by the set of edges colored i and j . Then, H is a subgraph of G such that each component of H is either a path or an even cycle. Since i occurs more times than j , there exists an $i - j$ path:

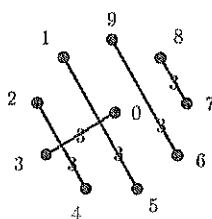


whose end edges are colored i . Now, by switching the colors on this path, we obtain a new edge coloring of G such that i occurs one less time and j occurs one more. It turns out that we can obtain an k -edge-coloring s.t. $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$. As a consequence, we are able to adjust all of them and obtain an equitable k -edge-coloring. ■

- (•) This Theorem is not difficult to prove, but very useful.
- (•) Without using 4CT, the proof of Theorem 70 is very difficult.

The complete graph K_n is of Class 2 if and only if K_n is overfull or equivalently n is odd.

First, we claim that for each $m \geq 1$, K_{2m} is of Class 1. It suffices to give a $(2m - 1)$ -edge-coloring of K_{2m} . For convenience, let $V(K_{2m}) = \mathbb{Z}_{2m} = \{0, 1, 2, \dots, 2m - 1\}$. For each color $i \in \{1, 2, \dots, 2m - 1\}$, let the set of edges colored i be $F_i = \{(0, i), (i + 1, i - 1), (i + 2, i - 2), \dots, (i + m - 1, i - m + 1)\} \pmod{2m - 1}$. See Figure 46 for an example of $m = 5$ and $i = 3$.



Since $\Delta(K_{2m}) = 2m - 1$, $\chi'(K_{2m}) = 2m - 1$.

Now, by deleting 0 in K_{2m} , we obtain a $(2m - 1)$ -edge-coloring of K_{2m-1} . On the other hand, it is not difficult to check that K_{2m-1} is overfull for $m \geq 2$, this concludes that $\chi'(K_{2m-1}) > \Delta(K_{2m-1}) = 2m - 2$.

■

- (•) This Theorem is not difficult to prove, but it is very useful in the construction of "Combinatorial Designs".
- (••) Equivalently, K_{2m} can be decomposed into $2m - 1$ 1-factors, which is also known as a 1-factorization of K_{2m} .
- (•••) If G is an r -regular graph and $\chi'(G) = r$, then G has a 1-factorization.
- (* * *) It was conjectured that if G is r -regular and $r \geq \frac{|G|}{2}$, then G has a 1-factorization or equivalently $\chi'(G) = r$.

Theorem 71' (D. Hoffman et al.)

A complete multipartite graph G is of Class 2 if and only if G is overfull.

(•) "Total coloring"

A k -total coloring of a graph G is a mapping

$\varphi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that

- (i) adjacent vertices receive distinct images,
- (ii) incident edges receive distinct images, and
- (iii) each vertex has a distinct image with the images of its incident edges.

e.g.