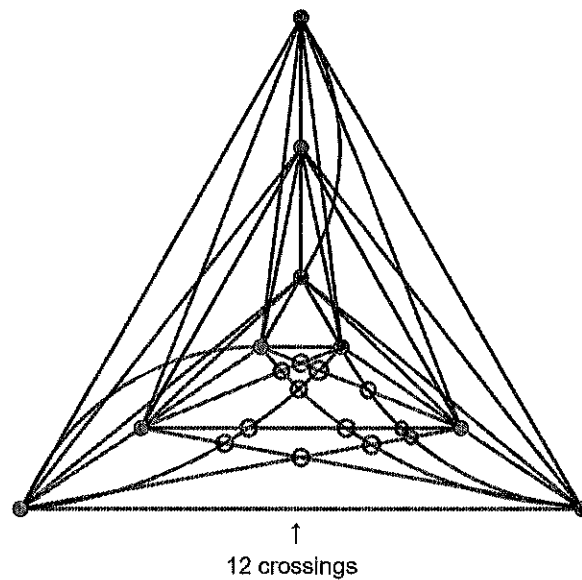


Figure 36. $cr(K_9)$ **Conjecture**

$$cr(K_p) = \frac{1}{4} \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{p-3}{2} \right\rfloor. \quad (\text{True for } 1 \leq p \leq 10.)$$

Vertex-Coloring

- (•) k -coloring (proper): $\varphi : V(G) \rightarrow \{1, 2, 3, \dots, k\}$ s.t.
 $uv \in E(G) \Rightarrow \varphi(u) \neq \varphi(v)$.
- (•) $\chi(G) = \min\{k \mid G \text{ has a } k\text{-coloring}\}$ (Chromatic number of G)
- (•) G is n -critical (chromatically) if $\chi(G - v) < \chi(G)$ for each
 $v \in V(G)$.
- (••) Every graph G has an n -critical induced subgraph H .

Theorem 56

Every critically n -chromatic (n -critical) graph, $n \geq 2$, is $(n - 1)$ -edge-connected.

Proof.

First, if $n = 2$, then $G \cong K_2$ and thus G is 1-edge-connected. If $n = 3$, then $G \cong C_{2m+1}$, $m \geq 1$, (?) and G is 2-edge-connected. Let $n \geq 4$ and assume that G is not $(n - 1)$ -edge-connected. Hence, $V(G) = V_1 \cup V_2$ such that $|\langle V_1, V_2 \rangle| < n - 1$ ($\leq n - 2$). Let $G_1 = \langle V_1 \rangle_G$ and $G_2 = \langle V_2 \rangle_G$. Now, both of them are $n - 1$ colorable since G is $(n - 1)$ -critical. (See Figure 37.) Let the colorings be ϕ_1 and ϕ_2 resp.

Now, consider the vertices incident to edges in $\langle V_1, V_2 \rangle$. If for each edge $uv \in \langle V_1, V_2 \rangle$, $\phi_1(u) \neq \phi_2(v)$, then G has an $(n - 1)$ -coloring, a contradiction. Thus, assume that for some edges $uv \in \langle V_1, V_2 \rangle$, $\phi_1(u) = \phi_2(v)$. (We shall permute the colors of G_1 (ϕ'_1) in order that for each $uv \in \langle V_1, V_2 \rangle$, $\phi_1(u) \neq \phi_2(v)$).

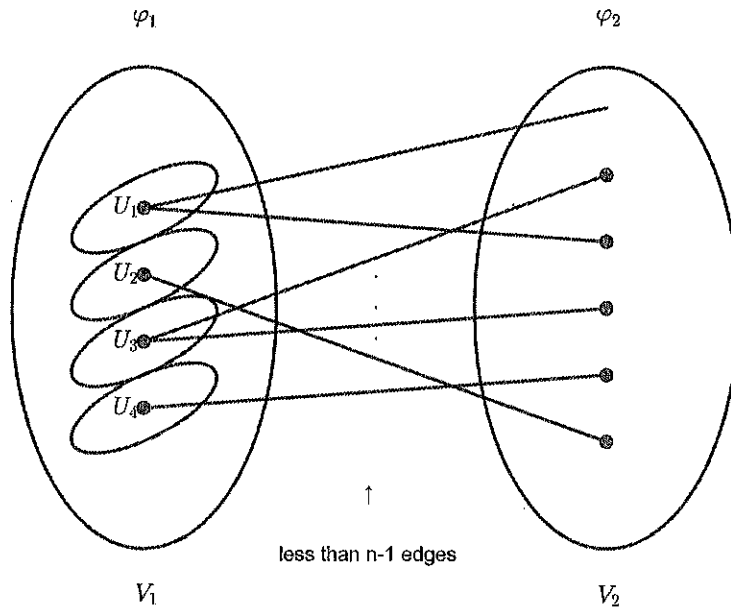


Figure 37.

Let $U_1, U_2, \dots, U_m$ be the subsets of V_1 such that $\varphi_1^{-1}(i) = U_i$, $i = 1, 2, \dots, m$ ($\leq n - 2$) and there is at least one edge joining U_i and $V(G_2)$ for each i . Furthermore, let n_i be the number of vertices in U_i which are incident to a vertices of $V(G_2)$. Hence, $\sum_{i=1}^m n_i \leq n - 2$.

Now, we start a process to recolor the vertices in V_1 . Starting U_1 . If $\forall x \in U_1$, $\varphi_1(x)$ has distinct colors with the colors of these vertices in V_2 which are incident to U_1 , then we go to consider U_2 . Otherwise, $\varphi_1(x) = \varphi_2(y)$ for some $x \in U_1$, and $xy \in \langle V_1, V_2 \rangle$. In this case, we permute the colors of $U_1, U_2, \dots, U_m$ (It was $1, 2, \dots, m$) such that the color used for the vertices in U_1 is distinct from the colors of vertices in V_2 which are incident to U_1 , there are n_1 of them. Since $\sum_{i=1}^m n_i \leq n - 2$, $n - 1 - n_1 > 0$ and thus there exists a color for U_1 .

Following this idea we consider U_2 . If there are vertices x in U_2 such that $\varphi_1(x) = \varphi_2(y)$ for some $xy \in \langle V_1, V_2 \rangle$ then permute the colors used

in $U_2, U_3, \dots, U_m$ where the color for U_1 is fixed. Again, since $n - 2 - n_2 \geq (n - 1) - n_1 - n_2 > 0$, a color for U_2 is available. Continuing this process, we end it up with an $(n - 1)$ -coloring of G , a contradiction to $\chi(G) = n$. ■

(•) If G is n -critical, then $\delta(G) \geq n - 1$. ✓

Question

Is this graph $(n - 1)$ -connected?

Theorem 57

Let $k = \max_{H \leq G} \delta(H)$. Then, $\chi(G) \leq k + 1$.

Proof.

(1st)

Let $\chi(G) = n$ and H' be an n -critical induced subgraph of G . Then, $\delta(H') \geq n - 1$. Since $\max_{H \leq G} \delta(H) \geq \delta(H') \geq n - 1$, $k \geq n - 1$ and thus $\chi(G) = n \leq k + 1$. ■

(2nd) (G is a (p, q) -graph.)

Since $k = \max_{H \leq G} \delta(H)$, $\delta(G) \geq k$. Let $x_p \in V(G)$ and $\deg_G(x_p) = \delta(G) \geq k$. Moreover, let $G_{p-1} = G - x_p$. Again, G_{p-1} has a vertex of degree at most k . So, we obtain a sequence of induced subgraphs, $G = G_p \geq G_{p-1} \geq \dots \geq G_1$ such that $\delta(G_i) \leq k$ for $i = p, p - 1, \dots, 1$ such that $x_i \in V(G_i)$. Hence, we obtain a sequence $\langle x_1, x_2, \dots, x_p \rangle$ such that x_{j+1} is incident to at most k vertices in $\langle \{x_1, x_2, \dots, x_j\} \rangle_G$. This implies that we can use greedy algorithm to

color G starting from x_1 , and then $x_2, \dots, x_p$. All we need is at most $k + 1$ colors. Hence, $\chi(G) \leq k + 1$. ■

Theorem 58 (Brooks, 1941)

Let G be a connected graph which is neither a complete graph nor an odd cycle. Then, $\chi(G) \leq \Delta(G)$.

Proof.

By induction on $|G| = p$. We may assume that the graph we consider is 2-connected and Δ -regular where $\Delta \geq 3$. (?) (Note that a 2-regular connected graph G with $\chi = 3$ is an odd cycle.)

First, if G is 3-connected, let x_p be any vertex such that $\langle N_G(x_p) \rangle_G$ is not a complete subgraph of G . Such an x_p does exist since G is not a complete graph. Let x_1 and x_2 be two vertices in $N_G(x_p)$ such that $x_1 \not\sim_G x_2$. Now, we may construct a sequence corresponding to $V(G)$. Choose $x_{p-1} \in N_G(x_n) \setminus \{x_1, x_2\}$ (3-connected). Then, x_{p-2} is adjacent to either x_n or x_{n-1} . As a consequence, we have a sequence $\langle x_1, x_2, \dots, x_p \rangle$ such that " x_i is incident to at least one vertex in $\{x_{i+1}, x_{i+2}, \dots, x_p\}$ ". Now, we use the greedy algorithm to color all the vertices starting "from x_1 and x_2 with a common color".

Second, let G be 2-connected (but not 3-connected), i.e. $k(G) = 2$. Let S be a cut set with two vertices and $x_p \in S$. Hence, $G - x_p$ has a cut vertex, see Figure 38. Let x_1 and x_2 be two vertices in distinct blocks (2-connected maximal subgraph of G). Again, we use the idea mentioned above to construct a sequence $\langle x_1, x_2, \dots, x_n \rangle$ and the proof

follows by using the greedy algorithm for vertex coloring. The proof for $k(G) = 1$ is similar. ■

- (•) $\Delta(G) - \chi(G)$ can be arbitrarily large.
- (•) There are also graphs such that $\Delta(G) = \chi(G)$, for example even cycles, non-bipartite 3-regular graphs, say, Peterson graph.

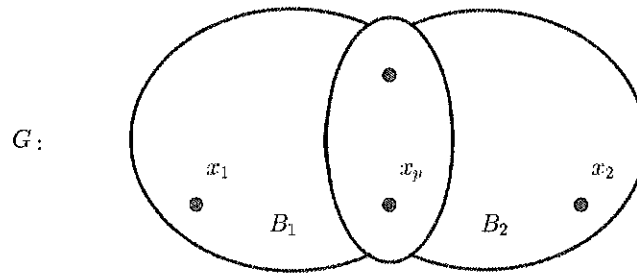
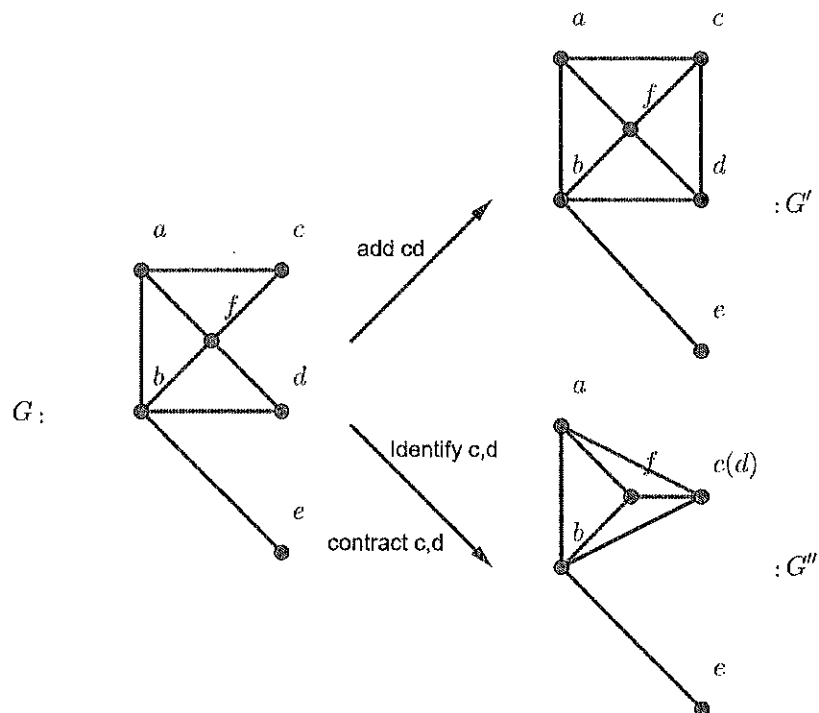


Figure 38. $k(G) = 2$

Another coloring algorithm



Observation

A coloring φ of G has two outcomes:

- (1) $\varphi(c) \neq \varphi(d)$ and (2) $\varphi(c) = \varphi(d)$.

Theorem 59

Let $p_H(k)$ be the number of distinct k -colorings of H . Then,
 $p_G(k) = p_{G'}(k) + p_{G''}(k)$ where G' and G'' are graphs obtained from G
 by adding xy and contracting x and y respectively for $x \neq y$. ($p_G(k)$ is
 known as the chromatic polynomial of G with k -colorings.)

Proof.

The proof follows by the fact that $\varphi(x) = \varphi(y)$ or $\varphi(x) \neq \varphi(y)$ but
 not both. ■

- (•) G is k -colorable if and only if $p_G(k) \geq 1$.
- (•) $\chi(G) = \min\{\chi(G'), \chi(G'')\}$.

Theorem 60

Let G be a (p, q) -graph with k components. Then,
 $p_G(x) = \sum_{i=0}^{p-k} (-1)^i \cdot a_i \cdot x^{p-i}$, where $a_0 = 1$, $a_1 = q$ and a_i is a positive
 integer for $0 \leq i \leq p - k$.

Proof.

By induction on $p + q$. Clearly, it's true for $p + q = 1$. Assume the
 assertion is true for the cases of smaller $p + q$ and let G be a
 (p, q) -graph with k components. First, if $m = 0$, then $p = k$, so

$p_G(x) = x^p$, thus $a_0 = 1$, $a_1 = q = 0$. Now, consider $m \geq 1$. Let uv be an edge of G and $G_0 = G - uv$. By induction,

$p_{G_0}(x) = x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i x^{p-i}$ where b_i is a non-negative integer for each i . (G_0 has at least k components.) Also,

$p_{G'_0}(x) = x^{p-1} - \sum_{i=2}^{p-k} (-1)^i c_i x^{p-i}$ where c_i is a positive integer for each i .

Note that $G'_0 \cong G$ (adding uv back).

This implies that

$$\begin{aligned} p_G(x) &= p_{G_0}(x) - p_{G'_0}(x) \\ &= x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i x^{p-i} - x^{p-1} + \sum_{i=2}^{p-k} (-1)^i c_i x^{p-i} \\ &= x^p - q \cdot x^{p-1} + \sum_{i=2}^{p-k} (-1)^i (b_i + c_i) x^{p-i} \\ &= x^p - q \cdot x^{p-1} + \sum_{i=2}^{p-k} (-1)^i a_i x^{p-i}, \quad a_i > 0 \text{ for each } i. \end{aligned}$$

■

(•) If T is a tree of order p , then for each $k \geq 1$, there are

$$k \cdot (k-1)^{p-1} = k^p - \binom{p-1}{1} \cdot k^{p-1} + \dots = k^p - (p-1)k^{p-1} + \dots \text{ different } k\text{-colorings of } T, \text{ i.e., } p_T(k) = k \cdot (k-1)^{p-1}.$$

Theorem 61 (Nordhaus and Gaddum, 1956)

If G is a graph of order p , then

- (1) $2\sqrt{p} \underset{\textcircled{1}}{\leq} \chi(G) + \chi(\bar{G}) \underset{\textcircled{1}}{\leq} p+1$, and
- (2) $p \underset{\textcircled{2}}{\leq} \chi(G) \cdot \chi(\bar{G}) \underset{\textcircled{2}}{\leq} [(p+1)/2]^2$.

Proof.

First, we claim that $\chi(G) \cdot \chi(\bar{G}) \geq p$. For each vertex v of K_p let $\varphi(v) = (\varphi_1(v), \varphi_2(v))$ where φ_1 and φ_2 are chromatic colorings of G

and $\bar{G}$ respectively. Since two vertices of K_p are either adjacent in G or $\bar{G}$, all ordered pairs of $v \in V(K_p)$ are distinct. Hence,

$\chi(G) \cdot \chi(\bar{G}) \geq p$. (We need p colors for K_p .)

This implies that $\frac{\chi(G) + \chi(\bar{G})}{2} \geq \sqrt{\chi(G) \cdot \chi(\bar{G})} \geq \sqrt{p}$, ① holds.

Now, let $k = \max_{H \leq G} \delta(H)$. We claim that every induced subgraph H' of $\bar{G}$ has minimum degree $p - k - 1$, i.e. $\max_{H' \leq \bar{G}} \delta(H') \leq p - k - 1$. Suppose not. Let H'' be an induced subgraph of $\bar{G}$ such that $\delta(H'') = p - k$.

Since H'' is an induced subgraph of $\bar{G}$, $H'' \cong \bar{H}$ for some induced subgraph H of G . Let $|H| = h$. Since $\delta(H'') = \delta(\bar{H}) = p - k$,

$\deg_H(v) \leq (h - 1) - (p - k)$ for each $v \in V(H)$. Therefore, in G ,

$\deg_G(v) \leq (h - 1) - (p - k) + (p - h) = k - 1$. On the other hand,

$k = \max_{H \leq G} \delta(H)$ and thus we have an induced subgraph $H''' \leq G$ such

that $\delta(H''') = k$. This implies that $V(H) \cap V(H''') = \emptyset$. By the fact

$|V(H''')| \geq k + 1$, $|H| = h \leq p - (k + 1)$ and thus $|\bar{H}| \leq p - (k + 1) = p - k - 1$.

$\delta(\bar{H}) = p - k$ is not possible. This concludes that

$\max_{H' \leq \bar{G}} \delta(H') \leq p - k - 1$ and thus $\chi(\bar{G}) \leq p - k - 1 + 1 = p - k$ (and $\chi(G) \leq 1 + k$), the proof of ①' follows.

Now, for ②', it follows by

$$\sqrt{\chi(G) \cdot \chi(\bar{G})} \leq \frac{\chi(G) + \chi(\bar{G})}{2} \leq \frac{p + 1}{2}. \quad \blacksquare$$

(•) A graph is said to be *self-complementary* if $G \cong \bar{G}$.

In this situation $\sqrt{p} \leq \chi(G) \leq \frac{p + 1}{2}$. $p = 5 \Rightarrow \chi(G) = 3 \rightarrow G \cong C_5$.

Problem.

Let $\omega(G)$ denote the order of a maximum clique, i.e., the order of

complete subgraphs with maximum order. Then, $\chi(G) \geq \omega(G) \rightarrow$
 Clique number of G . *When does the equality holds?*

(•) A subgraph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G .

(*) $\chi(H) - \omega(H)$ can be very large!

Theorem 62 (Mycielski's graph)

For every integer n , there exists a triangle-free graph G such that $\chi(G) = n$. ($\chi(G) - \omega(G) = n - 2$.)

Proof.

By induction on n and K_1, K_2, C_5 do have the property respectively for $n=1, 2$, and 3 . Now, assume that H is a triangle-free k -chromatic graph, i.e., $\chi(H) = k$. We construct a graph G based on H such that G is a triangle-free $(k+1)$ -chromatic graph.

Let $V(H) = \{v_1, v_2, \dots, v_p\}$ and $V(G) = V(H) \cup \{u_1, u_2, \dots, u_p, u_0\}$. Let $E(G) = E(H) \cup \{u_0 u_i \mid i = 1, 2, \dots, p\} \cup \{u_i v_j \mid v_j \in N_H(v_i)\}$. See Figure 38 for an example when $k=3$.

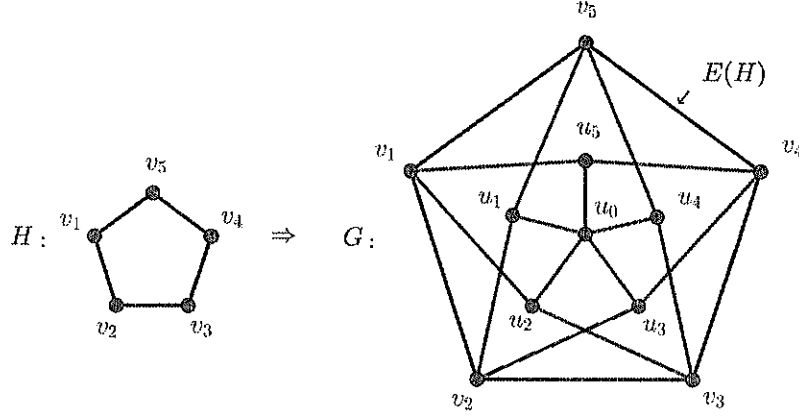


Figure 38. Grötzsch graph

Since $\langle \{u_1, u_2, \dots, u_p\} \rangle_G$ contains no edges, u_0 is not in any triangle. By assumption, $H \not\cong K_3$. So, the only possibility will be a triangle consists of u_i, v_j and v_k where $u_i v_j$ and $u_i v_k$ are edges of G . If they form a triangle, then $\langle \{v_i, v_j, v_k\} \rangle_H$ is a triangle in H . Hence, G is triangle-free.

Now, we claim $\chi(G) = k + 1$. Let φ be a k -coloring of H . Let $\tilde{\varphi} : V(G) \rightarrow \{1, 2, \dots, k + 1\}$ by letting $\tilde{\varphi}(u_i) = \varphi(v_i)$ and $\tilde{\varphi}(u_0) = k + 1$. Hence, we have a $(k + 1)$ -coloring of G , thus $\chi(G) \leq k + 1$. On the other hand, we show that $\chi(G) \geq k + 1$. Suppose not. Let φ' be a k -coloring of G and the colors used are $1, 2, \dots, k$. First, we assign u_0 the color k , i.e., $\varphi'(u_0) = k$. So, the colors used for $u_1, u_2, \dots, u_p$ must be in $\{1, 2, \dots, k - 1\}$. Since $\chi(H) = k$, k occurs somewhere in H , say v_i . (May have more vertices.) Now, we recolor v_i by using $\varphi'(u_i)$. Since u_i is adjacent to every vertex of $N_H(v_i)$, $\varphi'(u_i) \neq \varphi'(v)$ for each $v \in N_H(v_i)$ and thus we have a proper coloring of H using at most $k - 1$ colors. (?) $\rightarrow \leftarrow$ ■

(*) This theorem has been extended to a more general result obtained by Erdős and Lovász (1961): For any integers $m, n \geq 2$, there exists an n -chromatic graph whose girth exceeds m . (Theorem 62 is for $m = 3$.)

****Theorem 63 (Lovász, 1972) (Weakly Perfect Graph Theorem)**

A graph G is perfect if and only if $\bar{G}$ is perfect.

Note

The proof of this Theorem is not too long. But, the proof of next one is long.

****Theorem 63' (Maria Chudnovsky, Neil Rokertson, Paul Seymour and Robin Thomas, Annals of Mathematics, 164(2006), 15-229.)**

A graph G is perfect if and only if no induced subgraph of G or $\bar{G}$ is an odd cycle of length at least 5.

Proof of Theorem 63.

We prove a different version (stronger):

A graph G is perfect if and only if $|H| \leq \alpha(H) \cdot \omega(H) \longrightarrow (1)$

for all induced subgraphs H of G . ($\omega(H)$ is the clique number of H .)

(•) In $\bar{G}$, if A is an independent set, then in G , $\langle A \rangle_G$ is a clique (A is independent) $\langle A \rangle_G$ (a clique)

($\Rightarrow$) (Definition) If G is perfect, then for each induced subgraph H , $\chi(H) = \omega(H)$. Hence, the vertex set of H , $V(H)$, can be partitioned

Finally, let M and N ((0,1)-matrices) be defined as in Figure 39.

Let $V(G) = \{v_1, v_2, \dots, v_p\}$. Let $K_i \subseteq G - A_i$ (?) for each $i = 0, 1, \dots, \alpha\omega$.

$$\begin{array}{c}
 \begin{array}{cccc} & v_1 & v_2 & \dots & v_p \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} & & \begin{array}{cccc} & K_0 & K_1 & \dots & K_{\alpha\omega} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \\
 M: \begin{array}{c} A_0 \\ A_1 \\ \vdots \\ A_{\alpha\omega} \end{array} \left[\begin{array}{ccccc} & & & & \\ & M(i,j) = 1 & & & \\ & \text{iff } v_j \in A_i & & & \\ & & & & \\ & & & & \end{array} \right]_{(\alpha\omega+1) \times p} & N: \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_p \end{array} \left[\begin{array}{ccccc} & & & & \\ & N(i,j) = 1 & & & \\ & \text{iff } v_i \in K_j & & & \\ & & & & \\ & & & & \end{array} \right]
 \end{array}$$

(?) By induction $\chi(G - A_i) = \omega(G - A_i) = \omega(G)$ (otherwise, G is perfect.). Since $M \cdot N = J_{(\alpha\omega+1) \times (\alpha\omega+1)} - I_{\alpha\omega+1}$ is non-singular, the rank of $M \cdot N$ is $\alpha\omega + 1$ which is larger than $p = |G|$, a contradiction to the assumption when $H \cong G$. Hence, the proof follows. ■

Theorem 64

If G is a connected planar graph, then $\chi(G) \leq 5$.

Proof.

By induction on $|G|$. By Theorem 57, it suffices to consider an induced subgraph H whose minimum degree is 5.

Let $v \in V(H)$ such that $\deg_H(v) = 5$. By induction, $\chi(H - v) \leq 5$. Let φ be a 5-coloring of H and we consider the colors assigned on $N_H(v)$. Let them be $\varphi(v_1), \varphi(v_2), \dots, \varphi(v_5)$. Clearly, if any two of them are of the same color, then there is a color for v such that we have a proper 5-coloring of H . So, assume that $\varphi(v_i) = i, i = 1, 2, 3, 4, 5$ and the vertices are in clockwise order, see Figure 40.

Theorem 65 (The Heawood Map Coloring Theorem)

For every positive integer n , $\chi(S_n) = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$. ($\chi(S_n)$: the maximum chromatic number among all graphs that can be embedded on S_n .)

Proof. (Outline)

The upper bound $\chi(S_n) \leq \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$ was obtained by Heffter in 1890. At that time, he claimed that it's an equality. But, unfortunately, the correct proof came out many years later by the effort of considering the embedding of K_p since for sure K_p needs p colors.

So, define $p = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$. It follows that $n \geq \frac{(p-3)(p-4)}{12}$ and thus $n \geq \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$. By the fact $r(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$, $\chi(S_n) \geq p = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$. ■

(•) There exists a procedure to construct all the graphs G with

$$\chi(G) \geq k.$$

(••) $\mathcal{G}_k$ is a collection of k -constructible graphs if $G \in \mathcal{G}_k$ can be constructed recursively by the following steps.

(i) K_k is k -constructible.

(ii) If G is k -constructible and $x, y \in V(G)$ are non-adjacent, then $(G + xy)/xy$ is k -constructible.

(iii) If G_1 and G_2 are k -constructible such that

$$V(G_1) \cap V(G_2) = \{x\}, xy_1 \in E(G_1), xy_2 \in E(G_2), \text{ then}$$

$(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k -constructible, see Figure 41.

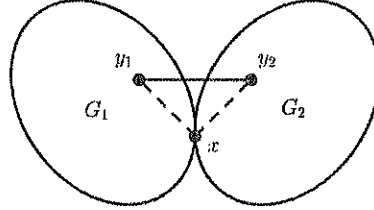


Figure 41. Hajós construction

(*) If G is k -constructible, then $\chi(G) \geq k$.

(i) is trivial.

(ii) If $(G + xy)/xy$ uses less than k colors, then by coloring x and y with the same color, we have $\chi(G) < k$.

(iii) If the graph obtained uses less than k colors, then either $\chi(G_1) < k$ or $\chi(G_2) < k$ depending on whether $\varphi(y_1) \neq \varphi(x)$ or $\varphi(y_2) \neq \varphi(x)$. Since y_1 and y_2 receive distinct colors, one of the above two conditions must hold.

Theorem 66 (Hajós, 1961)

Let G be a graph. Then, $\chi(G) \geq k$ if and only if G has a k -constructible subgraphs.

Proof.

($\Leftarrow$) It has been explained above.

($\Rightarrow$) Suppose not; then $k \geq 3$. Let G be a *maximal* counterexample, i.e. G is of maximum size such that G does not contain a

k -constructible subgraph. Now, G can not be a complete r -partite graph. For otherwise, $\chi(G) \geq k$ implies that $r \geq k$ and then G contains a k -constructible K_k . (Contract each partite set.) Hence there exist vertices x, y_1 and y_2 such that $xy_1 \notin E(G)$, $xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. By assumption of the maximality of G , both $G + xy_1$ and $G + xy_2$ contain k -constructible subgraphs, say H_1 and H_2 ; moreover, $xy_1 \in E(H_1)$ and $xy_2 \in E(H_2)$.

Let $H_2 - H_1$ denote the graph $\langle V(H_2) \setminus V(H_1) \rangle_{H_2}$ and H'_2 is an isomorphic copy of H_2 such that

$V(H'_2) \cap V(G) = \{x\} \cup (V(H_2) \setminus V(H_1))$, see Figure 42. So,

$V(H_1) \cap V(H'_2) = \{x\}$. Now, since $H'_2 \cong H_2$, let $\varphi : H_2 \rightarrow H'_2$ be an

isomorphism. By (iii) $H_1 \cup H'_2 - xy_1 - x \cdot \varphi(y_2) + y_1 \cdot \varphi(y_2)$ is

k -constructive, let this graph be H . Now, for each vertex v' in

$V(H'_2) \setminus V(G)$, there exists a v such that $v' = \varphi(v)$. Furthermore vv' is

not an edge of H . By (ii), we can identify v and v' and obtain a

k -constructible subgraph of G after identifying all vertices in

$V(H_2) \setminus V(H_1)$. ■