

(\*) Topological Graph Theory studies the drawings of graphs on surface.

(\*\*) A proper drawing on a surface of a graph  $G$  with  $p$  vertices and  $q$  edges follows the rules:

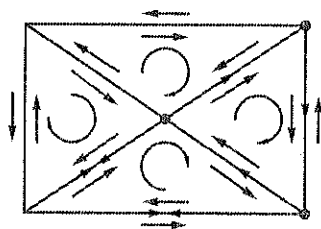
- (1) There are  $p$  points on the surface which corresponds to the set of vertices in  $G$ ; and
- (2) There are  $q$  curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

2-manifold: A connected topological space in which every point has a neighborhood homeomorphic to the open unit disk defined on  $\mathbb{R}^2$ .

Bound subspace: A subspace  $M$  of  $\mathbb{R}^3$  is bounded if  $\exists K \in \mathbb{R}^+$  such that  $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 = K\}$ .

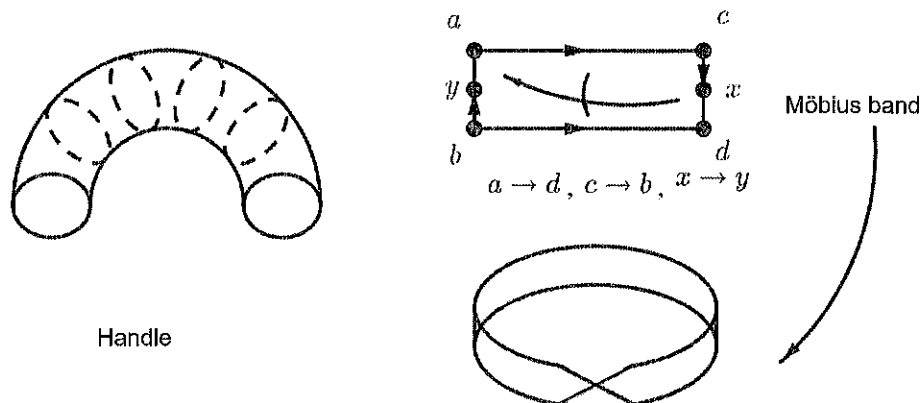
Closed:  $M$  is closed if its boundary  $\partial M$  coincides with  $M$ .

Orientable:  $M$  is orientable if for every simple closed curve  $C$  on  $M$ , a clockwise sense of rotation is preserved once around  $C$ . Otherwise,  $M$  is non-orientable.

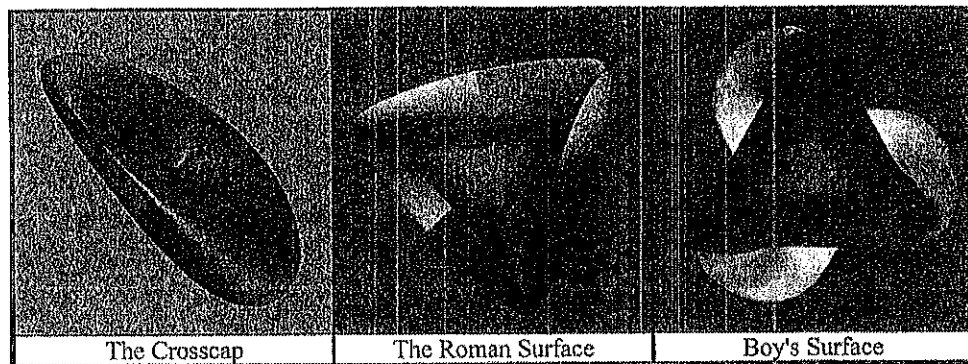


Orientable Surface: A surface  $S_k$  is a compact "orientable" 2-manifold that may be thought of as a sphere on which has been placed (inserted) a number  $k$  of "handles" (holes).

non-orientable Surface: A surface obtained by adding  $k$  cross-caps to a sphere ( $S_0$ ) is a non-orientable surface  $N_k$ .



- (•) Adding a crosscap: Attach the boundary of a Möbius band to a cycle on  $S_0$ .



### Definition (Embeddable)

A  $(p, q)$ -graph  $G$  is said to be embeddable on a surface if it is possible to draw  $G$  *properly* (drawing without crossings) on the surface.

### Definition (Planar graph)

A graph is planar if it can be embedded in the plane, equivalently, embedded on the sphere.

**Definition (2-cell embedding)**

A region is called a 2-cell if any simple closed curve in that region can be continuously deformed or contracted in that region to a single point, equivalently, a 2-cell is topologically homeomorphic to  $\mathbb{R}^2$ . An embedding of  $G$  on a surface is a 2-cell embedding of  $G$  if all the regions so determined are 2-cells.

The following figure shows embeddings of  $K_{3,3}$  on  $S_1$  and  $S_2$  respectively.  $S_k$  is a surface obtained by attaching  $k$  handles.

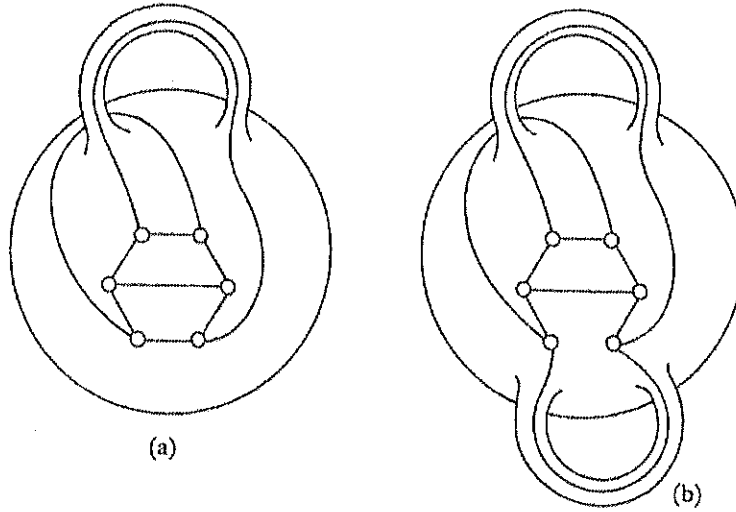


Figure 31. Embeddings of  $K_{3,3}$  on surfaces  $S_1, S_2$

- (•)  $S_0$ : Sphere
- (•)  $N_0 \simeq S_0$  (Homeomorphic),  $N_h$ : Attach  $h$  crosscaps to  $N_0(S_0)$ .

**Definition (Genus)**

The number of handles (resp. crosscaps) (on a surface) is referred to as genus of the orientable surface (resp. non-orientable surface). We use  $\gamma(G)$  (respectively  $\tilde{\gamma}(G)$ ) to denote the smallest genus of all

orientable surfaces (resp. non-orientable surfaces) on which  $G$  can be embedded.

(\*) If  $G$  is a planar graph, then  $\gamma(G)$  (so is  $\tilde{r}(G)$ ) is equal to zero.

But,  $G$  can also be embedded on a surface with genus larger than "0".

(\*\*) Given a graph  $G$ , determining  $\gamma(G)$  is a difficult problem.

### Theorem 46 (Euler)

Let  $G$  be a connected planar graph with  $p$  vertices,  $q$  edges and  $f$  faces (regions). Then  $p - q + f = 2$ .

#### Proof.

By induction on  $q$ . Since  $G$  is connected,  $G$  has at least  $p - 1$  edges.  
 (?) If  $G$  has  $p - 1$  edges and  $G$  is connected, then  $G$  is a tree which contains no cycles. This implies that  $f = 1$  and thus  $p - (p - 1) + 1 = 2$ .  
 The assertion is true for "minimal" graphs. Let the hypothesis is true for  $k = \|G\| \geq p - 1$ . Now, consider  $G$  with  $k + 1$  edges. Clearly,  $G$  contains a cycle. Let  $e$  be a cycle edge. Since  $G$  is a connected planar graph (with  $q$  faces),  $G - e$  is also a connected planar graph. Moreover,  $\|G - e\| = k$  and  $G - e$  has  $k$  edges and  $q - 1$  faces. By induction  $p - k + q - 1 = 2$  and thus  $p - (k + 1) + q = 2$ . This concludes the proof. ■

**Theorem 47**

If  $G$  is a planar graph with largest size, then  $\|G\| = 3|G| - 6$ .

**Proof.**

By observation, if  $G$  has maximum size, then each region of  $G$  is a triangle. Since each edge of  $G$  is in the boundary of exact two regions,  $3 \cdot f = 2 \cdot g$  where  $f$  is the number of regions and  $g$  is the size of  $G$ , i.e.,  $q = \|G\|$ . Now, by Euler's formula  $p - q + f = 2$  equivalently  $|G| - \|G\| + \frac{2}{3}\|G\| = 2 \Rightarrow 3 \cdot |G| - 6 = \|G\|$ . ( $G$  is a *maximal planar graph*!) ■

**Corollary**

If  $G$  is a planar graph, then  $\|G\| \leq 3 \cdot |G| - 6$ .

**Corollary**

In any planar graph, there exists at least one vertex of degree smaller than 6.

This corollary is very useful.

**Corollary**

The degree sum of a planar graph is at most  $6 \cdot |G| - 12$ .

(•) We can give a more accurate estimation of the above corollary

**Theorem 48**

Let  $G$  be a maximal planar graph (triangulated) of order  $p$ , and let

$p_i$  denote the number of vertices of degree  $i$  in  $G$  for  $i = 3, 4, \dots, \Delta(G) = d$ . Then

$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \dots + (d-6)p_d + 12.$$

**Proof.**

Since  $p = \sum_{i=3}^d p_i$  and  $2q = \sum_{i=3}^d i \cdot p_i$ , we have

$$\sum_{i=3}^d i \cdot p_i = 2(3p - 6) = 6 \cdot \sum_{i=3}^d p_i - 12.$$

This implies the conclusion. ■

### Theorem 49

There are exactly five regular polyhydra.

**Proof.**

Notice that a regular polyhydron is a polyhydron whose faces (regions) are bounded by congruent (全等) regular polygons and whose polyhydral angles are congruent (相等).

First, we convert a polyhydron into a regular planar graph. (See Figure 32.) Let the number of vertices, edges and faces be  $p, q$  and  $f$  respectively.

By Theorem 46,  $p - q + f = 2$ . Hence,

$$\begin{aligned} -8 &= 4q - 4p - 4f \\ &= 2q + 2q - 4p - 4f \\ &= \sum_{i \geq 3} i f_i + \sum_{i \geq 3} i p_i - 4 \sum_{i \geq 3} p_i - 4 \sum_{i \geq 3} f_i \quad (f_i: \# \text{ of } i\text{-face}) \\ &= \sum_{i \geq 3} (i-4) f_i + \sum_{i \geq 3} (i-4) p_i. \end{aligned}$$

Since the polyhedron is regular, all degrees and face sizes are the same, let them be  $k$  and  $h$  respectively. Therefore,

$$-8 = (h - 4)f_h + (k - 4)p_k.$$

By the fact that every planar graph contains a vertex of degree less than six, we only have nine cases to consider:  $3 \leq h \leq 5$  and  $3 \leq k \leq 5$ .

From direct checking, only 5 cases are possible, namely,

- (1)  $f_3 = p_3 = 4$  (Tetrahedron) 四面體
- (2)  $f_3 = 8$  and  $p_4 = 6$  (Octahedron) 八面體
- (3)  $f_3 = 20$  and  $p_5 = 12$  (Icosahedron) 二十面體
- (4)  $f_4 = 6$  and  $p_3 = 8$  (Cube) 六面體
- (5)  $f_5 = 12$  and  $p_3 = 20$  (Dodecahedron) 十二面體. ■

See Figure 32 for regular polyhedra.

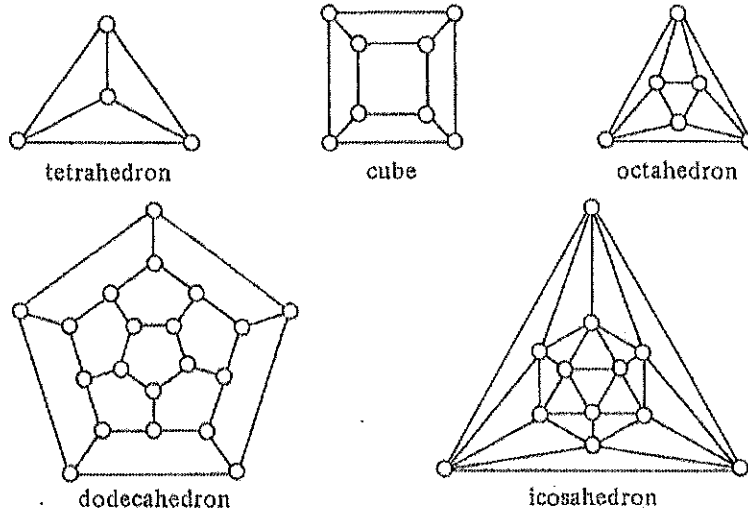


Figure 32. The graphs of the regular polyhedra

### Theorem 50 (Fáry (1948), Wagner(1936))

A planar graph  $G$  can be embedded in the plane so that each edge is a straight line segment.

**Proof.**

The proof is by induction on the order of  $G$ . It suffices to prove the case when  $G$  is a connected maximal planar graph. Clearly, it is true for small orders. Assume the hypothesis is true for order  $k$  and let  $G$  be a connected maximal planar graph of order  $k + 1$ .

Since  $G$  is maximal,  $3 \leq \delta(G) \leq 5$ .

Case 1.  $\delta(G) = 3$

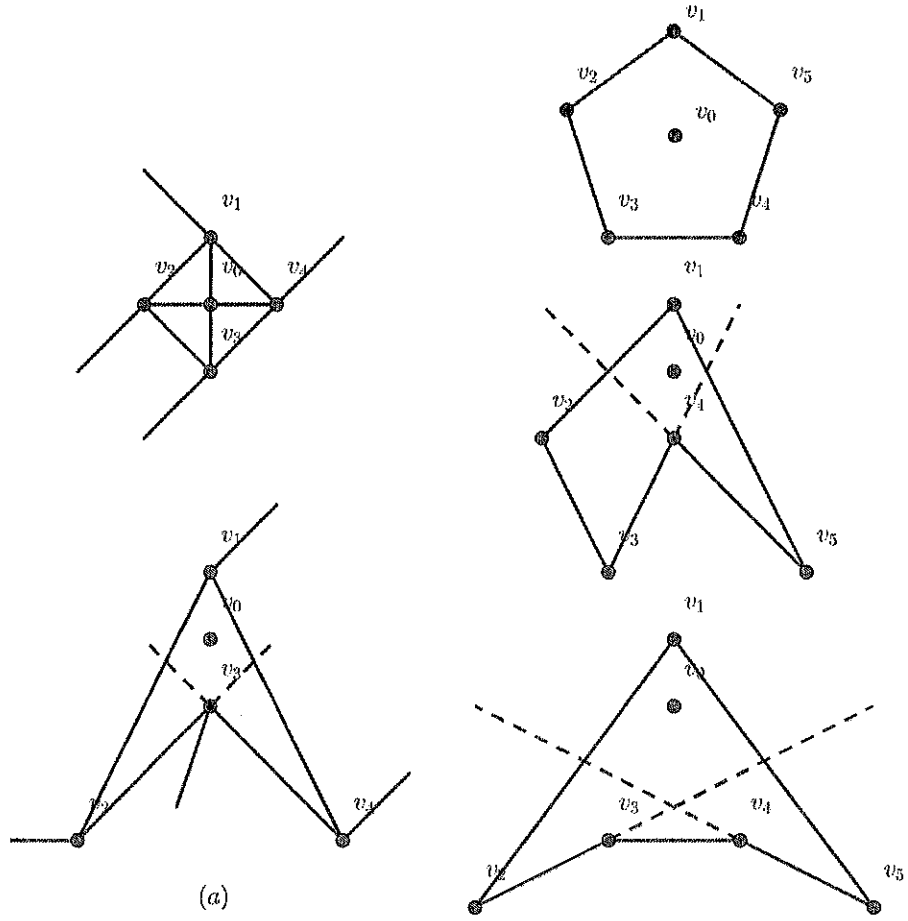
Let  $v_0 \in V(G)$  such that  $\deg_G(v_0) = 3$  and  $v$  is adjacent to  $v_1, v_2$  and  $v_3$ . Since  $G$  is maximal,  $\{\{v_1, v_2, v_3\}\}_G \cong K_3$ . This implies that  $G - v_0$  is also a maximal planar graph. By induction  $G - v_0$  has a straight line segment embedding. Now, put  $v_0$  back to the graph  $G - v_0$  such that  $v_0$  is inside the region bounded by  $\{\{v_1, v_2, v_3\}\}$  and connect  $v_0$  to the three vertices by straight line segment. This concludes the proof of this case.

Case 2.  $\delta(G) = 4$

The proof follows by a similar process by letting  $N(v_0) = \{v_1, v_2, v_3, v_4\}$ . Now,  $G - v_0 + v_1v_3$  is a maximal planar graph and thus it has a straight line segment embedding. The proof follows by placing  $v_0$  back to  $G - v_0 + v_1v_3 - v_1v_3$ . By considering the drawing of the embedding (Figure 33(a)), we are able to put  $v_0$  back and connect  $v_0$  to its neighbors in  $G$  by straight line segment.

Case 3.  $\delta(G) = 5$

Again, we use the same technique and the drawing can be seen in Figure 33(b). ■

Figure 33. Location of  $v_0$ 

- (\*) The following Theorem consider pseudographs, i.e., loops and multiedges are allowed.

### Theorem 51 (Euler-Poincaré Theorem)

Let  $G$  be a  $(p, q)$ -pseudograph which has a 2-cell embedding on  $S'_n$ . Then,  $p - q + f = 2 - 2n$  where  $f$  is the number of faces in the embedding.

**Proof.**

By induction on  $n$  and it's true when  $n = 0$  (by Euler's planar graph formula). Assume that the assertion is true when  $n = k \geq 0$  and  $G$  is a  $(p, q)$ -pseudograph which has a 2-cell embedding on  $S_{k+1}$ . Since  $k + 1 \geq 1$ , there exists a handle in the embedding, see Figure 34(a). It suffices to consider the embedding such that there exists at least one edge which passes through the handle (on the surface). Note that if we can pull back an edge without passing the handle, then pull it back, see Figure 34(b). Now, we apply the idea of "cut and past" to obtain a 2-cell embedding of  $\tilde{G}$  on  $S_k$ .

By using a circle around the handle, we can cut the handle through the circle and obtain  $\tilde{G}$ , see Figure 34(c). As a consequence the graph  $\tilde{G}$  is embedded in  $S_k$ . If there are  $t$  edges passing through the handle then  $|\tilde{G}| = p + 2t$ ,  $\|\tilde{G}\| = q + 3t$  and the embedding in embedding in  $S_k$  has  $f + t + 2$  faces. Hence,

$$(p + 2t) - (q + 3t) + (f + t + 2) = 2 - 2k.$$

This implies that  $p - q + f = 2 - 2(k + 1)$ . ■

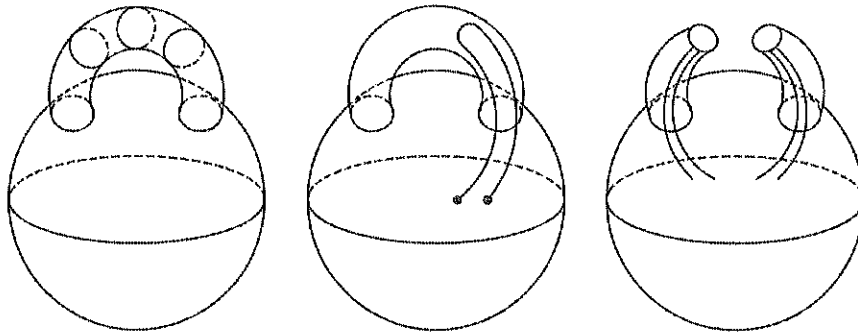
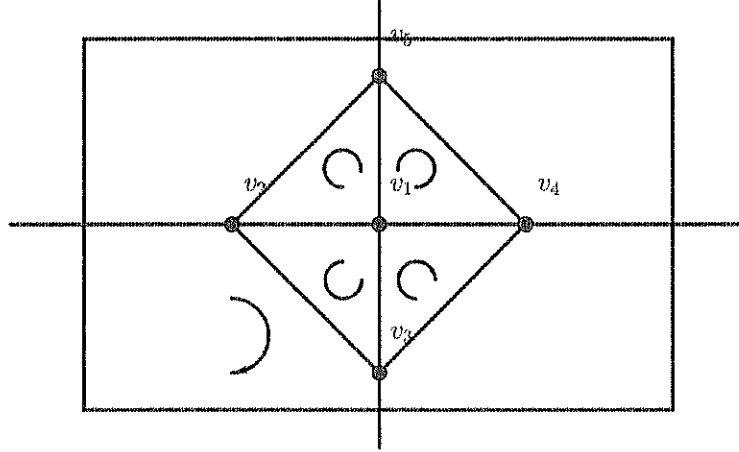


Figure 34(a)   Figure 34(b), pull edges back   Figure 34(c)

**Exercise** Ex. 3-1.

Give a more precise proof of Euler-Poincaré Theorem.

(•) How to find a 2-cell embedding on  $S_k$ ?



The above figure provide an embedding of  $K_5$  on  $S_1$ , their regions are  $((v_1, v_2, v_5))$ ,  $((v_1, v_3, v_2))$ ,  $((v_1, v_4, v_3))$ ,  $((v_1, v_5, v_4))$ , and  $((v_2, v_3, v_5, v_2, v_4, v_5, v_3, v_4))$ .

(•) Each are of  $D_5$  occurs exactly once in a region (face).

(••) By considering each vertex, we observe that there are five permutations to determine this embedding.

$$\pi_1 = (2, 3, 4, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix} \text{ (Defined on } v_1.)$$

$$\pi_2 = (3, 1, 5, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{pmatrix} \text{ (Defined on } v_2.)$$

$$\pi_3 = (4, 1, 2, 5)$$

$$\pi_4 = (3, 2, 5, 1)$$

$$\pi_5 = (1, 4, 3, 2)$$

On the other hand, if we are given five permutations one from each vertex, then we have a 2-cell embedding. For example,  $\pi_1 = (3, 2, 4, 5)$ ,  $\pi_2 = (3, 1, 5, 4)$ ,  $\pi_3 = (4, 1, 2, 5)$ ,  $\pi_4 = (3, 2, 5, 1)$ , and  $\pi_5 = (1, 4, 3, 2)$ .

Now, if we start from the arc  $(v_1, v_2)$ , then the permutation of  $\pi_2$  will give the next vertex of the oriented 2-cell containing  $(1, 2)$ .

$$v_1 - v_2 - v_{\pi_2(1)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_{\pi_1(5)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_3 \cdots \Rightarrow \cdots \Rightarrow ((v_1, v_2, v_5, v_1, v_3, v_2, v_1, v_4, v_3)). \quad (v_{\pi_3(4)} = v_1 \text{ and } v_{\pi_1(3)} = v_2)$$

(\*) Therefore, given a  $(p, q)$ -graph  $G$ , we can define  $p$  permutations for  $p$  vertices  $v_i$  such that each permutation is a cycle using the indices of vertices in  $N_G(v_i)$ . Then, a 2-cell embedding will be obtained.

### Theorem 52 (Heffter's Rotational Embedding Scheme)

Let  $G$  be a nontrivial connected graph with  $V(G) = \{v_1, v_2, \dots, v_p\}$ . For each 2-cell embedding of  $G$  on a surface, there exists a unique  $p$ -tuple  $(\pi_1, \pi_2, \dots, \pi_p)$  where for  $i = 1, 2, \dots, p$ ,  $\pi_i$  is a cyclic permutation of  $V(i)$  that describes the subscripts of the vertices in  $N_G(v_i)$  in counterclockwise order about  $v_i$ . Conversely, for each  $p$ -tuple  $(\pi_1, \pi_2, \dots, \pi_p)$ , there exists a 2-cell embedding of  $G$  on some surface such that for  $i = 1, 2, \dots, p$ , the subscripts of the vertices adjacent to  $v_i$  are in counterclockwise order about  $v_i$  given by  $\pi_i$ . Moreover, the set  $\{\pi_1, \pi_2, \dots, \pi_p\}$  induces a mapping  $\pi$  such that  $\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$  for each pair of adjacent vertices  $v_i$  and  $v_j$ ,  $1 \leq i \neq j \leq p$ .

**Proof.**

The scheme was first observed and used by Dyck (1888) and Heffter (1891). A formalized version was obtained later in 1960 by Edmonds.

The main idea has been mentioned before the statement of Theorem 52. The genus of surface can be obtained after the number of face (regions) has been determined. ■

**Theorem 53**

$$\gamma(K_{2m,2n}) = (m-1)(n-1), \quad m \leq n.$$

**Proof.**

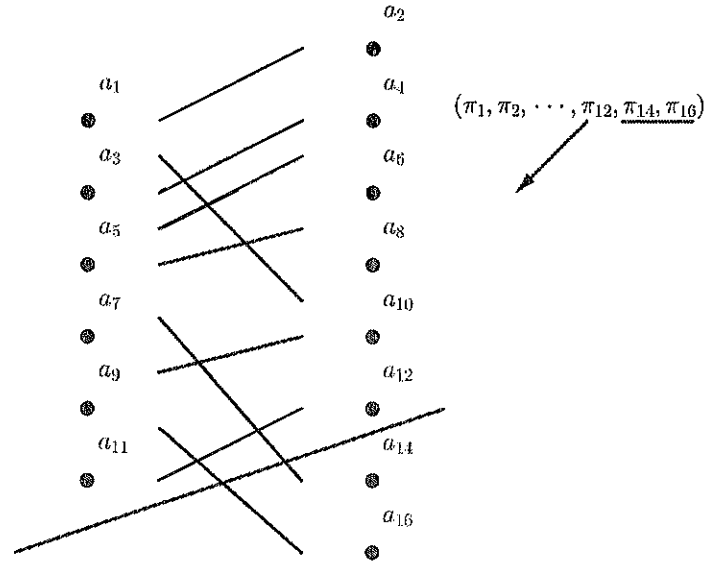
For convenience, let  $K_{2m,2n} = (A_o, A_e)$  where  $A_o = \{a_1, a_3, \dots, a_{4m-1}\}$  and  $A_e = \{a_2, a_4, \dots, a_{4n}\}$ . Note that  $m$  may not be equal to  $n$ . See Figure 35 for the case  $2m = 6$  and  $2n = 8$ . Now, let

$$\left\{ \begin{array}{l} \pi_1 = \pi_5 = \dots = \pi_{4m-3} = (2 \ 4 \ 6 \ \dots \ 4n); \\ \pi_3 = \pi_7 = \dots = \pi_{4m-1} = (4n \ 4n-2 \ \dots \ 6 \ 4 \ 2); \\ \pi_2 = \pi_6 = \dots = \pi_{4n-2} = (1 \ 3 \ 5 \ \dots \ 4m-1); \text{ and} \\ \pi_4 = \pi_8 = \dots = \pi_{4n} = (4m-1 \ 4m-3 \ \dots \ 5 \ 3 \ 1). \end{array} \right.$$

We may check that this embedding does have  $2mn$  regions each of them is bounded by a 4-cycle. Hence, by Theorem 51,

$$2\gamma(G) = 2 - p + q - r = 2 - (2m + 2n) + 4mn - 2mn = 2mn - 2m - 2n + 2.$$

Hence,  $\gamma(G) = (m-1)(n-1)$ . ■

Figure 35.  $K_{6,8}$ 

- (••) They are  $\prod_{i=1}^p (\deg_G(v_i) - 1)!$   $p$ -tuples of  $(\pi_1, \pi_2, \dots, \pi_p)$ .
- (•••) The 2-cell embedding with the largest number of faces gives the genus of  $G$ .
- (\*) The 2-cell embedding with the smallest number of faces gives the "maximum" genus of  $G$ , denoted by  $\gamma_M(G)$ .
- (\*\*) Finding  $\gamma(G)$  is a very difficult problem in general.
- (\*\*) Finding  $\gamma_M(G)$  is comparatively easier.

### Theorem 54

Let  $cr(G)$  denote the crossing number of  $G$ . Then,  
 $cr(K_5) = cr(K_{3,3}) = 1$  and  $cr(K_6) = 3$ .

**Proof.**

Since  $\gamma(K_5) = \gamma(K_{3,3}) = 1$ , the proof follows by a drawing with "1" crossing number. Now, we consider  $K_6$ . By Figure 35,  $cr(K_6) \leq 3$ . Let  $cr(K_6) = k$ . Then, we may convert the crossings into new vertices.

Hence, we have a planar graph  $G$  (obtained from above):  $|G| = 6 + k$  and  $\|G\| = 15 + 2k$ . By the fact  $15 + 2k \leq 3(6 + k) - 6$ , we have  $k \geq 3$ . ■

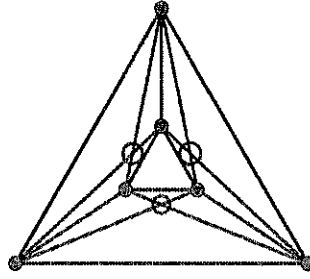


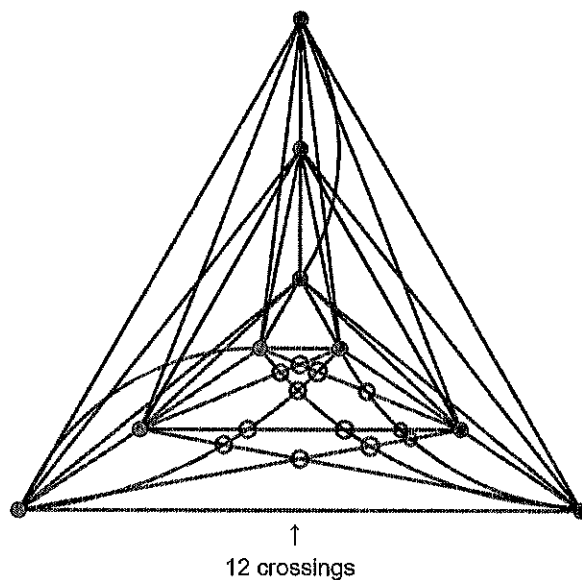
Figure 35.

**Theorem 55**

$$cr(K_9) = 36.$$

**Proof.**

For the upper bound, it suffices to give a drawing which has exactly 36 crossings. But, it is very technical to show the lower bound. Here, we provide a drawing for  $cr(K_9) \leq 36$ , see Figure 36.

Figure 36.  $cr(K_9)$ **Conjecture**

$$cr(K_p) = \frac{1}{4} \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{p-3}{2} \right\rfloor. \quad (\text{True for } 1 \leq p \leq 10.)$$


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**Vertex-Coloring**

- (•)  $k$ -coloring (proper):  $\varphi : V(G) \rightarrow \{1, 2, 3, \dots, k\}$  s.t.  
 $uv \in E(G) \Rightarrow \varphi(u) \neq \varphi(v)$ .
- (•)  $\chi(G) = \min\{k \mid G \text{ has a } k\text{-coloring}\}$  (Chromatic number of  $G$ )
- (•)  $G$  is  $n$ -critical (chromatically) if  $\chi(G - v) < \chi(G)$  for each  
 $v \in V(G)$ .
- (••) Every graph  $G$  has an  $n$ -critical induced subgraph  $H$ .