

Figure 29. Bipartite version

Ramsey Theory

- (•) The Ramsey number $R(s, t)$ is the smallest value "n" for which either a graph G of order n contains K_s or $\bar{G} \geq K_t$.
- (•) Edge-coloring version of Ramsey number. The Ramsey number $R(s, t)$ is the smallest value "n" for which any 2-edge-colored K_n (red and blue), either there exists a red K_s or a blue K_t . (A red K_s is a complete graph of order s such that all its edges are colored red.)
- (•) $R(3, 3) = 6$ (Do you know this fact?)

Theorem 37

The following statements are true:

- (1) $R(s, 2) = s$ and $R(2, t) = t$,
- (2) $R(s, t) = R(t, s)$,
- (3) For $s > 2, t > 2$, $R(s, t) \leq R(s, t-1) + R(s-1, t)$, and
- (4) $R(s, t) \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$.

(3) If $R(s, t-1)$ and $R(s-1, t)$ are even, then $R(s, t) \leq R(s, t-1) + R(s-1, t) - 1$. (proof.)

例如
 $R(4, t) = R(4, t-1) + R(3, t)$
 则存在一个反例
 它有 $R(4, t-1) + R(3, t) - 1$
 个顶点(奇数), 每个顶点
 旁边分别有 $R(4, t-1) - 1$
 蓝色边, $R(3, t) - 1$
 红色边, 但不可能。
 (?)

Proof.

(1) and (2) are easy to see.

Claim of (3)

Let $n = R(s, t-1) + R(s-1, t)$. Then, in K_n , each vertex is of degree $R(s, t-1) + R(s-1, t) - 1$. Therefore, if K_n is 2-edge-colored by red and blue, then the edges incident to a fixed vertex $x \in V(K_n)$ are either red edges or blue edges. By Pigeon-hole principle, either there are $R(s, t-1)$ blue edges or $R(s-1, t)$ red edges. If the first case holds, then in $\langle N_{K_n}(x) \rangle_{K_n}$ (a complete graph of order $R(s, t-1)$), either there exists a red K_s or blue K_{t-1} . Hence, we have a red K_s or a blue K_t in K_n . The other case can be obtained by a similar argument.

Claim of (4)

By *inductive* argument. (Or induction)

$$\begin{aligned}
 R(s, t) &\leq R(s, t-1) + R(s-1, t) \\
 &\leq \binom{s+t-1-2}{s-1} + \binom{s-1+t-2}{t-1} \\
 &= \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2} \\
 &= \binom{s+t-3+1}{s-1} \\
 &= \binom{s+t-2}{s-1}.
 \end{aligned}$$

■

Theorem 38 (Erdős and Szekeres, 1935)

For each $s \geq 2$, $R(s) \leq \frac{2^{2s-2}}{s^{1/2}}$. ($R(s) =_{\text{def}} R(s, s)$.)

Proof.

$R(s, s) \leq \binom{2s-2}{s-1}$. We claim $\binom{2s-2}{s-1} \leq \frac{2^{2s-2}}{s^{1/2}}$ by induction on s .

First, if $s = 2$, $2 \leq \frac{4}{\sqrt{2}}$, the assertion is true. Assume that the assertion

is true for $s = k$, thus $\binom{2k-2}{k-1} \leq \frac{2^{2k-2}}{k^{1/2}}$. Now, we calculate

$$\begin{aligned} \binom{2k}{k} &= \frac{(2k)!}{k!k!} = \frac{2k \cdot (2k-1) \cdot (2k-2)!}{k^2(k-1)! \cdot (k-1)!} = \frac{2k(2k-1)}{k^2} \cdot \binom{2k-2}{k-1} \leq \\ &= \frac{4k^2 - 2k}{k^2} \cdot \frac{2(2k-2)}{k^{1/2}} = \frac{4k-2}{4k} \cdot \frac{2(2k)}{k^{1/2}}. \quad \text{---(1)} \end{aligned}$$

Since $(k+1)^{1/2} \leq \frac{4k \cdot k^{1/2}}{4k-2}$, we conclude that $\binom{2k}{k} \leq \frac{2^{2k}}{(k+1)^{1/2}}$. ■

(•) The result has been there for almost 50 years before the improvement due to Thomason in 1988:

$$R(s) \leq 2^{2s}/s.$$

(••) The original proof by Ramsey shows that

$$R(s) \leq 2^{2s} - 3 = \frac{2^{2s-2}}{2}. \quad (1930)$$

Theorem 39

$$R(k) \geq \lceil 2^{\frac{k}{2}} \rceil. \quad (k \geq 3)$$

Proof. (Probabilistic method)

Consider a random red-blue coloring of the edges of K_n . For a fixed set T of k vertices, let A_T be the event that $\langle T \rangle_{K_n}$ is monochromatic.

Hence $P(A_T) = \left(\frac{1}{2}\right)^{\binom{k}{2}} \cdot 2(\text{red or blue}) = 2^{1-\binom{k}{2}}$. Since there are $\binom{n}{k}$ $\binom{k}{2}$ $\mathcal{C}$

possible sets for T , the probability that at least one of A_T occurs is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$. Now, if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then no event A_T occurs is of positive probability, i.e., there exists a coloring of edges such that no monochromatic K_k occurs. Therefore, for such n , $R(k) > n$.

Let $n = \lfloor 2^{\frac{k}{2}} \rfloor$. (It suffices to show that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.)

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \leq \frac{(2^{\frac{k}{2}})^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \leq \frac{2^{1+\frac{k}{2}}}{k!} < 1 \quad (k \geq 3).$$

$(1 - \binom{k}{2}) = 1 - \frac{k^2}{2} + \frac{k}{2}$ Hence, $R(k) \geq \lfloor 2^{\frac{k}{2}} \rfloor$. ■

(•) Combining Theorems obtained above

$$2^{\frac{s}{2}} \leq R(s) \leq 2^{2s-3} \text{ for } s \geq 2.$$

(••) Open problem: $R(s) = 2^{(c+o(1))s}$, (c may be equal to 1).

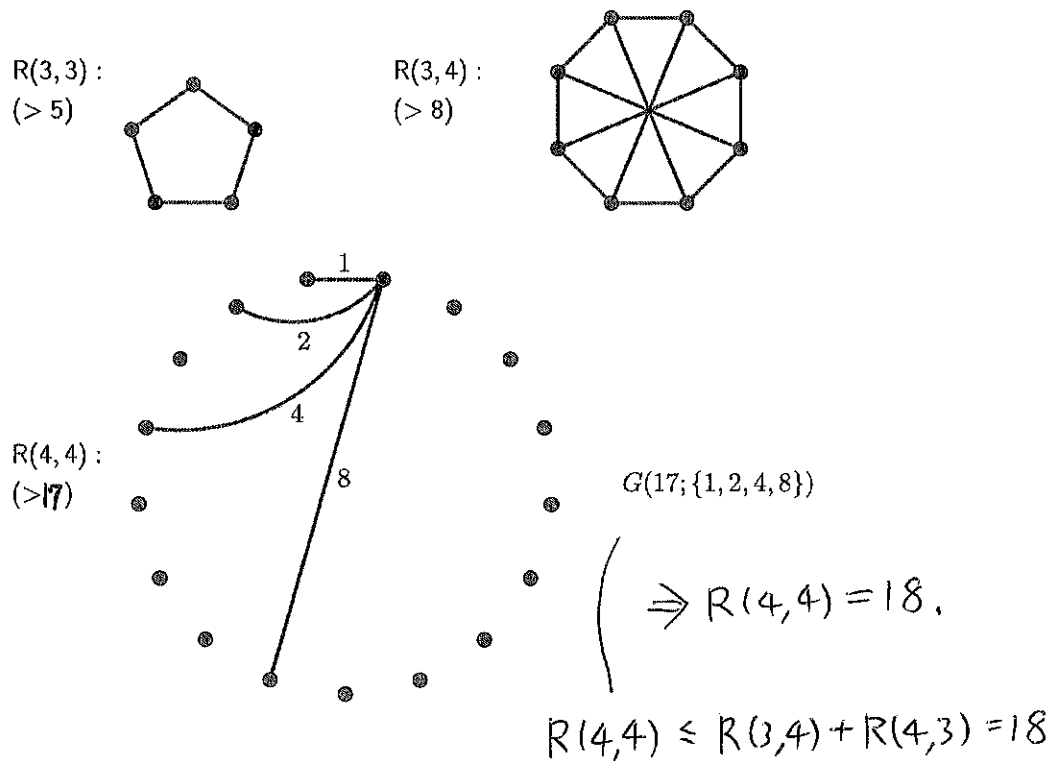
Theorem 40

Known results of $R(s, t)$.

t \ s	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	36-41	49-61	59-84	73-115
5			43-48	58-87	80-143	101-216	133-316
6				102-165	115-298	134-495	183-780
7					205-540	217-1031	252-1713
8						282-1870	329-3583
9							565-6588

$$R(s, t) = R(t, s)$$

- (•) The results of lower bounds are obtained by "a special edge-coloring" with two colors. Corresponding to the coloring we have G and $\bar{G}$ of order (prescribed).



Bonus

Find as many vertices (n) as possible such that a graph G of order n satisfying $G \not\supset K_5$ and $\bar{G} \not\supset K_5$. (Try 43!)

Exercise C-1

Find a better upper bound for $R(s)$. (Do your best!)

Theorem 41

$$R(p_1, p_2, \dots, p_t) \leq R(p_1 - 1, p_2, \dots, p_t) + R(p_1, p_2, \dots, p_{t-1}) - t + 2.$$

Proof.

By a similar argument as the proof

$$R(s, t) \leq R(s-1, t) + R(s, t-1), \quad \blacksquare$$

$$(\bullet) \quad R(3, 3, 3) \leq 6 + 6 + 6 - 3 + 2 = 17 \text{ (Theorem 41).}$$

($\bullet\bullet$) There exists a 3-edge-coloring of K_{16} such that no mono-chromatic triangles occur. (?)

Theorem 42

$$R(3, 3, \dots, 3) =_{def} R_k(3) \leq \lfloor e \cdot k! \rfloor + 1.$$

k -tuples

Proof.

Since $R(3, 3) = 6$, $R(3, 3, 3) = 17$, the assertion is true for $k = 2$ and

3. Assume that it holds for $k-1$ when $k > 3$. Hence,

$$R_{k-1}(3) \leq \lfloor e(k-1)! \rfloor + 1. \text{ By Theorem 41,}$$

$$\begin{aligned} R_{k-1}(3) &\leq k(\lfloor e(k-1)! \rfloor + 1) - k + 2 \\ &= k\lfloor e(k-1)! \rfloor + 2. \end{aligned}$$

$$\begin{aligned} \text{Now, } k\lfloor e(k-1)! \rfloor &= k \left\lfloor (k-1)! \cdot \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(k-1)!} + \frac{1}{k!} + \dots\right) \right\rfloor = \\ &k \left\lfloor M + \frac{1}{k} + \frac{1}{(k+1)k} + \frac{1}{(k+2) \cdot (k+1)k} + \dots \right\rfloor = \dots = \lfloor (k!) \cdot e \rfloor - 1. \quad (?) \quad \blacksquare \end{aligned}$$

($\bullet$) Instead of $R(s, t)$, we use $R(H_1, H_2)$ to denote the smallest integer n such that any 2-edge-coloring (red, blue) of K_n , either there exists a red H_1 or a blue H_2 .

Theorem 43

$$R(C_4, C_4) =_{def} R(C_4) \leq 8.$$

Proof.

Consider a graph $G \not\geq C_4$ and $|G| = 8$. By Theorem 35-1, $\|G\| \leq \frac{1}{2}z(8, 8; 2, 2) \leq \frac{n \cdot (1 + \sqrt{4n-3})}{4} = 2(1 + \sqrt{29}) < 14$. That is, if a graph G is of order 8 and size 14, then $G \geq C_4$. Now, in a 2-edge-colored K_8 , either red or blue edges induce such a graph, the proof follows. ■

Theorem 44

$$R_k(C_4) \leq \underline{k^2 + k + 2}.$$

Proof.

Let $n = k^2 + k + 2$ and consider a k -edge-colored K_n . Since $ex(n; C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}) = \frac{k^2 + k + 2}{4} \cdot (1 + \sqrt{4k^2 + 4k + 5}) \leq \frac{k^2 + k + 2}{2} \cdot \frac{1 + \sqrt{4k^2 + 4k + 5}}{2}$. Now, compare $(k^2 + k + 2)$ and $\frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2}$. By direct calculation, we have $(k^2 + k + 2) > \frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2}$. This implies that $k \cdot ex(n; C_4) \leq \frac{k^2 + k + 2}{2} \cdot \frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2} < \frac{(k^2 + k + 2)(k^2 + k + 1)}{2} = \binom{n}{2}$. ■

$$(\bullet) R(H_1, H_2) = \min\{n \mid |G| = n, G \geq H_1 \text{ or } \bar{G} \geq H_2\}.$$