

Theorem 28 (Tutte's 1-factor theorem)

A nontrivial graph G has a 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G - S$, $o(G - S) \leq |S|$.

Proof.

($\Rightarrow$) Assume that F is a 1-factor of G and there exists a proper subset W of $V(G)$ such that $o(G - W) > |W|$. Since an odd component H has an odd number of vertices, one of the vertices in H incident to F must be joining a vertex of W . But, we have more odd components than $|W|$. One of the vertices in W will be incident to at least two edges in F , a contradiction.

($\Leftarrow$) Since $o(G - \emptyset) \leq 0$, G contains only even components. Hence, $|G|$ is even. Furthermore, if $|S|$ is odd (even), $o(G - S)$ must be odd (even). So, $|S|$ and $o(G - S)$ are of the same parity. We shall prove the sufficiency by induction on $|G| = n$. Clearly, if $n = 2$, then $G \cong K_2$.

Assume for all graphs H of even order less than n that if

$o(H - W) \leq |W|$ for every proper subset of $V(H)$, then H has a 1-factor. Let G be a graph of order n and $o(G - S) \leq |S|$ for each proper subset S of $V(G)$. We claim that G has a 1-factor.

Case 1. $\forall S \subsetneq V(G)$, $|S| \geq 2$ and $o(G - S) < |S|$. —(*)

The fact of parity shows that $o(G - S) \leq |S| - 2$ for all S . Let $e = uv$ be an edge of G and consider $G - \{u, v\}$. By the fact that for each proper subset T of $V(G - \{u, v\})$, $o(G - \{u, v\} - T) \leq |T|$ and induction

hypothesis, $G - \{u, v\}$ has a 1-factor, so is G . (If

$o(G - \{u, v\} - T) > |T| = |T \cup \{u, v\}| - 2$, then

$o(G - (\{u, v\} \cup T)) \geq |T \cup \{u, v\}|$, a contradiction to $(*)$.)

Case 2. $\exists R \subseteq V(G)$, s.t. $o(G - R) = |R|$ where $2 \leq |R| \leq n$.

Among all such R 's, let S be the one of maximum cardinality

$|S| = h$. Now, let $G_1, G_2, \dots, G_h$ denote the odd components of $G - S$.

Note that the h odd components are the only components in $G - S$.

For otherwise, let G_0 be an even component of $G - S$ and $v_0 \in V(G_0)$.

Then, $o(G - S \cup \{v_0\}) \geq h + 1 = |S \cup \{v_0\}|$. In fact,

$o(G - S \cup \{v_0\}) = |S \cup \{v_0\}|$ by the assumption. Now, we have a larger

" R " for S , a contradiction.

For $i = 1, 2, \dots, h$, let S_i be the set of vertices in S which are adjacent with vertices in G_i . None of S_i 's will be empty. For otherwise, G_i is an odd component of G and it is not possible. (G has only even components.)

Now, for $1 \leq k \leq h$, consider the union T of "any" k sets in

$\{S_1, S_2, \dots, S_h\}$. Suppose that $|T| < k$. Since $o(G - T)$ is at least k ,

$o(G - T) \geq k > |T|$ which violates the assumption $o(G - S) \leq |S|$. So,

$\{S_1, S_2, \dots, S_h\}$ has an SDR $(v_1, v_2, \dots, v_h)$ where $v_i \in S_i$. Moreover, in

G_i , let $u_i \sim_G v_i$.

For showing that G has a 1-factor, it's left to show that for each

$i = 1, 2, \dots, h$, $G_i - u_i$ has a 1-factor. Therefore, let W be a proper

subset of $V(G_i - u_i)$ and we claim $o(G - u_i - W) \leq |W|$. (This will

imply the existence by induction.)

Suppose not. Let $o(G - u_i - W) > |W|$. Again, since $o(G - u_i - W)$ and $|W|$ are of the same parity, we have $o(G - u_i - W) \geq |W| + 2$. Now, combining with S , $o(G - u_i - W - S) = o(G - S) + o(G - u_i - W) - 1$.

$$\begin{aligned}
 o(G - u_i - W - S) &= o(G - S) + o(G - u_i - W) - 1 \\
 &\geq |S| + |W| + 2 - 1 \\
 &= |S| + |W| + 1 \\
 &= |\{u_0\} \cup W \cup S|.
 \end{aligned}$$

Hence, we conclude that $o(G - (\{u_0\} \cup W \cup S)) = |\{u_0\} \cup W \cup S|$. Since $\{u_0\} \cup W \cup S$ is larger than S (in size), this contradicts to the choice of S . As a consequence, we have the fact: $G_i - u_i$ contains a 1-factor and thus G has a 1-factor. ■

Theorem 29 (Petersen)

Every 2-edge-connected (bridge less) cubic graph G has a 1-factor F and $G - F$ is a 2-factor.

Proof.

Let $S \subseteq V(G)$ and consider an odd component in $G - S$. (Notice that if $o(G - S) = 0$, then $o(G - S) \leq |S|$.) Since G is cubic, the number of edges between S and C must be odd. (Otherwise, the degree sum of $V(C)$ in $G - S$ is odd.) By the assumption that G is 2-edge-connected, there are at least three edges in $\langle S, C \rangle$. This implies that the number of edges between S and $G - S$ is at least $3 \cdot o(G - S)$. By the fact that G is cubic, such edges are at most $3 \cdot |S|$. Hence, $3 \cdot o(G - S) \leq 3 \cdot |S|$.

By Tutte's 1-factor theorem, G has a 1-factor F and $G - F$ is clearly a 2-factor. ■

Theorem 30 (Petersen's 2-factor theorem)

Let k be an even integer. Then, a k -regular graph contains $\frac{k}{2}$ edge-disjoint 2-factors.

Proof.

It suffices to consider a connected k -regular graph G . Let $k = 2h$.

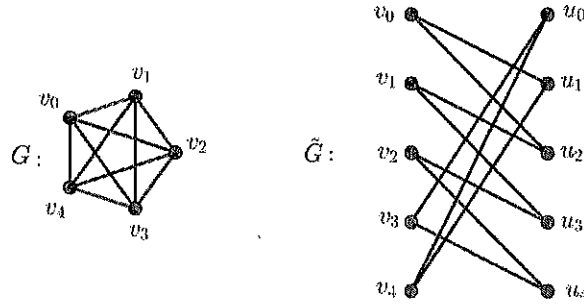
By Euler's circuit theorem, G has an eulerian circuit

$$Z : ((v_0, v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_t, v_0)).$$

Now, we defined a bipartite graph $\tilde{G} = (A, B)$ such that $|A| = |B| = |G|$, (See Figure 26.) and $v_i \sim_{\tilde{G}} v_j$ if v_i, v_j are two consecutive vertices in Z . Since G is $2h$ -regular, $\tilde{G}$ is h -regular. By König's Theorem, $\tilde{G}$ contains h edge-disjoint perfect matchings. It is not difficult to see that a perfect matching in $\tilde{G}$ gives a 2-factor in G . This concludes the proof. ■

(•) Unfortunately, we are not able to control the type of 2-factors we are going to obtain.

(••) A perfect matching in $\tilde{G}$ can be represented as a permutation.



$((v_0, v_1, v_2, v_3, v_4, v_0, v_2, v_4, v_1, v_3))$ is an eulerian circuit of G .

- A graph F (or a class $\mathcal{F}$) is said to be forbidden in a class of graphs $\mathcal{G}$ if for each $G \in \mathcal{G}$, $G \not\supseteq F$ (or $G \not\supseteq F$ for each $F \in \mathcal{F}$).
- $ex(n; F) = \max\{\|G\| \mid G \text{ is a graph of order } n \text{ such that } G \not\supseteq F\}$.
 $ex(n; \mathcal{F})$ can be defined accordingly.
- The graph G of order n with $\|G\| = ex(n; F)$ is called an extremal graph of order n with forbidden graph F .
- The class of bipartite graphs with partite sets of sizes m and n respectively is denoted by $G_2(m, n)$.
- The extremal size of graphs in $G_2(m, n)$ which do not contain $K_{s,t}$ is denoted by $z(m, n; s, t)$. (The notation is in honor of Zarankiewicz.)
- Notice that $ex(n; K_{s,t})$ is different from $z(m, n; s, t)$.
- $z(m, n; s, t) \geq 2ex(n; K_{s,t})$. (?)
- $T_r(n) =_{def} K_{\lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor, \dots, \lfloor \frac{n}{r} \rfloor}$ and $\|T_r(n)\| = t_r(n)$.

Theorem 31 (Turán, 1941)

$ex(n; K_{r+1}) = t_r(n)$ and $T_r(n)$ is the unique extremal graph.

Proof.

(1st)

By induction on n . (To show $ex(n; K_{r+1}) = t_r(n)$.)

Since $T_r(n)$ does not contain K_{r+1} , $ex(n; K_{r+1}) \geq t_r(n)$. We claim $ex(n; K_{r+1}) \leq t_r(n)$. Let G be a graph such that $G \not\supseteq K_{r+1}$ and G is of maximum size. Then, $G \geq K_r$. For otherwise, we may add more edges to G . Let $W \subseteq V(G)$ and $\langle W \rangle_G \cong K_r$. Let $U = V(G) \setminus W$.

Now, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + \|\langle U \rangle_G\|$. The term $(r-1)(n-r)$ comes from the fact that each vertex of U is incident to at most $r-1$ vertices of W . By induction hypothesis, $\|\langle U \rangle_G\| \leq t_r(n-r)$. Hence, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + t_r(n-r) = t_r(n)$. This is a direct consequence of adding one vertex of W to one partite set of $T_r(n-r)$ and $\left\lfloor \frac{n-r}{r} \right\rfloor + 1 = \left\lfloor \frac{n}{r} \right\rfloor \left(\left\lfloor \frac{n-r}{r} \right\rfloor + 1 = \left\lfloor \frac{n}{r} \right\rfloor \right)$.

Next, we claim the uniqueness. The proof is also by induction on n . Let $y \in V(G)$ such that $\deg_G(y) = \delta(G)$. Clearly, $G - y$ does not contain K_{r+1} and thus $\|G - y\| = \|G\| - \delta(G) \geq t_r(n-1)$ by the proof of the first part. By induction, $T_r(n-1)$ is the unique graph which is isomorphic to $G - y$. This implies that in $G - y$ the smallest partite set is of size $\left\lfloor \frac{n-1}{r} \right\rfloor$. Since $T_r(n-1)$ contains a K_r from r -partite sets, y is incident to at most $r-1$ partite sets of $T_r(n-1)$. Therefore, y can be recognized as a vertex in one of the partite sets, and thus the number of edges between y and $G - y$ is $(n-1) - \left\lfloor \frac{n-1}{r} \right\rfloor = n - \left\lfloor \frac{n}{r} \right\rfloor$. This implies that $G \cong T_r(n)$. ■

(2nd) (Zykov)

Only $ex(n; K_{r+1}) = t_r(n)$.

Let $v_1 \in V(G)$ such that $\deg_G(v_1) = \Delta(G)$ and let $W = N(v_1)$. Let $G_1 = G - \langle N(v_1) \rangle_G + T_{r-1}(\Delta(G))$, and $U_1 = V(G_1) \setminus (W \cup \{v_1\})$. See Figure 26.

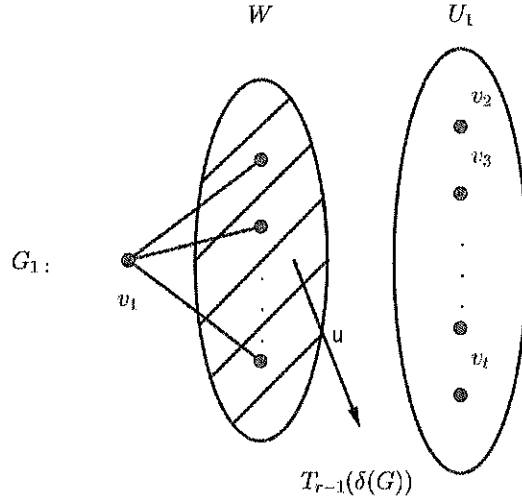


Figure 26. G_1

If U_1 is an empty set, then we stop and evaluate $\|G_1\|$. Otherwise, if $U_1 \neq \emptyset$, let $v_2 \in U_1$. Now, delete all edges with $U_2 = V(G) \setminus (W \cup \{v_1, v_2\})$ in G_1 which are incident to v_2 , E_2 and add v_2u for each $u \in W$ to $G_1 - E_2$. The new graph is defined as G_2 . Since $T_{r-1}(\Delta(G))$ defined on W does not contain K_r , G_2 does not contain K_{r+1} . By continuing this process (until U_t is empty), we shall obtain a complete r -partite graph H such that $\|H\| \geq \dots \geq \|G_2\| \geq \|G_1\| \geq \|G\|$. (Notice that $v_1, v_2, \dots, v_t$ is a new partite set.)

(3rd)

We can replace all the vertices of U_1 at the same time by deleting all the edges incident to U_1 and add $\langle W, U_1 \rangle$ to obtain a complete r -partite graph. ■

Theorem 32 (Erdős, 1970)

Let $G \not\preceq K_{r+1}$. Then, there exists an H satisfying (1) H is an r -partite graph, (2) $V(H) = V(G)$, and (3) $\forall x \in V(G)$, $\deg_G(x) \leq \deg_H(x)$. Moreover, if G is not a complete r -partite graph, then there exists a vertex $z \in V(G)$, s.t. $\deg_G(z) < \deg_H(z)$.

Proof.

By induction on r for the whole statement, and $r = 1$ is true. Let the assertion be true for $r' < r$.

Let $x \in V(G)$ s.t. $\deg_G(x) = \Delta(G)$, $N(x) = W$ and $\langle W \rangle_G = G_0$. Clearly, $G_0 \not\preceq K_r$. By induction, there exists an $(r - 1)$ -partite graph H_0 , s.t., $V(H_0) = W$, $\forall y \in W$, $\deg_{G_0}(y) \leq \deg_{H_0}(y)$, moreover, if G_0 is not a complete $(r - 1)$ partite graph, then there exists a $y' \in W$, s.t. $\deg_{G_0}(y') < \deg_{H_0}(y')$.

Now, let $H = H_0 \vee (V \setminus W)$ (join), see Figure 27. So, H is an r -partite graph. For $z \in V \setminus W$, $\deg_G(z) \leq \Delta(G) = |W| = \deg_H(z)$, and if $z \in W$, $\deg_G(z) \leq \deg_{G_0}(z) + n - |W| \leq \deg_{H_0}(z) + n - |W| = \deg_H(z)$. This concludes the first part. For the second part, assume that $\deg_G(x) = \deg_H(x)$ for all $x \in V(G)$. Hence, $\|G\| = \|H\|$ and thus $\|G_0\| = \|H_0\|$. (For otherwise, $\|G\| > \|H\|$.) Moreover, $\deg_{G_0}(x) = \deg_{H_0}(x)$ for each $x \in W$. Suppose not. Let

$\deg_{G_0}(x') < \deg_{H_0}(x')$ for some vertex $x' \in W$. This implies that $\deg_G(x') < \deg_H(x') = \deg_{H_0}(x) + n - |W|$, a contradiction. As a consequence, G_0 (H_0) is a complete $(r-1)$ -partite graph and G is a complete r -partite graph as well. ■

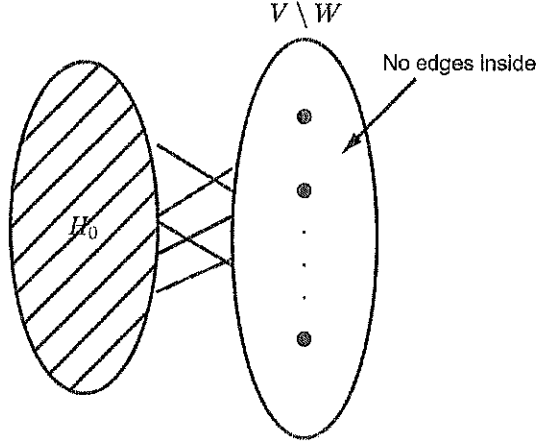


Figure 27.

(••) Try to estimate $z(m, n; s, t)$

Theorem 33 (Important Lemma)

Let $2 \leq s \leq m$, $2 \leq t \leq n$, $0 \leq r \leq m$, $z = km + r$ and $z = my$. $y \in \mathbb{R}^+$

Let G be a bipartite graph, $G \in G_2(m, n)$ and $G \not\geq K_{s,t}$. Then,

$$m \cdot \binom{y}{t} \leq (m-r) \binom{k}{t} + r \binom{k+1}{t} \leq (s-1) \binom{n}{t}.$$

(Remark. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex if

$$tf(x) + (1-t)f(y) \geq f(xt + y \cdot (1-t)), \quad 0 \leq t \leq 1.)$$

Proof.

Let $G = (A, B)$ where $|A| = m$ and $|B| = n$. Define a graph

$H = \left(A, \binom{B}{t} \right)$. $\binom{B}{t}$ is the collection of all t -subset of B . And $x \sim_G y$ for each $y \in T$. Figure 28 is an example $|A| = 5$, $|B| = 6$ and $t = 3$.

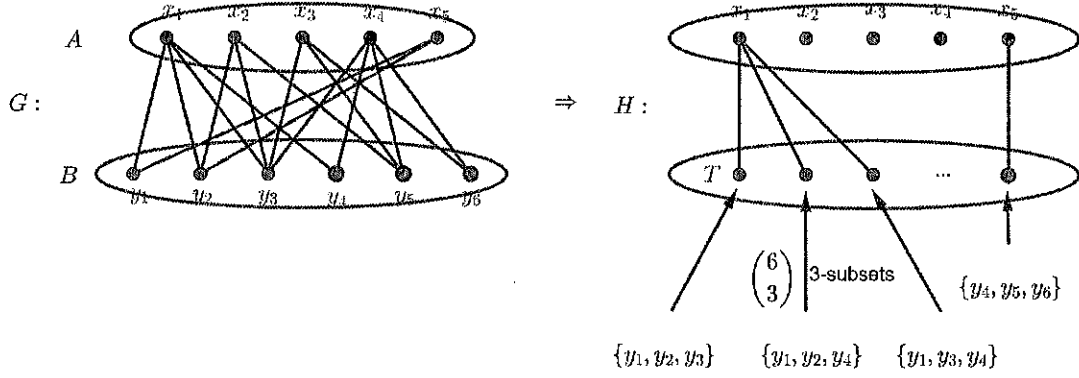


Figure 28. H induced by G

Hence, we have

- (1) $\|H\| = \sum_{x \in A} \binom{\deg_G(x)}{t}$, (For example, in Figure 28, $\|H\| = 10$.)
- (2) $\forall T \in \binom{B}{t}$, $\deg_H(T) \leq s - 1$, ($G \not\cong K_{s,t}$)
- (3) $\|H\| \leq (s - 1) \binom{n}{t}$. (From (2).)

Now, since $z = km + r = my = \sum_{x \in A} \deg_G(x)$, (*)
 $m \binom{y}{t} \leq (m - r) \binom{k}{t} + r \binom{k+1}{t} \leq \sum_{x \in A} \binom{\deg_G(x)}{t} \leq (s - 1) \binom{n}{t}$. ■

(*) comes from the property of combination number. For example,

$$z = 16, k = 3, m = 5, r = 1 \text{ and } t = 2. \text{ Then,}$$

$$5 \cdot \binom{3.2}{2} \leq 4 \binom{3}{2} + \binom{4}{2} \leq \binom{a}{2} + \binom{b}{2} + \dots + \binom{c}{2}.$$

Theorem 34

$$z(m, n; s, t) \leq (s - 1)^{\frac{1}{t}} \cdot (n - t + 1) \cdot m^{1 - \frac{1}{t}} + (t - 1)m.$$

Proof.

By Theorem 33, $m \binom{y}{t} \leq (s-1) \binom{n}{t}$, $\frac{\binom{y}{t}}{\binom{n}{t}} \leq \frac{s-1}{m}$. Hence,
 $\frac{y(y-1)\cdots(y-t+1)}{n(n-1)\cdots(n-t+1)} \leq \frac{s-1}{m}$. By the fact $\frac{y-i}{n-i} \geq \frac{y-t+1}{n-t+1}$ for each
 $0 \leq i \leq t-1$, we have $\left(\frac{y-t+1}{n-t+1}\right)^t \leq \frac{s-1}{m}$, i.e.,
 $(y-t+1)^t \leq (s-1) \cdot (n-t+1)^t \cdot m^{-1}$. This implies that
 $y-t+1 \leq (s-1)^{\frac{1}{t}} \cdot (n-t+1) \cdot m^{-\frac{1}{t}}$ and
 $y \leq (s-1)^{\frac{1}{t}} \cdot (n-t+1) \cdot m^{-\frac{1}{t}} + (t-1)$.
Hence, $z = my \leq (s-1)^{\frac{1}{t}} \cdot (n-t+1) \cdot m^{1-\frac{1}{t}} + (t-1)m$. ■

Theorem 35-1

$$z(m, n; 2, 2) \leq \frac{n}{2} [1 + 4n - 3]^{\frac{1}{2}}.$$

Proof.

By Theorem 33, $n \cdot \binom{y}{2} \leq \binom{n}{2}$. Hence, $n \cdot y(y-1) \leq n(n-1)$ and we
have $y^2 - y - (n-1) \leq 0$. A direct calculation shows that
 $y \leq \frac{1 + \sqrt{4n-3}}{2}$. This implies that $z = m \cdot y = n \cdot y \leq \frac{n(1 + \sqrt{4n-3})}{2}$. ■

Theorem 35-2

If $n = q^2 + q + 1$ and q is a prime power, then

$z(m, n; 2, 2) = \frac{n}{2} [1 + 4n - 3]^{\frac{1}{2}}$. (Proof. By the existence of a
projective of order q .)

Theorem 36

$$ex(n; K_{s,t}) \leq \frac{1}{2}(s-1)^{\frac{1}{t}} \cdot n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n, \text{ i.e.,}$$

$$ex(n; K_{s,t}) \leq \frac{1}{2}z(n, n; s, t).$$

Proof.**(1st)**

Let G be an extremal graph such that $\|G\| = ex(n; K_{s,t})$. Define a bipartite graph $H = (A, B)$ based on G . Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$ and $a_i \sim_H b_j$ if and only if $v_i \sim_G v_j$, see Figure 29 for an example. Now, clearly, $\|H\| = 2\|G\|$ and $a_i \not\sim_H b_j$ for $i = 1, 2, \dots, n$. Moreover, if $G \not\leq K_{s,t}$, then $H \not\leq K_{s,t}$. (?) This concludes that $\|G\| = \frac{1}{2}\|H\| \leq \frac{1}{2}z(n, n; s, t)$. ■

(2nd) (Two-way counting)

Consider the number of stars $K_{1,t}$. Since $G \not\leq K_{s,t}$, every set of t vertices of $V(G)$ has at most $s-1$ centers of stars whose pendent vertices are these vertices. So, there are at most $(s-1)\binom{n}{t}$ t -stars. The number of stars $K_{1,t}$ can be obtained by $\sum_{i=1}^n \binom{d_i}{t}$ where $d_i = \deg_G(v_i)$. Now, let $m = \|G\|$. $\sum_{i=1}^n d_i = 2m$. By a similar technique as Theorem 34, we conclude the proof ■

$$(*) \quad n \cdot \binom{2m/n}{t} \leq (s-1) \binom{n}{t}.$$