

Theorem 23 (Menger, 1927)

Let s and t be two nonadjacent vertices of a graph G . Then, the minimal number of vertices separating s from t is equal to the maximal number of vertex-disjoint (Internally) $s - t$ paths.

Proof.

(1st)

Let the number of vertices separating s and t be k . Then, it is easy to see that there are at most k independent (vertex-disjoint) paths connecting s and t . Also, if $k = 1$, then we have a path joining s and t . Now, suppose the assertion is not true, i.e., we can find less than k independent $s - t$ paths for certain k . Now, take the minimal k in which we have a counterexample. Then, among all such examples, let G be the one with minimum size (number of edge).

First, we notice that s and t have at most $k - 1$ independent paths and no common neighbors. For otherwise, let sx and xt be edges of G . Then, $G - x$ will be a counterexample for " $k - 1$ " (smaller than k).

Let W be a separating set of s and t and $|W| = k$. Suppose, neither $N_G(s) = W$ nor $N_G(t) = W$. (Figure 22-1)

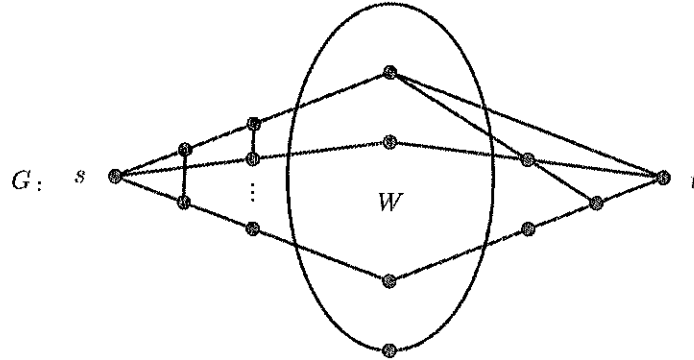


Figure 22-1.

Let G_s be obtained by deleting all the vertices to the left of G in Figure 22-1 and adding a replacing s' with edges joining to W , see Figure 22-2. Now, G_s has fewer edges than G and thus there are k independent $s' - t$ paths. Hence, we have k $W - t$ independent paths. With the same technique, we derive k $s - W$ independent paths (by changing s to t).

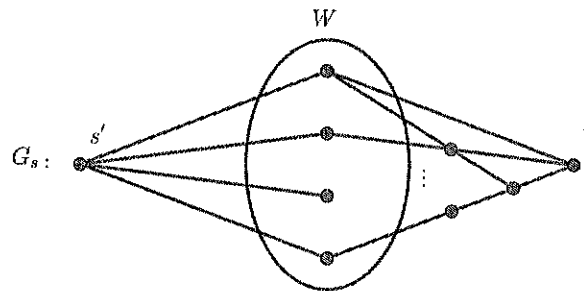


Figure 22-2.

So, as a conclusion, either s or t must have their neighbors W . Let $N_G(s) = W$ and $P = \langle s, x_1, x_2, \dots, x_l, t \rangle$ be a shortest $s - t$ path. Then, $l \geq 2$. Consider $G - x_1x_2$.

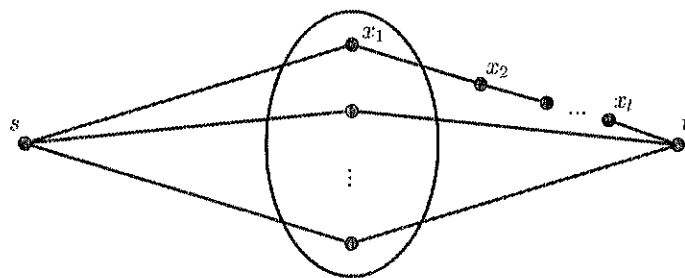


Figure 22-3.

In $G - x_1x_2$, there exists an $s - t$ separating set W_0 of size $k - 1$. Then, both $W_1 = W_0 \cup \{x_1\}$ and $W_2 = W_0 \cup \{x_2\}$ are $s - t$ separating sets of G . By the fact that P is a shortest $s - t$ path, s is not adjacent to x_2 and t is not adjacent to x_1 . This implies that $N_G(s) = W_1$ since t is not adjacent to a vertex of the separating set W_1 . Similarly, $N_G(t) = W_2$. Hence, $N_G(s) \cap N_G(t) = W_0$ (s and t have common neighbors), a contradiction. ($|W_0| = k - 1 \geq 1$.) ■

(2nd)

Apply Theorem 22.

(•) The "Edge" version of Menger's Theorem can be started as follows:

Let s and t be two vertices of G . Then, the minimal number of edges separating s from t is equal to the maximal number of edge-disjoint $s - t$ paths. (We can prove this part by using Theorem 22. Replace each edge xy of G by (x, y) and (y, x) and assign capacity "1" to each arcs.)

Theorem 24

If S and T are arbitrary subsets of $V(G)$, then the maximal number of vertex-disjoint (including endvertices) $S - T$ paths is $\min\{|W| \mid W \subseteq V(G) \text{ and } G - W \text{ has no } S - T \text{ paths}\}$.

Proof.

By adding to new vertices s and t as in Figure 23, we have a new graph $\tilde{G}$. Now, by Menger's Theorem, the maximal number of $S - T$ paths is the same as that of $s - t$ paths in $\tilde{G}$. Hence, we have the proof. ■

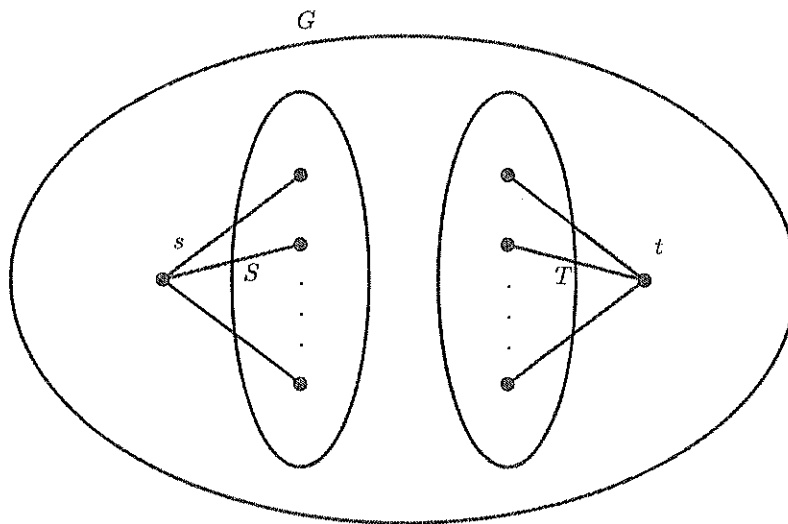


Figure 23. graph $\tilde{G}$

Definition (System of Distinct Representatives, SDR)

Let $A = \{A_1, A_2, \dots, A_n\}$ be a collection of subsets of a given set X . Then, an ordered n -tuple $(a_1, a_2, \dots, a_n)$ is called an SDR of A if $a_i \in A_i$, $i = 1, 2, \dots, n$ and all elements a_i 's are distinct.

Hall's Theorem(1935)

$A = \{A_1, A_2, \dots, A_n\}$ has an SDR if and only if for each $1 \leq k \leq n$, the union of any k subsets in A contains at least k distinct elements, i.e., $|\bigcup_{j=1}^k A_{i_j}| \geq k$. (Hall's condition)

We can use a bipartite graph to depict the above idea.

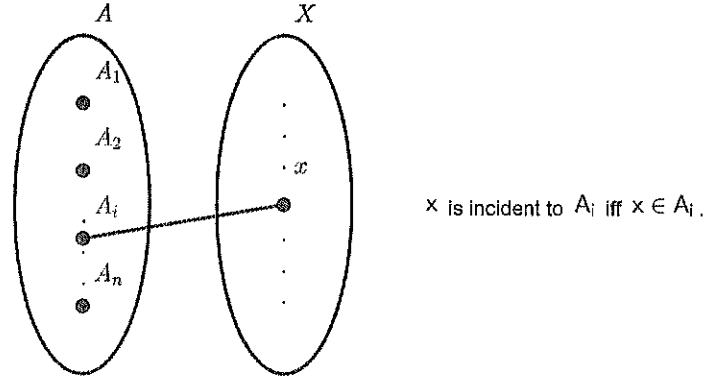


Figure 24. (Marriage problem)

Theorem 25

A bipartite graph $G = (A, B)$ contains a matching saturates A if and only if for every $S \subseteq A$, $\Gamma(S) = \bigcup_{x \in S} N_G(x)$ contains at least $|S|$ elements of B , i.e., $|\Gamma(S)| \geq |S|$.

Proof.

(1st)

($\Rightarrow$) By the existence of a matching saturates A .

($\Leftarrow$) By Theorem 24, it suffices to prove that there are $|A|$ vertex-disjoint $A - B$ paths (and thus a matching saturates A).

Suppose not. Then, there exists a subset A_1 of A and a subset B_1 of B such that there is no edge between $A \setminus A_1$ and $B \setminus B_1$, see Figure 24, and $|A_1| + |B_1| < |A|$. (The number of $A - B$ paths is less than $|A|$.)

Hence, there are no edges between $A \setminus A_1$ and $B \setminus B_1$, equivalently $\Gamma(A \setminus A_1) \subseteq B_1$. Then, $|\Gamma(A \setminus A_1)| \leq |B_1| < |A| - |A_1| = |A \setminus A_1|$. $\rightarrow \leftarrow$ ■

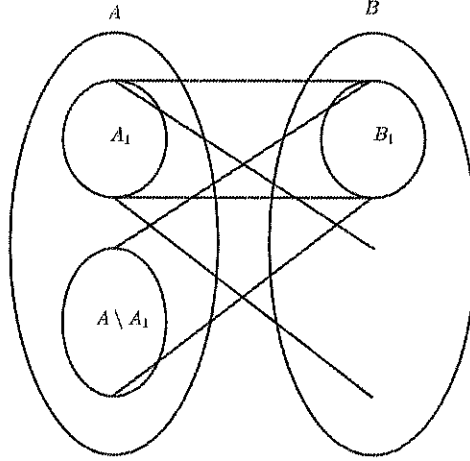


Figure 24.

(2nd)

By induction on $|A|$. Clearly, it's true for $|A| = 1$.

First, if for each $S \subsetneq A$, $|\Gamma(S)| \geq |S| + 1$, then set $a_1 \in A$ and $a_1 b_1 \in E(G)$ where $b_1 \in N_G(a_1)$. Now, consider the bipartite graph $G' = (A_1 \setminus \{a_1\}, B_1 \setminus \{b_1\})$. Since for each $S' \subseteq A_1 \setminus \{a_1\}$, $|\Gamma(S')| \geq |S'|$, there exists a matching saturates $A_1 \setminus \{a_1\}$. Combining with $a_1 b_1$, we have a matching needed.

Second, if there exists a proper subset S of A such that $|\Gamma(S)| < |S|$. By induction, we have a matching M_S saturates S . Now, consider $(A - S, B - T)$ where T is the set of vertices used in M_S . If there exists an $S' \subseteq A - S$ such that $|\Gamma(S')| < |S'|$, then $|\Gamma(S \cup S')| < |S \cup S'|$ a contradiction. Hence, the Hall's condition holds for the graph $(A - S, B - T)$. By induction, we have a matching saturates $A - S$. As a consequence, G has a matching saturates A .

(3rd) (Rado)

Let $G = (A, B)$ be a minimal (size) graph satisfying the condition. It suffices to claim that G contains $|A|$ independent edges (matching of size $|A|$). Suppose not. There exist two vertices a_1 and a_2 in A and b in B such that a_1b and a_2b are edges of G . Since both $G - a_1b$ and $G - a_2b$ violate Hall's condition, there exist two subsets A_1 and A_2 of A such that $|\Gamma(A_1)| = |A_1|$, $|\Gamma(A_2)| = |A_2|$ and a_i ($i = 1, 2$) is the only vertex of A_i which is adjacent to b .

$$\begin{aligned} \text{Hence, } |\Gamma(A_1) \cap \Gamma(A_2)| &\geq |\Gamma(A_1 - a_1) \cap \Gamma(A_2 - a_2)| + 1 \\ &\geq |\Gamma(A_1 \cap A_2)| + 1 \\ &\geq |A_1 \cap A_2| + 1. \end{aligned}$$

$$\begin{aligned} \text{On the other hand, } |\Gamma(A_1 \cup A_2)| &= |\Gamma(A_1) \cup \Gamma(A_2)| \\ &= |\Gamma(A_1)| + |\Gamma(A_2)| - |\Gamma(A_1 \cap A_2)| \\ &\leq |A_1| + |A_2| - |A_1 \cap A_2| - 1 \\ &= |A_1 \cup A_2| - 1. \quad (\rightarrow \leftarrow). \end{aligned}$$

■

(•) Can you find another proof?

Theorem 26 (König)

Every r -regular bipartite graph contains r edge-disjoint perfect matching.

Proof.

By induction on r . Clearly, it is true for $r = 1$. Let $r \geq 2$ and G be the r -regular bipartite graph where $G = (A, B)$. Then $|A| = |B|$. So, it

suffices to find a matching saturates A . Now for any subset S of A , $\Gamma(S) = \bigcup_{x \in S} N_G(x)$. If $|S| = k$, then S is incident to $k \cdot r$ edges. Since each vertex of B is of degree r , it takes at least k vertices of B to join with these $r \cdot k$ edges. This implies that $|\Gamma(S)| \geq |S|$. So, by Hall's Theorem, a matching saturates A can be obtained. Following the same process we conclude proof. ■

Theorem 27

Let $G = (A, B)$ be a bipartite graph such that for each $S \subseteq A$, $|\Gamma(S)| \geq |S| - d$, $d < |A|$. Then, G contains a matching with $|A| - d$ edges.

Proof.

Clearly, if $d = 0$, then we have a matching with $|A|$ edges. Now, let $d > 0$ and $B' = B \cup D$ where $D = \{y_1, y_2, \dots, y_t\}$ and $D \cap B = \emptyset$. Let $G' = (A, B')$ such that $E(G') = E(G) \cup \{y_i a_j | i = 1, 2, \dots, d; j = 1, 2, \dots, |A|\}$. (Join each vertex in D to every vertex of A .)

Now, for each $S \subseteq A$, $|\Gamma(S)| \geq S$ (in G'). Hence G' has a matching saturates A . This implies G has a matching of size at least $|A| - d$. ■

Remark

The following results can be obtained by applying Hall's Theorem.

1. A Latin rectangle can be extended to a Latin square.
2. An $n \times n$ matrix (non-negative) $A = (a_{ij})$ is said to be doubly stochastic if $\sum_{i=1}^n a_{ij} = 1$ for every j and $\sum_{j=1}^n a_{ij} = 1$ for every i .

Then, there exist $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda_k = 1$, and permutation matrices

$P_1, P_2, \dots, P_m$ such that $A = \sum_{k=1}^m \lambda_k P_k$.

3. More ...

Appendix

Use max-flow min-cut theorem to prove Hall's Theorem.

Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, $m \leq n$.

Claim: There exists a matching in $G = (X, Y)$ saturates X if and only if for each subset $A \subseteq X$, $|\Gamma(A)| \geq |A|$.

(Note) We shall prove the theorem for both $(\Rightarrow)$ and $(\Leftarrow)$ by using network argument.

First, we construct a network N by (1) adding s and t such that (s, x_i) and (y_j, t) are arcs in N , $1 \leq j \leq n$, (2) orienting $x_i y_j \in E(G)$ with (x_i, y_j) , and (3) $c(s, x_i) = c(y_j, t) = 1$ and $c(x_i, y_j) = M > |X| = m$. (See Figure 1 below.)

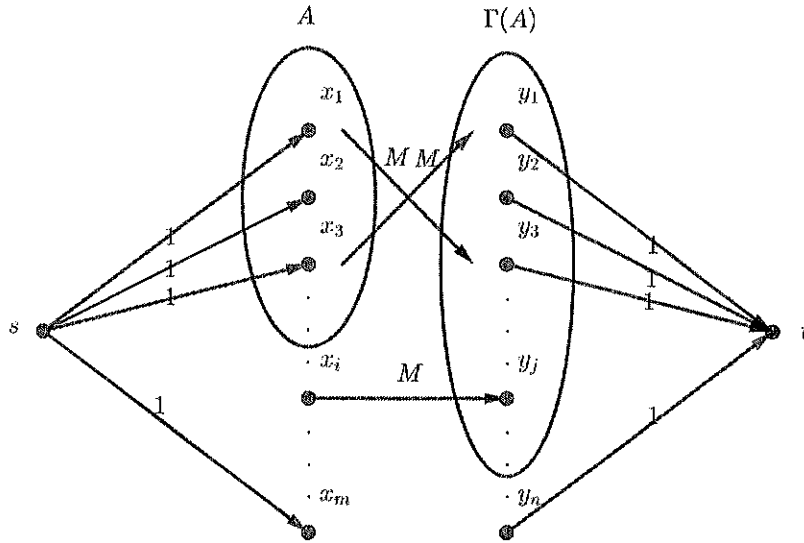


Figure 1.

Now, it is easy to see that a flow of value $|X| = m$ will provide a matching saturates X since each (y_j, t) can take at most one flow value through the flow.

($\Rightarrow$) Suppose that there exists a subset A of X such that $|A| > |\Gamma(A)|$.

Let $S = \{s\} \cup A \cup \Gamma(A)$. Then, $\langle S, \bar{S} \rangle$ is a cut with capacity

$(|X| - |A|) + |\Gamma(A)|$ since there are no arcs from the vertices of A to the $(s \rightarrow \bar{S}) \quad (S \rightarrow t)$

vertices of $Y \setminus \Gamma(A)$. This capacity of cut is less than $|X|$. Hence, there exists no flow with value $|X|$ and thus no matching saturates X .

($\Leftarrow$) Assume that $|\Gamma(A)| \geq |A|$ for each $A \subseteq X$. It suffices to claim that all cuts have capacity at least $|X|$. Let $S = \{s\} \cup A \cup B$, see Figure 2.

($A \subseteq X$ and $B \subseteq Y$.) $\leftarrow$ They can be empty. (B is not necessary be $\Gamma(A)$).

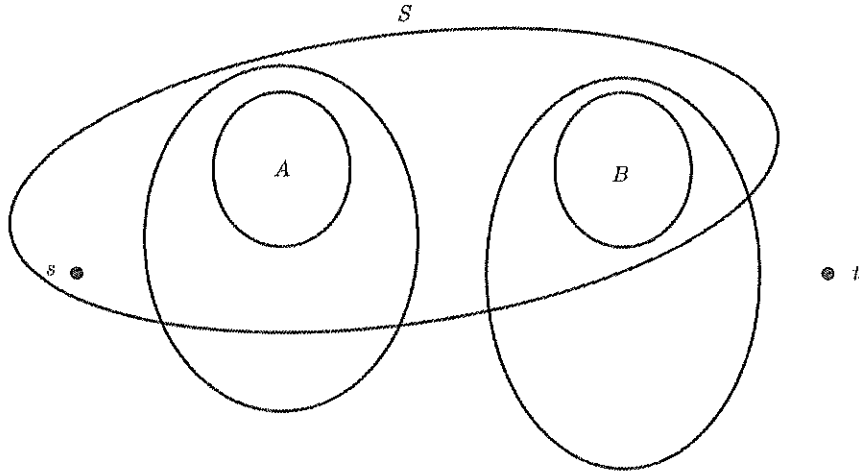


Figure 2.

Now, if there exists an arc from the vertices of A into $Y \setminus B$, then

$c(S, \bar{S}) \geq M > |X|$. On the other hand, if $\Gamma(A) \subseteq B$, then

$c(S, \bar{S}) = (|X| - |A|) + |B| \geq |X| - |A| + |\Gamma(A)| \geq |X|$.

$$s \rightarrow \bar{S}$$

Since $c(S, \bar{S}) = |X|$ in the case $S = \{s\}$, we obtain a min-cut with capacity $|X|$ and thus there exists a flow with maximum value $|\bar{X}|$.

The proof follows. ■

Use max-flow min-cut theorem to prove Menger's Theorem.

Proof.

First, we proof a directed version of Menger's Theorem.

(•) If s and t are distinct vertices of a digraph D such that $s \not\stackrel{D}{\sim} t$, then the maximum number of internally disjoint $u - v$ directed paths in D equals the maximum number of vertices in a $u - v$ separating set in D .

(••) For the undirected version, we replace each edge uv by a pair of arcs (u, v) and (v, u) .

Proof of (•).

Let $\tilde{D}$ be the digraph obtained as follows:

- (1) $\forall x \in V(D) \setminus \{s, t\}$, split x into two vertices x' and x'' , also let $(x', x'') \in A(\tilde{D})$;
- (2) $\forall (x, y) \in A(D)$ such that $\{x, y\} \cap \{s, t\} = \emptyset$, replace (x, y) with (x'', y'') ;
- (3) For $x \neq t$, $(s, x) \in A(D)$, $(x, s) \in A(D)$, replace them with (s, x') and (x'', s) respectively, and

- (4) Replace (t, x) with (t, x') and (x, t) with (x'', t) if $x \neq s$. As a consequence, we have a network N defined on D by assigning each arc a capacity "1".

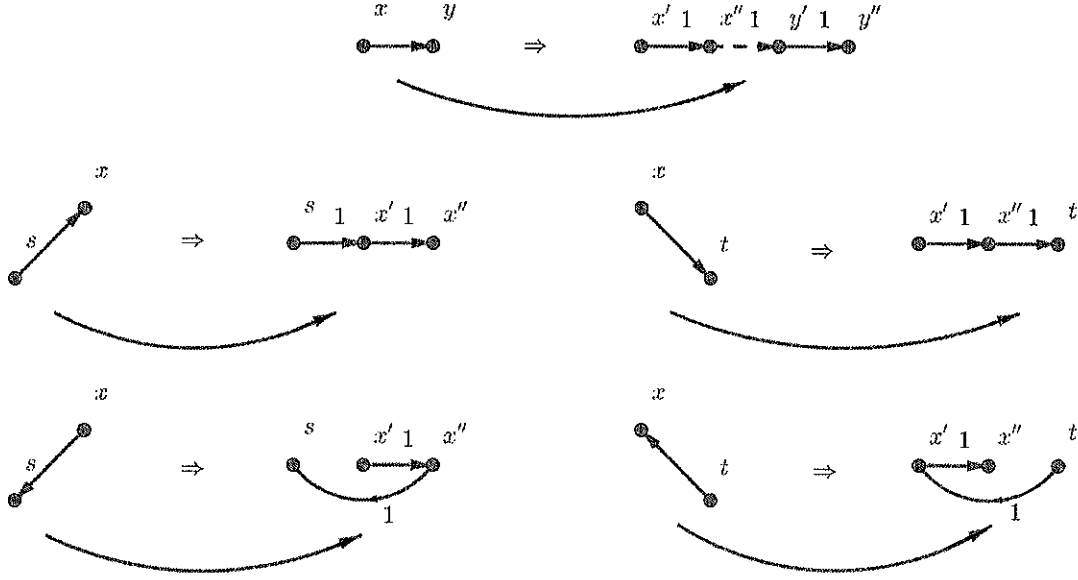


Figure 3.

Let m be the maximum number of internally disjoint $s - t$ paths in D and n be the minimum number of vertices in an $s - t$ separating set in D . Let A be a $u - v$ separating set of arcs in $\tilde{D}$ and $|A| = n'$. First, we observe that if $(S, \bar{S})$ is a cut, then $\text{cap}(S, \bar{S}) \leq n'$. Here, S contains s if $(s, x') \in A$, S contains x if $(x', x'') \in A$ and $(x'', t) \in A$. Moreover, $\text{cap}(S, \bar{S}) \geq n$. As a matter of fact, the min-cut is of capacity n and thus N has a maximum flow n . By the way, the construction of network shows that a flow value 1 will give a path (directed) from s to t . Since each arc has capacity one, these paths are internally disjoint. The proof follows. ■

- (•) A 1-factor of a graph G is a 1-regular spanning subgraph of G .