Figure 19. G_{i+1} **Theorem 20**

Every non-trivial graph G contains at least two vertices which are not cutvertices.

Proof.

Review that if v is a cutvertex of G , then the number of components of G , $c(G)$ is smaller than that of $c(G - v)$. Now, consider u and v such that $d(u, v) = \text{diam}(G)$. We show both u and v are not cutvertices. Suppose not. Let u be a cutvertex, then $G - u$ is disconnected. Let w be a vertex which is in a component different from v belongs. Since u is a cutvertex and v, w are in different components, all $v - w$ path must pass through u . This implies that $d(v, w) > d(v, u) = \text{diam}(G)$. Hence, u can not be a cutvertex. Similarly, v is not a cutvertex either. ■

Definition (Network)

A network N is a digraph D with two distinguished vertices u and v , called the *source* and *sink* of N , respectively, and a non-negative integer-valued function c on $E(D)$. The digraph is the underling

digraph of N and the function c is the capacity function on N . For convenience, $c(\vec{a}) = c((x, y)) = c(x, y)$ for each arc $\vec{a} = (x, y)$, is the capacity of $\vec{a}$.

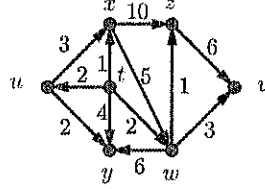


Figure 20. A network

$$N^+(x) = \{y \mid y \in V(D) \text{ and } (x, y) \in E(D)\}$$

$$N^-(x) = \{y \mid y \in V(D) \text{ and } (y, x) \in E(D)\}$$

(•) A flow in a network N , f , is a function on $E(D)$, s.t.

① $0 \leq f(\vec{a}) \leq c(\vec{a})$ for every $\vec{a} \in E(D)$ (capacity bound) and

② $\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x)$ for every $x \in V(D) \setminus \{u, v\}$.

(Conservation law)

• The net flow into x is equal to $\sum_{y \in N^-(x)} f(y, x) - \sum_{y \in N^+(x)} f(x, y)$

which is zero except $x \in \{u, v\}$.

• $(V_1, V_2) = \{(x, y) \in E(D) \mid x \in V_1 \text{ and } y \in V_2\}$ (digraph version!)

(••) A cut in N is $(X, V(D) \setminus X)$ such that $u \in X$ such that $u \in X$ and

$v \in V(D) \setminus X$.

||

X'

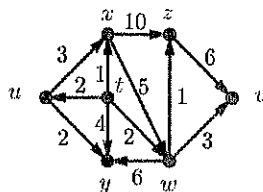


Figure 20. A network

Definition

Let $K = (X, X')$ be a cut in N . Then, the capacity of K , $\text{cap } K = c(X, X') = \sum_{(x,y) \in K} c(x, y)$. e.g. In Figure 20, let $X = \{u, x, t, y\}$, then $c(X, X') = 17$.

(*) Definition

The value of a flow f in N is defined as the net value flow out the source and therefore the net value flow into the sink. (Denoted by $\text{val } f$.)

Theorem 21

Let f be a flow in a network N and $K = (X, X')$ be a cut in N . Then, $\text{val } f \leq \text{cap } K$.

Proof.

Note that $\text{val } f = f(u, \{u\}') - f(\{u\}', u)$ and $f(x, \{x\}') - f(\{x\}', x) = 0, \forall x \in X \setminus \{u\}$. ($u \in X, v \in X'$) This implies that $\sum_{x \in X} [f(x, \{x\}') - f(\{x\}', x)] = \text{val } f$

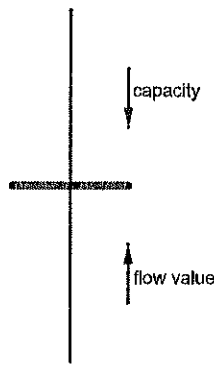
$$\parallel$$

$f(X, X') - f(X', X)$ (Calculation).

Now, $f(X, X') \leq \text{cap}(X, X')$ and $f(X', X) \geq 0$. Hence, $\text{val } f \leq \text{cap } K$.

■

- A minimum cut is a cut K in N such that for every cut K' in N , $\text{cap } K \leq \text{cap } K'$.
- A minimum flow is a flow f in N such that for every flow f' in N , $\text{val } f \geq \text{val } f'$.



min-max problem

$\Rightarrow$ If there exists a K and an f s.t. $\text{cap } K = \text{val } f$, then K is a minimum cut and f is a maximum flow.

Theorem 22 (Ford and Fulkson, 1956-1962)

In any network N defined on D , the value of a maximum flow equals the capacity of a minimum cut.

Proof.

Clearly, if there exist no cuts such that its capacity of the cut is $\text{val } f$, then f does not exist. (Theorem 21) So, it suffices to claim that

if the value of a maximum flow f is v , then there exists a cut K , such that $\text{cap } K = \text{val } f = v$.

Define a subset $S \subseteq V(D)$ recursively as follows. Let $s \in S$. If $x \in S$, and $c(x, y) > f(x, y)$ or $f(y, x) > 0$, then let $y \in S$. We shall prove that (S, S') is a cut with capacity v . First, we claim $t \notin S$. Suppose not, i.e., $t \in S$. Hence, we can find a sequence of vertices in N such that $s = x_0, x_1, \dots, x_l = t$. Moreover, if we let $\varepsilon_i = \max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\}$, $i = 0, 1, \dots, l-1$, then $\varepsilon_i > 0$. Let $\varepsilon = \min\{\varepsilon_i\}$. Now, let $f^*(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \varepsilon$ if $c(x_i, x_{i+1}) - f(x_i, x_{i+1}) = \varepsilon_i > 0$ and $f^*(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \varepsilon$ if $f(x_{i+1}, x_i) = \varepsilon_i > 0$. As a consequence, f^* is a flow from s into t with value $\text{val } f^* = v + \varepsilon$, a contradiction. (See Figure 21) By the definition of a flow,

$$\text{val } f = v = \sum_{x \in S, y \in S'} f(x, y) - \sum_{x \in S', y \in S} f(x, y). \quad (1)$$

Again, by the definition of S , if $x \in S$ and $y \in S'$, then $c(x, y) = f(x, y)$ and $f(y, x) = 0$. This implies that (1) =

$\sum_{x \in S, y \in S'} c(x, y) = \text{cap}(S, S') = v$, the proof follows. ■

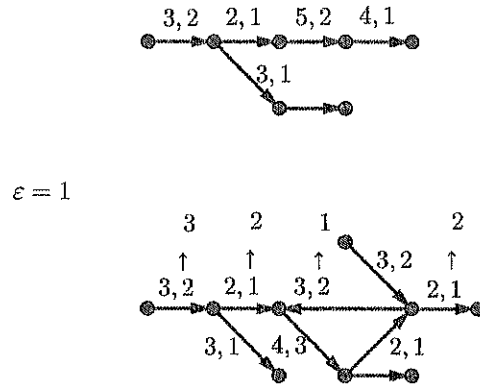


Figure 21. augmenting flow f^*