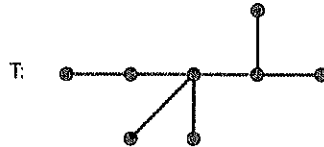


Tree Packing problem

Given a tree T of size $\|T\| = n$. Can we pack $2n + 1$ copies of T into the complete graph of order $2n + 1$?

For example,



Fact 1

K_{2n+1} has $(2n + 1) \cdot n$ edges.

Fact 2

If we can pack T ($\|T\| = n$) into K_{2n+1} , then it is a *decomposition* of K_{2n+1} into T 's.

Review

An $\mathcal{H}$ -decomposition of G is a partition of $E(G)$ such that each part induces a graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$, then we simply say G has an H -decomposition.

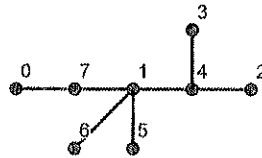
Review

A packing or $\mathcal{H}$ -packing of G is a collection of edge-disjoint subsets of $E(G)$ such that induces a graph in $\mathcal{H}$.

e.g. A K_3 -packing of K_5 .

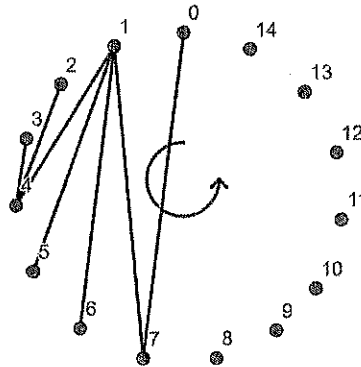


An idea of packing.



(Find a good labeling!)

[Graceful labeling or β -labeling]



Definition (Graceful labeling) (or a β -labeling)

A vertex-labeling $\varphi : V(G) \xrightarrow{1-1} \{0, 1, 2, \dots, \|G\|\}$ is a graceful labeling if all the weights of uv , $|\varphi(u) - \varphi(v)|$ are distinct.

Graceful labeling tree conjecture

:Trees are graceful!

$k(G)$: Connectivity of G , $k'(G)$: edge-connectivity of G

Theorem 14

For each connected graph G , $k(G) \underset{\textcircled{1}}{\leq} k'(G) \underset{\textcircled{2}}{\leq} \delta(G)$.

Proof.

② Let v be a vertex of G such that $\deg_G(v) = \delta(G)$. Since the deletion of all edges incident v will provide a disconnected graph, $\delta(G) \geq k'(G)$.

① Let F (Figure 14) be an edge cut of G which has $k'(G)$ edges.

Case 1. $\exists v \in V(G)$, v is not incident any edge of F . Let C be the component in $G - F$ which contains v . Then, C contains exactly one vertex of each edge of F . (?) Hence, deleting all such vertices results in a disconnected graph. Hence $k(G) \leq k'(G)$.

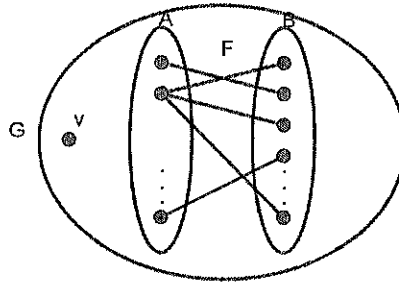


Figure 14. Edge - cut F

Case 2. $\forall u \in V(G)$, u is incident to an edge of F .

Let C be the component of $G - F$ which contains an arbitrary vertex v . Now, if $wv \notin F$, then w is incident to an edge of F . Then,

$\deg_G(v) \leq F$. This implies (again) $k(G) \leq |F| \leq k'(G)$. On the other hand all, $wv \in F$, $w \in N_G(v)$, then (W.L.O.G.) we have a complete graph of order $|F| + 1 = |G|$. $k(G) = |G| - 1 = |F| = k'(G)$. $\blacksquare$

(•) It is interesting to know the graphs G satisfying $k(G) = \delta(G)$.

Theorem 15 (Mader, 1972)

Let $k \in \mathbb{N}$. Every graph G with $d(G) \geq 4k$ has a $(k+1)$ -connected subgraph H such that $\varepsilon(H) > \varepsilon(G) - k$.

Proof.

For convenience, let $\varepsilon = \varepsilon(G)$ and consider $G' \leq G$ s.t. $|G'| \geq 2k$ and $\|G'\| > \varepsilon \cdot (|G'| - k)$. Since $\|G\| = \varepsilon \cdot |G| > \varepsilon(|G| - k)$, such graphs G' do exist. Let H be the one with minimum order.

Clearly, $|H| > 2k$, for otherwise, $\|H\| > \varepsilon \cdot k \geq 2k^2 > \binom{|H|}{2} = k \cdot (2k - 1)$.

$\rightarrow \leftarrow$ The minimality of H implies that $\delta(H) > \varepsilon$, by Theorem 13 (we can choose a proper induced subgraph). Hence, $|H| \geq \varepsilon$. By the choice of H , we have $\|H\| > \varepsilon(|H| - k)$ mentioned above, thus $\varepsilon(H) > \varepsilon - k$.

$\left(\frac{\|H\|}{|H|} > \frac{\varepsilon(|H|) - \varepsilon \cdot k}{|H|} = \varepsilon - \frac{\varepsilon \cdot k}{|H|} \geq \varepsilon - \frac{|H| \cdot k}{|H|} = \varepsilon - k \right)$ Therefore, H satisfies the condition we need. It's left to check that H is

$(k+1)$ -connected. Suppose not. Let K be a set of k vertices in H such that $H - K$ is disconnected, see Figure 15. Now, let $V(H) = U_1 \cup U_2$ such that $U_1 \cap U_2 = K$, $H_1 = \langle U_1 \rangle_H$, $H_2 = \langle U_2 \rangle_H$ and no vertices of $U_1 \setminus U_2$ is adjacent to vertices in $U_2 \setminus U_1$.

Let $v \in U_1 \setminus U_2$. Since $\deg_H(v) \geq \delta(H) > \varepsilon$, this implies that $|H| > \varepsilon \geq 2k$, so is $|H_2| \geq 2k$. Because of the fact that H is the choice of

G' with minimum order, $\|H_1\| \leq \varepsilon(|H_1| - k)$ and $\|H_2\| \leq \varepsilon(|H_2| - k)$.

Hence $\|H\| \leq \|H_1\| + \|H_2\| \leq \varepsilon(|H_1| + |H_2| - 2k) = \varepsilon(|H| - k)$. A

contradiction to the choice of H . ■

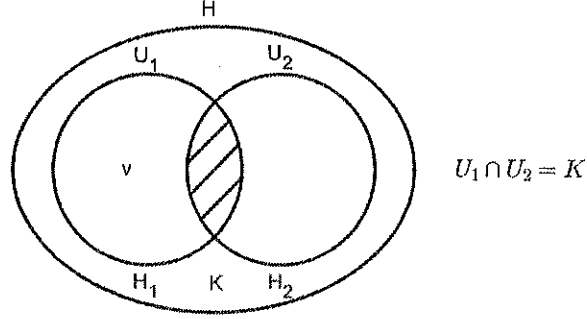


Figure 15. $H_1 = \langle U_1 \rangle_H$ and $H_2 = \langle U_2 \rangle_H$

- A maximal connected subgraph without a "cut vertex" is call a "block".



increase the # of components

- (••) Every block of a graph G is either a maximal 2-connected subgraph or a bridge (with its endvertices), or an isolated vertex.
(See Figure 16.)

- A block graph of G is a bipartite graph $bc(G) = (A, B)$ where A is the set of cutvertices of G and B is the set of blocks, and $a \in A$ is incident to $B_i \in B$ if $a \in V(B_i)$.

Theorem 16

The block graph of a *connected* graph is a tree.

Proof.

Observe that each block is an induced subgraph and any two blocks have at most one cutvertex in common.

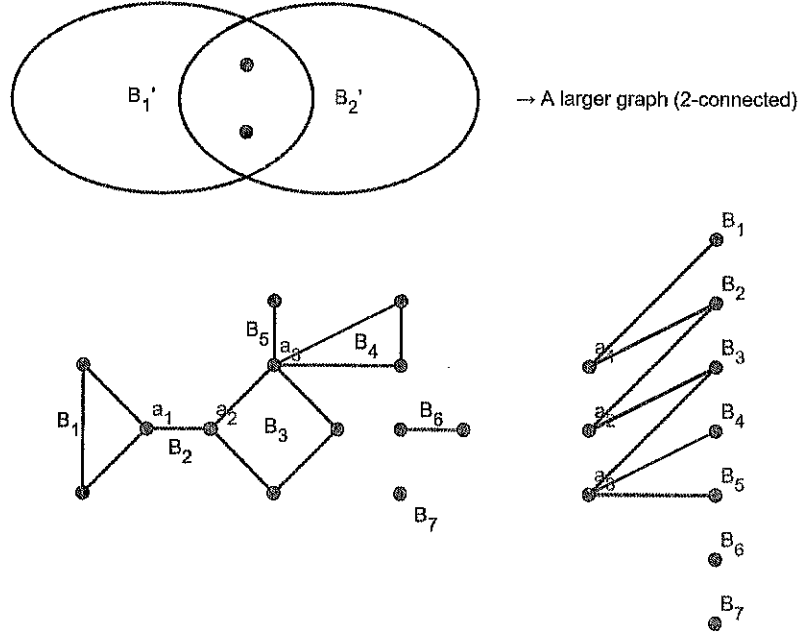


Figure 16. A block graph

Since G is connected, if G has only one block and thus contains no cutvertex, $bc(G)$ is a single vertex. The proof is trivial. Assume that G contains more than one block. Then, each vertex is either in a block or a cutvertex itself. Now, consider a cutvertex v and a block B_i . Let $u \in B_i \setminus \{v\}$. ($B_i \setminus \{v\}$ is non-empty since G is connected.) Then, we have a path P connecting v and u . Clearly, v is going to connect to a block which contains some vertices of P . If this block B_i , then vB_i is an edge of $bc(G)$, done. Otherwise, this path will contain a cutvertex following the block and travels to another block and finally to B_i . The

other two cases, cutvertex to cutvertex and block to block can be verified similarly.

Now, for the acyclic part, a cycle in $bc(G)$ will produce a cycle in G which passes all cutvertices involved. But, in that case, none of these cutvertices are cutvertices anymore, a contradiction. This completes the proof. ■

Theorem 17

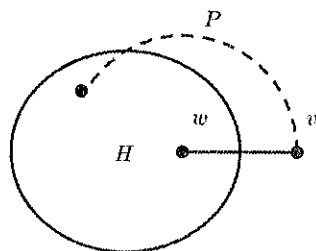
A graph G is 2-connected if and only if can be constructed from a cycle by successively adding H -paths to graphs H already constructed.

Proof.

($\Leftarrow$) From the construction, it is clear that G contains no cutvertices. Hence, G is 2-connected.

($\Rightarrow$) Assume that G is 2-connected and H is a maximum (size) subgraph following the construction. This is possible, since G contains a cycle. Infact, H is an *induced* subgraph, since for any two vertices x and y in $V(H)$ and $xy \in E(G) \setminus E(H)$, we have an H -path and a larger subgraph will be obtained.

Now, assume that $H \neq G$. $\exists v \in V(G) \setminus V(H)$, $w \in V(H)$ and $vw \in E(G) \setminus E(H)$. Since G is 2-connected, $G - w$ contains a $v - H$ path P . (See Figure 17) This implies that $\langle w, v, P \rangle$ is an H -path. Hence, a larger subgraph is obtained, a contradiction. ■

Figure 17. An extra H -path

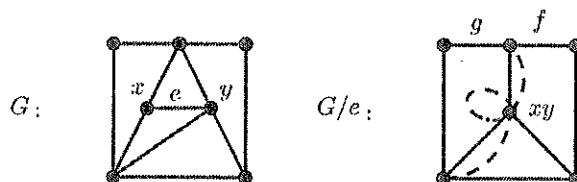
Definition (Graph minors)

A graph M is called a minor of G if M can be obtained from G by contracting edges, deleting vertices and edges.

(*) Determine whether a graph G contains a minor H is considered as the most important problem in the study of graph structure.

Review (Edge-contraction)

Given an edge $xy = e$ of a graph G , the graph G/e is obtained from G by contracting e ; that is to identify the vertices x and y and deleting resulting loops and duplicate edges.

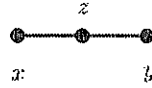


Example

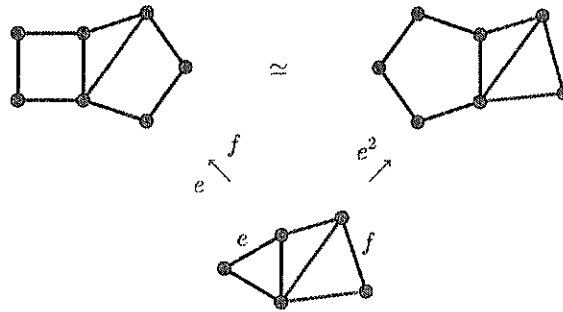
K_4 is a minor of the above G . (Contracting e, f and g .)

Definition (Subdivision)

A subdivision of an edge xy is obtained by adding a new vertex z such that we have edges xz and zy .

**Definition (Homeomorphic Graphs)**

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

Remark

Two cycles are homeomorphic.

(•) The reverse of subdivision can be "*considered*" as *contraction*.

Theorem 18

If G is 3-connected and $|G| > 4$, then G has an edge such that G/e is 3-connected.

Proof.

Suppose not. Then, for each $xy \in E(G)$, the graph G/xy contains a separator S with $|S| \leq 2$. ($G - S$ is disconnected.) Since $k(G) \geq 3$, the contracted vertex $v_{xy} \in S$ and $|S| = 2$, i.e., $\exists z \in V(G)$, s.t. $S = \{v_{xy}, z\}$. Therefore, $T = \{x, y, z\}$ is a separator set of G . Since one proper subset of T can separate G , each vertex of T is incident to every component of $G - S$. (See Figure 18)

Among all edges of G , we choose an edge xy and its corresponding vertex z such that the component C has minimum size. (*) Let $v \in V(C)$ and $vz \in E(G)$. By assumption, G/vz is again not 3-connected and there exists a corresponding vertex w such that $\{v, z, w\}$ separates G .

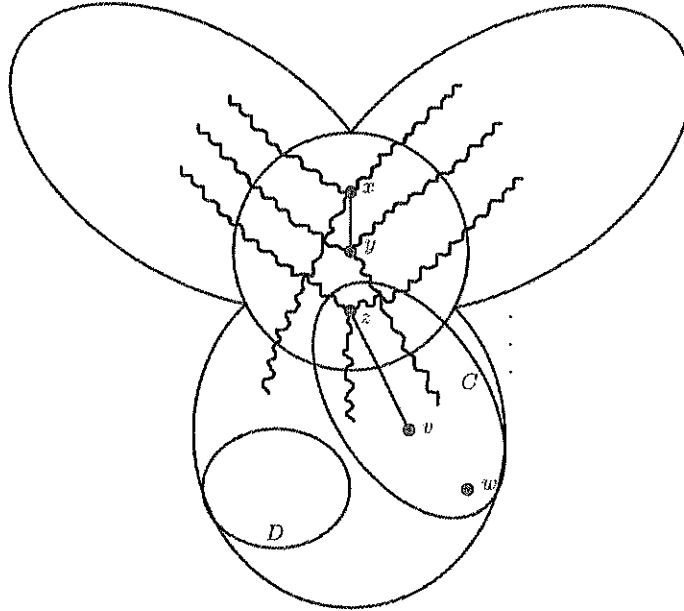


Figure 18. 3-connected graph

Moreover, each vertex of $\{v, z, w\}$ is incident to every component of $G - \{v, z, w\}$. Since $xy \in E(G)$, $G - \{v, z, w\}$ has a component D s.t.

$D \cap \{x, y\} = \emptyset$. By the fact $v \in V(C)$, the neighbor of v in D is also in C . Hence, $D \cap C \neq \emptyset$. This implies that D is a proper subset of C , i.e., $|D| < |C|$, a contradiction to the choice of C . ($|C|$ is minimum.) ■

Theorem 19 (Tutte, 1961)

A graph G is 3-connected if and only if there exists a sequence $G_0, G_1, \dots, G_n$ of graphs satisfying :

- (a) $G_0 = K_4$ and $G_n = G$; and
- (b) G_{i+1} has an edge xy (with degree of x, y at least 3) such that

$$G_i = G_{i+1}/xy, 1 \leq i < n.$$

Proof.

($\Rightarrow$) By Theorem 18, we start with G as G_n and end at $K_4 = G_0$.

($\Leftarrow$) Let $G_0, G_1, \dots, G_n$ be a sequence of graphs satisfying (a) and (b).

It suffices to show that if $G_i = G_{i+1}/xy$ is 3-connected, then G_{i+1} is also 3-connected, for all $1 \leq i < n$.

Suppose not. Let S be a separator with $|S| \leq 2$. Also, let C_1 and C_2 be two components of $G_{i+1} - S$. Since $xy \in E(G_{i+1})$, let $\{x, y\} \cap V(C_1) = \emptyset$. (Figure 19) Now, if $\{x, y\} \subseteq C_2$, then $G_i - S$ is disconnected, a contradiction. Hence, at most one of $\{x, y\}$ is in C_2 , either x or y , but not both. Furthermore, if $v \notin \{x, y\}$ and $v \in V(G)$, then $G_i - S$ is also disconnected, a contradiction to the fact G_i is 3-connected. Hence, C_2 contains exactly one vertex of degree at most 2, $\rightarrow \leftarrow$. ■