

Theorem 10

A (d, g) -cage is a d -regular graph with girth g and minimum number of vertices. Prove that a (d, g) -cage is 2-connected. ($g \geq 3$)

Proof.

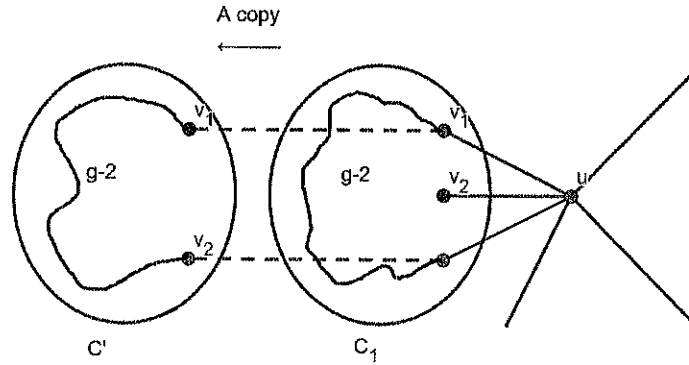
First, we claim that if $g_1 > g_2$, then a (d, g_1) -cage contains more vertices than the order of a (d, g_2) -cage. Suppose not. Let G_1 and G_2 be two cages respectively and $|G_1| < |G_2|$. It suffices to consider the case $g_1 = g_2 + 1$. Let $\|G_1\| = f(d; g_1)$ and $\|G_2\| = f(d; g_2)$.

(a) d is even

Let C be a cycle (in G_1) of length g_1 and $uv_1 \in E(C)$ where $N_{G_1}(u) = \{v_1, v_2, \dots, v_d\}$. Let $E' = \{v_1v_2, v_3v_4, \dots, v_{d-1}v_d\}$. Now, consider $G_1 - u + E'$ and denote the component contains v_1 by G'_1 . Clearly, G' is a simple graph and G'_1 contains a cycle of length $g_2 : C - u + v_1v_2$. Further, if C' is a cycle of G'_1 and $E(C') \cap E' = \emptyset$, then C' is a cycle of G_1 and thus of length at least g_1 . On the other hand, if $E(C') \cap E' \neq \emptyset$, then let $v_i v_j$ be one of the edges. Let P be a $\langle v_i, \dots, v_j \rangle$ path on C' satisfying $E(P) \cap E' = \emptyset$. So, $P + \{uv_i, uv_j\}$ is a cycle of G_1 . This implies that $|E(C')| \geq g_1 - 1 = g_2$. This concludes that G'_1 is a d -regular graph with girth at least g_2 and $|G_2| \leq |G'_1| = f(d; g_1) - 1 \leq |G_1|$.

(b) d is odd

Let C be a cycle (in G_1) of length g_1 and $uv_1, uv_2 \in E(C)$. Let $N_{G_1}(u) = \{u_1, u_2, \dots, u_{d-1}, w\}$. Clearly, $w \notin V(C)$. For otherwise, we have a cycle of length less than g_1 . (See Figure 11.)

Figure 12. Construction of H

This implies that G is not a (d, g) -cage, a contradiction. ■

Facts

1. It has been proved that a $(3, g)$ -cage is 3-connected.
2. It is conjectured that a (d, g) -cage is d -connected.

Reference

H.L. Fu, K.C. Huang and C.A. Rodger, Connectivity of cages, JGT, Vol. 24, No. 2, 187-191(1997).

Theorem 11

$$(V(G) = \{v_1, v_2, \dots, v_n\})$$

The vertices of a connected graph G can be enumerated so that $G_i =_{def} \langle v_1, v_2, \dots, v_i \rangle_G$ is connected for every $i = 1, 2, \dots, n$.

Proof.

We construct all G_i 's recursively. Define $\{\{v_1\}\}_G$ be G_1 where v_1 is an arbitrary vertex of G . Assume that $G_1, G_2, \dots, G_i$ have been

constructed for $i \geq 1$ and $i < n$. Now, let $v \in V(G) \setminus V(G_i)$. Since G is connected, there exists a path connecting v and v_1 . Let v_{i+1} be the last vertex on the path not in G_i . (See Figure 13.) Then, $\{v_1, v_2, \dots, v_i, v_{i+1}\}$ induces a connected graph, G_{i+1} . ■

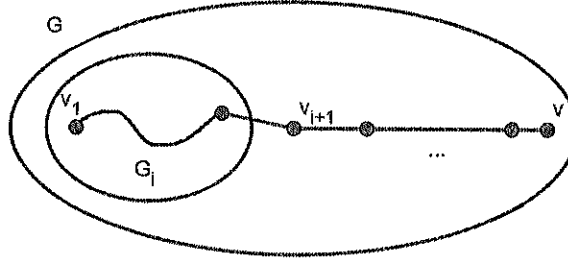


Figure 13. Graph enumeration

- (•) The vertices of a tree T can be enumerated such that every v_{i+1} has a unique neighbor in $\{\{v_1, v_2, \dots, v_i\}\}_T$. ($v \sim v_1$ is unique.)

Theorem 12

If T is a tree and G is any graph with $\delta(G) \geq \|T\|$, then $T \leq G$.

Proof.

(1st)

By induction on $\|T\|$. Clearly, the assertion is true for $\|T\| = 1$. Assume that the assertion is true for $\|T\| = k \geq 2$. Consider a tree T with $\|T\| = k + 1$. Since $\delta(G) \geq k + 1 > k$, G contains a tree (arbitrarily given) of size k , T' . (Let $T' = T - v_0$ where v_0 is a pendant vertex of T , moreover $v_0 v' \in E(T)$.) Now, consider T' in G . v' corresponds to a vertex u in $V(G)$. Since $\delta(G) \geq k + 1$, u has a neighbor u_0 which is not

a vertex of T' . Note here that T' has k vertices not including u . By attaching u_0 to T' , we have a subgraph T .

(2nd)

Direct construction

Let T be a tree defined on $\{v_1, v_2, \dots, v_n\}$ such that for each i , every vertex v_{i+1} has a unique neighbor in $\{v_1, v_2, \dots, v_i\}_T$. Now, we can construct directly on G the tree T by following the enumerating order. (Pick the vertices and edges recursively.) Since $\delta(G) \geq \|T\|$, there is always a vertex in G available for the next choice. ■

Theorem 13

Every graph G with at least one edge has an (induced) graph with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.

Proof.

We construct a sequence $G = G_0 \geq G_1 \geq G_2 \geq \dots$ of induced subgraphs of G as follows. If G_i has a vertex v_i of degree $\deg_{G_i}(v_i) \leq \varepsilon(G_i)$, let $G_{i+1} =_{def} G_i - v_i$; if not, we terminate our sequence and set $H =_{def} G_i$.

Now, by the selection process, $\delta(H) > \varepsilon(H)$. Furthermore,

$$\varepsilon(G_{i+1}) \geq \frac{\|G_{i+1}\|}{\|G_i\| - 1} = \frac{\|G_i\| - \deg_{G_i}(v_i)}{|G_i| - 1} \geq \frac{\|G_i\| - \varepsilon(G_i)}{|G_i| - 1} =$$

$$\frac{\|G_i\| - \|G_i\|/|G_i|}{|G_i| - 1} = \frac{\|G_i\| \left(1 - \frac{1}{|G_i|}\right)}{|G_i| - 1} = \frac{\|G_i\|}{|G_i|} = \varepsilon(G_i).$$

Hence, $\varepsilon(H) \geq \varepsilon(G)$. ■

(•) If there exists no v_i to delete, then G is the subgraph we need.