

Graph Theory - An Introduction

In this lecture note of Graph Theory we select “theorems” to introduce the basic properties of graphs.

Theorem 1 (Veblen, 1912)

The edge set of a graph can be partitioned into cycles if and only if every vertex has even degree.

Proof.

($\Rightarrow$) A vertex contained in t cycles has degree $2t$.

($\Leftarrow$) The cycles can be obtained recursively. We start with finding the first cycle. Let $\{x_0, x_1, \dots, x_l\}$ be a path of *maximal* length l in G . Since $x_0x_1 \in E(G)$, $\deg_G(x_0) \geq 2$. Let $y (\neq x_1)$ be a neighbor of x_0 , i.e., $x_0y \in E(G)$. Now, $y \in \{x_2, x_3, \dots, x_l\}$. For otherwise, we have a longer path. So, if $y = x_i$, then we have a cycle $C = (x_0, x_1, \dots, x_i)$. The process continues in $G - E(C)$. (Each vertex is of even degree in the graph $G - E(C)$.) ■

Theorem 2 (Mantel, 1907)

Every graph of order n and size greater than $\left\lfloor \frac{n^2}{4} \right\rfloor$ contains a triangle (C_3 or K_3).

Proof.

(1st)

Since $K_3 \not\leq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \emptyset$. This implies that $\deg_G(x) + \deg_G(y) \leq |G| = n$. (Figure 1) Now, consider

$$\begin{aligned} \sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) &= \sum_{x \in V(G)} (\deg_G(x))^2 \text{ (Two-way counting)} \\ &\leq n \cdot \|G\| \\ &= n \cdot e(G). \end{aligned}$$

By Cauchy's inequality,

$$\begin{aligned} (2e(G))^2 &= (\sum_{x \in V(G)} \deg_G(x))^2 \\ &\leq n \cdot \sum_{x \in V(G)} (\deg_G(x))^2 \\ &\leq (n)^2 \cdot e(G). \end{aligned}$$

Hence, $e(G) \leq \frac{n^2}{4}$. ■

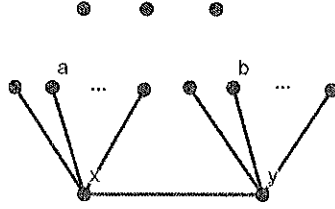


Figure 1. $\deg_G(x) = a + 1$, $\deg_G(y) = b + 1$

(2nd)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2) Since $K_3 \not\leq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies that $\|G\| \leq \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G))$. $\|G\|$ will take a maximum when $\Delta(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Hence, we have the proof. ■

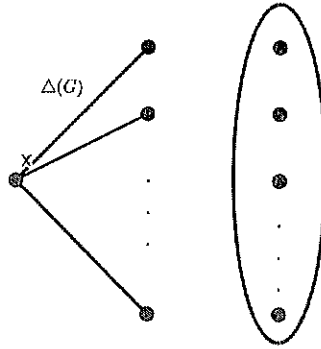


Figure 2.

Theorem 3

A graph is bipartite if and only if it does not contain an odd cycle.

Proof.

($\Rightarrow$) Let $G = (A, B)$ where A and B are its partite sets. If

$(x_0, x_1, \dots, x_l)$ is a cycle of G , then x_0 and x_l must be in different partite sets. Hence, the index l must be odd, thus the cycle is of even length.

($\Leftarrow$) W.L.O.G, let G be a connected graph. Let $x \in V(G)$ and

$V_1 = \{y \mid y \in V(G) \text{ and } d(x, y) \text{ is even}\}$. Hence, $x \in V_1$. Let

$V_2 = V(G) \setminus V_1$. It suffices to claim that both V_1 and V_2 are

independent sets. First, consider V_2 . Clearly, for each $z \in V_2$, $d(x, z)$ is

odd. Suppose that $z_1, z_2 \in V_2$ and $z_1 \sim z_2$ ($z_1 z_2 \in E(G)$). (Figure 3) Let

P_1 and P_2 be the two paths such that $P_1 = \langle x, \dots, z_1 \rangle$ and

$P_2 = \langle x, \dots, z_2 \rangle$, moreover they are the shortest paths connecting x to z_1 and x to z_2 respectively.

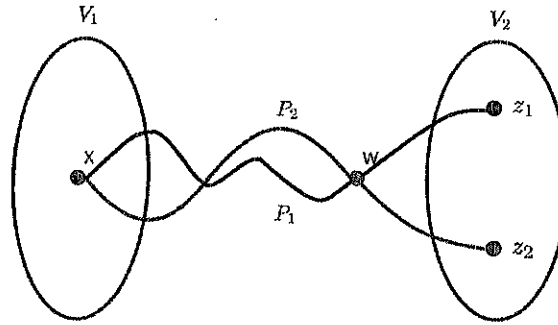


Figure 3.

Let w be the last vertex in which P_1 and P_2 intersect. Also, let $\|P_1\| = 2s + 1$ and $\|P_2\| = 2t + 1$. (Note that if $V(P_1) \cap V(P_2) = \{x\}$ (Figure 4), then we have an odd cycle (x, P_1, z_1, z_2, P_2) (length $2s + 2t + 3$).) Now, if w does exist, then $\langle x, P_1, w \rangle$ and $\langle x, P_2, w \rangle$ are of the same length, let the length be h .(?) So, the cycle $(w, \dots, z_1, z_2, \dots)$ is of length $(2s + 1 - h) + (2t + 1 - h) = 2s + 2t - 2h + 3$, an odd integer. Thus, an odd cycle exists, a contradiction. Hence, V_2 is an independent set. A similar argument can be applied to show that V_1 is also an independent set. (x is not adjacent to any vertex of $V_1 \setminus \{x\}$.)

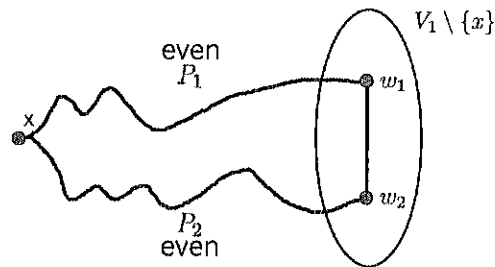


Figure 4.

Open problem

How many edges can a graph G of order n have such that $G \not\cong C_4$?

Ex. 1-3. (a) Find $ex(10; C_4)$. (b) Find $ex(91; C_4)$.
 (***) Find $ex(n; C_4)$.

Theorem 4

The following statements are equivalent for a graph G .

- (a) G is a tree. (G is connected and acyclic.)
- (b) G is connected and every edge of G is a bridge.
- (c) G is a maximal acyclic graph. (If x and y are not adjacent, then $G + xy$ contains a cycle.)

Proof.

(a) $\Rightarrow$ (b)

Let xy be an edge of G and $G - xy$ is connected. Then, there exists a path P connecting x and y in $G - xy$. Clearly, G contains a cycle (x, P, y) in G , a contradiction.

(b) $\Rightarrow$ (c)

If G is not a maximal acyclic graph, then there exists a pair of vertices z_1 and z_2 in G such that $G + z_1z_2$ is also acyclic. Since G is connected, there exists a path connecting z_1 and z_2 , say P . This implies that (z_1, P, z_2) is a cycle in $G + z_1z_2$, a contradiction.

(c) $\Rightarrow$ (a)

If G is not connected, then there exists a pair of vertices x_1 and x_2 (in different components) such that $G + x_1x_2$ is also acyclic. $\rightarrow\leftarrow$ ■

Theorem 5 (Euler, 1741)

A nontrivial connected graph (multigraph) has an eulerian circuit

(Euler circuit) if and only if each vertex has even degree. Moreover, a connected graph has an eulerian trail from a vertex x to a vertex $y \neq x$ if and only if x and y are the only two vertices of odd degree.

Proof.

The second statement follows directly from the first one. We prove the first statement.

($\Rightarrow$) If a circuit passes a vertex x h times, then $\deg_G(x) = 2h$.

($\Leftarrow$) By induction on $\|G\|$. Since $\|G\| \geq 1$, $\delta(G) \geq 2$ (G is not a tree!) and thus G contains a cycle. Let Z be a circuit in G with the maximum number of edges. If Z is an eulerian circuit, then we are done. Suppose not. Let H be a nontrivial component of $G - E(Z)$. Since G is connected, $V(H) \cap V(Z) \neq \emptyset$. Now, H is a nontrivial connected graph (even graph). Hence, H contains an eulerian circuit Y . By using x , we can attach Z and Y together to obtain a larger circuit. (Figure 5) This contradicts to the maximality of $|E(Z)|$. Hence, Z must be an eulerian circuit of G . ■

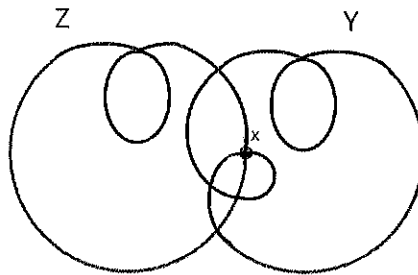
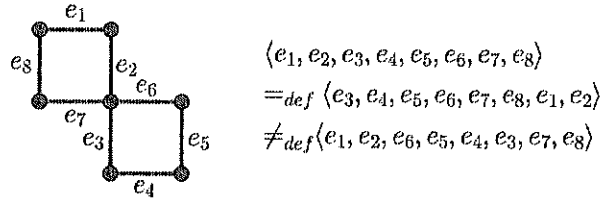


Figure 5.

Open problem

Find the number of distinct eulerian circuits of an eulerian graph G .

(Two circuits are the same if they can be obtained each other by a cyclic shift of edges.)



Theorem 5.5 (*BEST Theorem*)

A digraph D has an eulerian (directed) circuit if and only if D is strongly connected and for each vertex $v \in V(D)$, $\deg_D^+(v) = \deg_D^-(v)$. Moreover, if D is an eulerian graph, $s(D)$ is the number of distinct eulerian circuits, then

$s(D) = t_i(D) \cdot \prod_{j=1}^n (\deg_D^+(v_j) - 1)!$ for every $i \in \{1, 2, \dots, n\}$ where $t_i(D)$ is the number of spanning trees oriented toward v_i .

(Note: The counting part of the theorem was proved by "de Bruijn and van Aardenne-Ehrenfest", Smith and Tutte. (two independent groups))

Proof.

The existence part can be obtained by a similar argument as the "multigraph" version. (We omit the proof of 2nd part.)

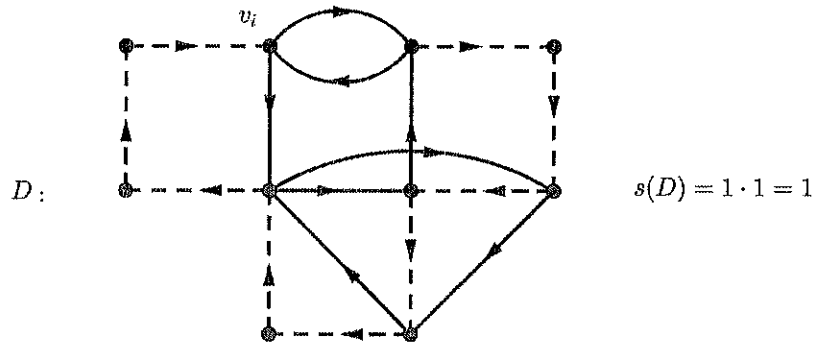


Figure 6. Spanning tree oriented toward v_i (dot arcs)

Theorem 6

Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$ (provided $\delta(G) \geq 2$).

Proof.

Let $\langle v_0, v_1, \dots, v_l \rangle$ be a longest path. Then, $N_G(v_l) \subseteq \{v_0, v_1, \dots, v_{l-1}\}$. For otherwise, we have a longer path. Since v_l has at least $\delta(G)$ neighbors $l \geq \deg_G(v_l) \geq \delta(G)$. This concludes the first part. Now, let i be the smallest index in $\{0, 1, 2, \dots, l-1\}$ such that $v_i v_l \in E(G)$. Hence, $(v_i, v_{i+1}, \dots, v_l)$ is a cycle of length at least $\delta(G) + 1$. ■

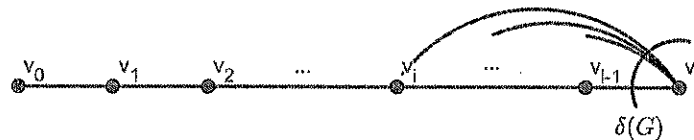


Figure 7. $l - i \geq \delta(G)$

Exercise (Bonus)

Every connected graph G contains a path or cycle of length at least $\min\{\underline{2\delta(G)}, |G|\}$.

Theorem 7

For each connected graph G , $rad(G) \leq diam(G) \leq 2rad(G)$.

Proof.

It suffices to consider the second inequality. Let u and v be two vertices in G such that $d(u, v) = diam(G)$. Let w be a vertex in the center of G , i.e., $ecc(w) = rad(G)$. By the fact that " d " is a metric, $d(u, v) + d(w, v) \geq d(u, w)$. This implies that

$$ecc(w) + ecc(w) = 2rad(G) \geq d(u, w) + d(w, v) \geq d(u, v) = diam(G). \quad \blacksquare$$

(Note. The eccentricity of $w \in V(G)$ is $\max\{d(x, w) \mid x \in V(G)\}$.)

Ex. 1-4, $a, b \in \mathbb{N}$, $a \leq b \leq 2a$, find graphs G with $diam(G) = b$ and $rad(G) = a$.

Theorem 8

A graph of minimum degree δ and girth (shortest cycle) g has at least $n_0(\delta, g)$ vertices where

$$n_0(\delta, g) = \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i, & g =_{def} 2r + 1, \text{ and} \\ 2 \cdot \sum_{i=0}^{r-1} (\delta - 1)^i, & g = 2r. \end{cases}$$

Proof.

Case 1. $g = 2r + 1, r \geq 1$.

Let v_0 be a fixed vertex in G , see Figure 9.

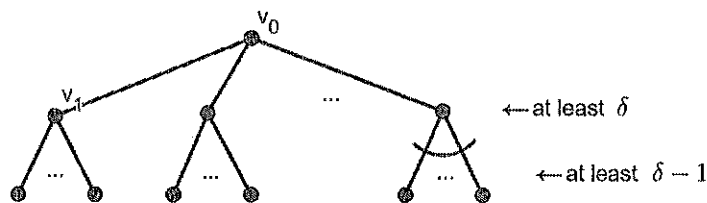


Figure 9.

Then, there are at least δ neighbors of v_0 , and for each neighbor, say v_1 , v_1 has at least $\delta - 1$ neighbors. Since $g = 2r + 1$, G contains at least $1 + \delta + \delta(\delta - 1) + \dots + \delta(\delta - 1)^{r-1}$ vertices. This concludes the proof of the Case 1.

Case 2. $g = 2r$

In this case, we start with an edge u_0v_0 , see Figure 10. By a similar argument, G contains at least $2 \cdot [(\delta - 1) + (\delta - 1)^2 + \dots + (\delta - 1)^{r-1}]$ vertices. ■

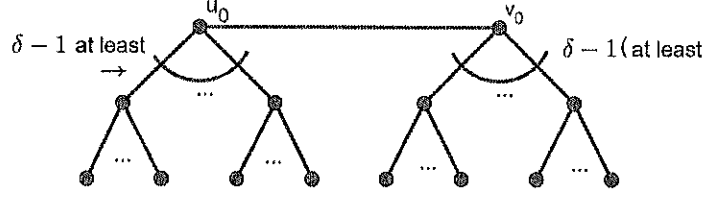


Figure 10.

Theorem 9

If $\delta(G) \geq 3$, then $g(G) < 2\log_2 |G|$.

Proof.

Note that if $\delta_1 \geq \delta_2 \geq 3$, then $n_0(\delta_1, g) \geq n_0(\delta_2, g)$. It suffices to consider $n_0(3, g)$. By Theorem 8,

$$|G| \geq n_0(3, g) = 2^r + 2^r - 2 > 2^r \quad (g = 2r) \text{ and}$$

$$|G| \geq n_0(3, g) = 1 + 3 \cdot \frac{2^r - 1}{2 - 1} = \frac{3}{\sqrt{2}} 2^{\frac{g}{2}} - 2 > 2^{\frac{g}{2}} \quad (g = 2r + 1).$$

This implies that $r < \log_2 |G|$ and thus $g < 2\log_2 |G|$. ■