

# Isomorphic Graphs

## Definition (Isomorphism)

Two graphs  $G$  and  $G'$  are isomorphic if there exists a bijection  $f$  from  $V(G)$  onto  $V(G')$  such that  $uv \in E(G) \iff f(u)f(v) \in E(G')$ .

Facts: If  $G$  and  $G'$  are isomorphic, then

1.  $G$  and  $G'$  are of the same order and size,
2.  $G$  and  $G'$  have the same “degree sequences”, and
3. the complements of  $G$  and  $G'$  respectively are also isomorphic.

## Exercise 1-2

- (a) List all non-isomorphic graphs of order 5. (Explain your answer.)
- (b) Give an idea of determining whether two given graphs are isomorphic or not.
- (\*) In general, to determine whether two given graphs are isomorphic or not is a very difficult problem in the sense of “Algorithm”.*
- (\*) That is, there does not exist a sufficient condition for two graphs which are isomorphic.

# Subgraphs

- Let  $G$  be a graph. If  $H$  is a graph such that  $V(H)$  is a subset of  $V(G)$  and  $E(H)$  is a subset of  $E(G)$ , then we call  $H$  a subgraph of  $G$ , denoted by  $H \leq G$ . Furthermore,  $H$  is an *induced subgraph* of  $G$  if  $u$  and  $v$  are vertices of  $H$  and  $uv \in E(G)$ , then  $uv \in E(H)$ . If  $V(H) = S$ , then  $H$  is denoted by  $\langle S \rangle_G$ . For convenience, we may also say  $H$  is generated by  $S$  where  $S$  is a subset of  $V(G)$ .
- Let  $E'$  be an edge subset of  $E(G)$ . Then, the *edge induced subgraph* induced by  $E'$  is the graph  $G' = (V', E')$  where  $V'$  is the set of vertices incident to an edge of  $E'$ .

# Graph Structures

- One of the most important part of learning a graph  $G$  is to discover the kind of subgraphs which are inside the graph  $G$  and the collection of graphs which are forbidden for  $G$ .
- For example, the well-known Petersen graph does contain two disjoint 5-cycles,  $C_5$ , but it contains no 4-cycles, 3-cycles and 10-cycles (also known as Hamilton cycles).
- As a matter of fact, an even non-trivial graph does contain many cycles.

# Veblen's Theorem

## Theorem

*The edge set of a graph can be partitioned into cycles if and only if every vertex has even degree, or the graph is an even graph.*

## Proof. (Necessity)

If a vertex  $v$  is contained in exactly  $h$  cycles, then the degree of  $v$  in the graph is  $2h$ .

(Sufficiency) The cycles can be obtained recursively. We start with finding the first cycle. Let  $\langle x_0, x_1, \dots, x_l \rangle$  be a path of maximum length  $l$  in  $G$ . Since  $x_0x_1$  is an edge of  $G$ ,  $\deg(x_0)$  is at least 2. Let  $y$  be a vertex which is incident to  $x_0$ . Clearly  $y$  is a vertex in  $\{x_2, x_3, \dots, x_l\}$ . Otherwise, we have a longer path, a contradiction. Therefore, we have a cycle  $C = (x_0, x_1, \dots, x_i)$  provided  $y = x_i$ . Now delete  $C$  from  $G$  and we can continue to find another cycles in  $G - C$ .



# *Eulerian Circuit Theorem*

## *Definition*

An *Eulerian circuit* of a graph  $G$  is a *closed trail* in  $G$  such that the trail passes all edges of  $G$ .

## *Eulerian Circuit Theorem*

*A connected multigraph has an Eulerian circuit if and only if every vertex of  $G$  is even, or  $G$  is an even graph.*

**Proof.** The necessity part is easy to see and the sufficiency can be proved by combining cycles together to obtain a circuit. 

# Directed Version of Eulerian Circuits

- A graph  $D$  is a *directed graph, digraph* in short, if each edge has an orientation, called arc. For convenience, an arc is represented by an order pair, say  $(u, v)$ . ( $u$  is a *tail* and  $v$  is a *head*.) Note that a directed walk has to follow the orientation.
- Let  $D = (V, A)$ . For a vertex  $v$  in  $V$ , the *out-degree* (resp. *in-degree*) of  $v$  denoted by  $\deg^+(v)$  (resp.  $\deg^-(v)$ ), is the number of arcs whose tail (resp. head) is  $v$ .
- A digraph is *balanced* if for each vertex, its in-degree is equal to out-degree.

# Directed Eulerian Circuit Theorem

## Theorem

*A digraph  $D$  has a directed Eulerian circuit if and only if  $D$  is balanced and strongly connected.*

Note that a graph is **strongly connected** if for any two vertices  $u$  and  $v$ , there exists a directed path from  $u$  to  $v$  and also a directed path from  $v$  to  $u$ .

Proof. Since  $D$  is balanced and strongly connected,  $D$  can be partitioned into directed cycles. The idea is similar to the proof of Veblen's Theorem. Then the circuit can be obtained by combining all those directed cycles. 

# Counting the Number of Distinct Eulerian Circuits

**BEST Theorem** (de-Bruijn and van Aardenne-Ehrenfest, Smith and Tutte)

If  $D$  is an Eulerian digraph, then the number of distinct Eulerian circuits  $s(D) = t_i(D) \cdot \prod_{j=1}^n (\deg^+(v_j) - 1)!$  for every  $i \in \{1, 2, \dots, n\}$  where  $t_i(D)$  is the number of spanning trees oriented toward  $v_i$ .

Note that in an Eulerian graph  $t_i(D) = t_{i'}(D)$ ,  $i$  is not equal to  $i'$ .

Proof. It follows from *Matrix Tree Theorem*. (For now, we omit the details of proof.)

**Open problem**

Find the number of distinct Eulerian circuits of an Eulerian graph.  
(Not digraphs!)

## *Bonus Exercise*

Give an idea of counting the number of distinct Eulerian circuits provided the Eulerian graph we consider is 4-regular, that is, all vertices have degree 4.

***Turning coffee into solutions!***

