

Basic Terms in Graph Theory

- A graph G is an ordered pair $G = (V, E)$ where V (or $V(G)$) is the vertex set of G and E (or $E(G)$) is the edge set of G .
- If G contains only edges with 1-subsets or 2-subsets of $V(G)$, G is known as a *general graph*. Otherwise, G is a *hypergraph*.
- A general graph G is called a *simple graph* if G contains only 2-subsets (edges) without duplication. A graph with multi-edges and no 1-subsets (loops) is a *multi-graph*. A graph with loops and multi-edges is known as a *pseudo-graph*.
- *Without mention otherwise, we shall mainly study the properties of simple graphs.*

Notations

- Let G be a simple graph.
- The cardinality of $V(G)$ is called the order of G , denoted by $|G|$.
- The cardinality of $E(G)$ is called the size of G , denoted by $\|G\|$.
- The **neighborhood** of a vertex v in G , $N_G(v)$, is the set of vertices of $V(G)$ which are incident to v , i. e. $N_G(v) = \{u: uv \text{ is an edge of } G\}$. (We use uv to denote the set $\{u, v\}$.)
- $N_G[v]$ is the closed neighborhood of v , $N_G[v] = N_G(v) \cup \{v\}$.
- The degree of v in G is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$.

Important Property of Graphs

- The sum of all degrees in a graph is equal to twice the size of G .

Proof. An edge of G will provide two degrees to the sum.

(Shake hand principle)

- The above fact shows that the number of vertices in G with odd degree must be even.
- For simplicity, a vertex with odd degree (resp. even degree) is called an odd (resp. even) vertex.
- A graph is called **k -regular** if all vertices are of the same degree k . Therefore, if **k is odd and G is k -regular, then G is of even order.**

About Degrees

- To characterize a graph, the following notations are important.
 - $\Delta(G)$: Maximum degree of G
 - $\delta(G)$: Minimum degree of G
 - $d(G)$: Average degree of vertices in G , $d(G) = 2||G|| / |G|$
 - $\mathcal{E}(G)$: Average size of edges incident a vertex in G , which is $d(G)/2$.
- (*) If a graph G is randomly generated, then we do know $d(G)$ or corresponding $\mathcal{E}(G)$, but $\Delta(G)$ or $\delta(G)$ may remain unknown. For example, if we use the probability $1/3$ for the existence of all edges to construct a graph G , then $d(G) = (|G| - 1)/3$. (?)

Sperner's Lemma

- Let ABC be a triangle with **triangulation** inside the triangle, where A, B and C are labeled 1, 2 and 3 respectively. All the vertices on **AB** (resp. BC and CA) are labeled with **1 or 2** (resp. 2 or 3, 1 and 3). All the vertices inside ABC can be labeled with 1 or 2 or 3 arbitrarily. Then there exists a triangle inside ABC such that its three vertices are labeled with 1, 2 and 3 respectively. More precisely, there are odd number of such triangles.

Proof. We construct a new graph by assigning each region (or face) of the graph a vertex and connect two vertices with an edge if the two regions have a common boundary edge whose end vertices are of different labels, say 1 and 2 or 1 and 3 or 2 and 3. **Since the vertex of the outer region is of odd degree, the existence of such triangles do exist!** 

Tiling a Large Rectangle with Rectangles

- A large rectangle T is tiled with smaller rectangles $T_1, T_2, \dots, T_l$. Prove that if each smaller rectangle has an integer side, then T has an integer side.
- There are different ways to prove the above statement.
 1. Consider the function $F(x, y) = \sin 2\pi x \cdot \sin 2\pi y$. Then find the double integral defined on T . **By the fact that if the rectangular region has an integer side, then its double integral is equal to 0 and vice versa.**
 2. Graph model: Construct a graph (bipartite) with vertex set $A = (\mathbb{N} \cup \{0\})^2$ and $B = \{T_1, T_2, \dots, T_l\}$. Join an edge from T_i to a vertex in A if the vertex is in the corner of T_i . Now, by the fact $(0, 0)$ is incident to T_1 only, there exists another vertex of A with odd degree, they are in the corner of T if exists. 

Notes for Tiling Problem

- There are more ways to prove the above statement.
- The tiling problem can be extended to higher dimensions, for example 3-dimension.
- In fact, there is a stronger version for the tiling problem.

Theorem

Let B be a box in $\mathbf{R}^n$ tiled with boxes $B_1, B_2, \dots, B_l$. If each box B_i has at least k integer sides, then B has at least k integer sides.

(*) Can you prove this assertion?

Exercise 1-1

- A mountain range is a polygonal curve from $(a, 0)$ to $(b, 0)$ in the upper half-plane. Hikers A and B begin at $(a, 0)$ and $(b, 0)$ respectively.
(i) *Prove that A and B can meet by traveling on the mountain range in such a way that at all times their heights above the horizontal axis are the same.*



Continued

Exercise 1-1 (ii)

Let the heights of mountain peaks be $s_1, s_2, \dots, s_h$ and the lower intersections be of heights $t_1, t_2, \dots, t_{h-1}$ respectively. Moreover, let their corresponding coordinates on the ground be $(a_1, 0), (a_2, 0), \dots, (a_h, 0); (b_1, 0), (b_2, 0), \dots, (b_{h-1}, 0)$. Now, given a sequence $\langle 0, s_1, t_1, s_2, t_2, \dots, s_h, 0 \rangle$, determine the meeting point.

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(For example, $\langle 0, 2, 1, 3, 0 \rangle$, the meeting point is at the peak with height 3 located at $(a_2, 0)$.)