Graph Theory Lecture 12 (补充资料)

(•) There exists a procedure to construct all the graphs G with $\chi(G) \geq k$.

(••) $\mathcal{G}_k$ is a collection of k-constructible graphs if $G \in \mathcal{G}_k$ can be constructed recursively by the following steps.

(i) K_k is k-constructible.

(ii) If G is k-constructible and $x, y \in V(G)$ are non-adjacent, then $(G+xy)/xy$ is k-constructible.

(iii) If G_1 and G_2 are k-constructible such that $V(G_1) \cap V(G_2) = \{x\}$, $xy_1 \in E(G_1)$, $xy_2 \in E(G_2)$, then $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible, see Figure 41.

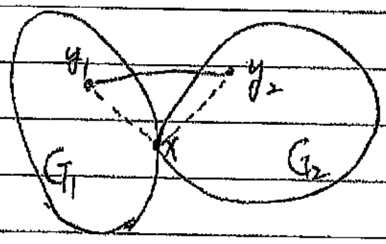


Figure 41. Hajós construction.

(*) If G is k-constructible, then $\chi(G) \geq k$.

(i) is trivial (ii) If $(G+xy)/xy$ uses less than k colors, then by coloring x and y with the same color, we have $\chi(G) < k$.

(iii) If the graph obtained uses less than k colors, then either $\chi(G_1) < k$ or $\chi(G_2) < k$ depending on whether $\varphi(y_1) \neq \varphi(x)$ or $\varphi(y_2) \neq \varphi(x)$. Since y_1 and y_2 receive distinct colors, one of the above two conditions must hold.

Theorem 70. (Hajós, 1961)

Let G be a graph. Then, $\chi(G) \geq k$ if and only if G has a k-constructible subgraphs.

Proof. ($\Leftarrow$) It has been explained above.

($\Rightarrow$) Suppose not; then $k \geq 3$. Let G be a maximal counterexample, i.e. G is of maximum size such that G does not contain a k-constructible subgraph. Now, G can not be a complete r-partite graph. For otherwise, $\chi(G) \geq k$ implies that $r \geq k$ and then G contains a k-constructible K_k. (Contract each partite set.) Hence there exist vertices x, y_1 and y_2 such that $xy_1 \notin E(G)$, $xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. By assumption of the maximality of G, both $G+xy_1$ and $G+xy_2$ contain k-constructible subgraphs, say H_1 and H_2;

moreover, $xy_1 \in E(H_1)$ and $xy_2 \in E(H_2)$.

Let $H_2 - H_1$ denote the graph $\langle V(H_2) \setminus V(H_1) \rangle_{H_2}$ and H_2' is an isomorphic copy of H_2 such that $V(H_2') \cap V(G) = \{x\} \cup (V(H_2) \setminus V(H_1))$, see Figure 42. So, $V(H_1) \cap V(H_2') = \{x\}$. Now, since $H_2' \cong H_2$, let $\varphi : H_2 \to H_2'$ be an isomorphism. By (iii) $H_1 \cup H_2' - xy_1 - x\cdot\varphi(y_2) + y_1\cdot\varphi(y_2)$ is k-constructive, let this graph be H. Now, for each vertex v' in $V(H_2') \setminus V(G)$, there exists a v such that $v' = \varphi(v)$. Furthermore vv' is not an edge of H. By (ii), we can identify v and v' and obtain a k-constructible subgraph of G after identifying all vertices in $V(H_2) \setminus V(H_1)$. ∎

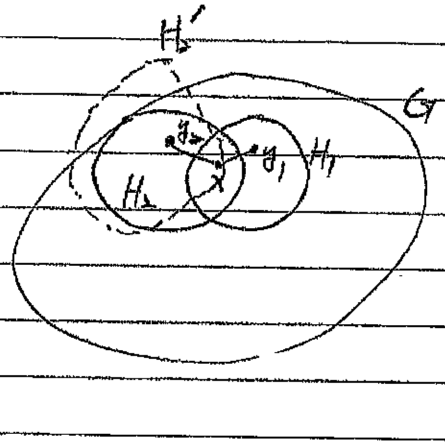


Figure 42

(◦) A k-edge-coloring is a mapping $\pi : E(G) \to \{1, 2, \cdots, k\}$ such that incident edges receive distinct images (colors).

(◦) $\chi'(G) = \min\{k \mid G$ has a k-edge-coloring$\}$. (Chromatic index of G.) If $\chi'(G) = k$, then G is h-edge-colorable for each $h \geq k$.

Theorem 7'. (Vizing, 1964)

If G is a simple graph, then $\Delta(G) \leq \chi' \leq \Delta(G)+1$.

Proof. The left hand inequality is easy to see, we prove the right hand inequality. By induction on $\|G\|$. We shall prove that G has a $(\Delta(G)+1)$-edge-coloring (coloring in short) for G and the assertion is true for smaller sizes, i.e, for each $e \in E(G)$, $G-e$ has a coloring π. (Let $e = xy$.)

First, we observe that since each vertex v is of degree at most $\Delta(G)$, a color is missing around v. Second, if α and β be two colors used in the (a) coloring, then α and β induce a subgraph with components either paths or even cycles.

Finally, if G has no colorings using $\Delta(G)+1$ colors, then for each edge xy and any coloring of $G-xy$, there exists an β-α path from y ends in x provided α is missing at x and β is missing at y. See Figure 43 for missing colors.

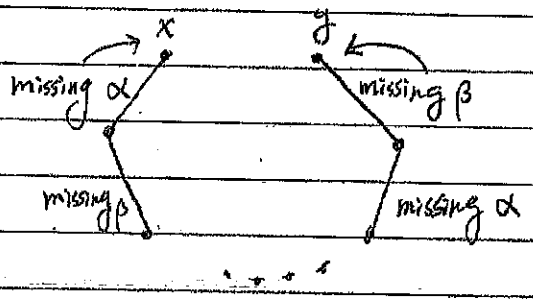


Figure 43

(*) If α-β path does not connect x and y, then we may recolor one of the path (α, β), to obtain a coloring of G using $\Delta(G)+1$ colors.

Clearly, if x and y are missing the same color, then we can use that color to color xy and obtain a $\Delta(G)+1$ coloring of G. (By induction)

(*) Claim: There is a way to recolor some edges in $G-xy$ such that x and y miss the same color.

Outline of proof

Let $M(y)$ denote the colors missing at y, and $c_1 \in M(y)$. Now, consider $M(x)$. If $c_1 \in M(x)$, then color xy by c_1 results in a $\Delta(G)+1$ coloring of G. (The claim holds.)

Hence, $c_1 \notin M(x)$, let $c_0 \in M(x)$ and $\pi(xy_1) = c_1$. (See Figure 44)
Then, consider $M(y_1)$ and let $c_2 \in M(y_1)$. Note that $c_2 \notin M(x)$.
If $c_2 \in M(x)$, then we let $\pi(xy_1) = c_2$. Thus, c_1 becomes a missing color in $M(x)$, the coloring for xy is available, $\pi(xy) = c_1$. This fact will continue: $c_2 \notin M(x) \Rightarrow \exists\, y_2$, s.t. $\pi(xy_2) = c_2$; and then $c_3 \in M(y_2)$, $\pi(xy_3) = c_3$, $\cdots$, $c_{i+1} \in M(y_i)$, $\pi(xy_{i+1}) = c_{i+1}$.
Since we only have $\Delta(G)+1$ colors, there exists an l such that $\pi(xy_{l+1}) = c_{l+1} \in \{c_1, c_2, \cdots, c_l\}$. W.L.O.G., let $c_{l+1} = c_k$, $k \in \{1, 2, \cdots, l\}$.

Now, we have several cases to consider depending on whether $c_0 \in M(y_l)$ or $c_0 \notin M(y_l)$.

(0) If $c_0 \in M(y_1)$, then color xy_1 by c_0. This implies that $c_1 \in M(x)$.

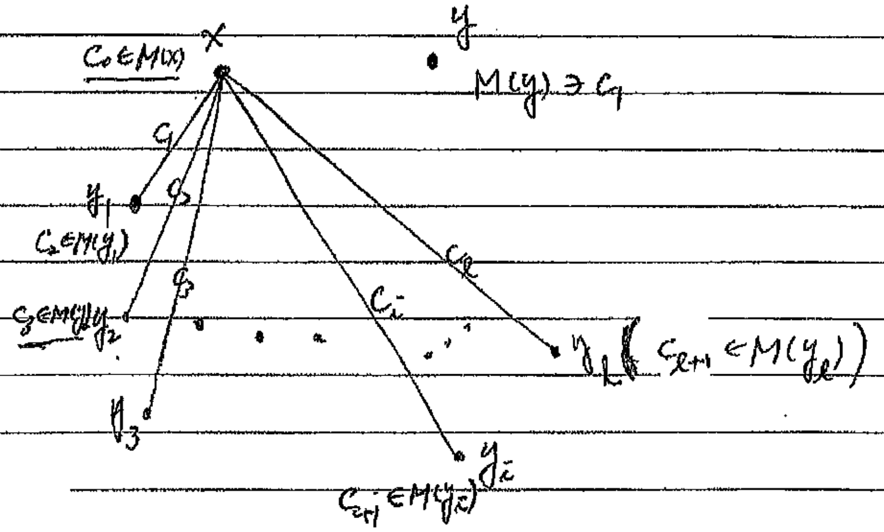


Figure 44 (Fan sequence!)

(a) $c_0 \in M(y_\ell)$.

Then, let $\pi(xy_\ell) = c_0$, $\pi(xy_{\ell-1}) = c_\ell$, $\cdots$, $\pi(xy_1) = c_2$. This implies that $c_1 \in M(x)$ and the proof follows.

(b) $c_0 \notin M(y_\ell)$.

Since $c_{\ell+1} = c_k$, $c_k \in M(y_\ell)$. Now, consider $c_k - c_0$ path starting from y_ℓ.

(i) It is a $y_\ell - y_k$ path. Since $\pi(xy_k) = c_k$, we may recolor them to a $c_0 - c_k$ path starting from y_k. (Note here that c_0 occurs in an edge incident to y_ℓ. By the fact that the last color is c_k, both c_0 and c_k occur an even number of times.) Now, since $\pi(xy_k) = c_0$, the recoloring of $xy_1, xy_2, \cdots, xy_{k-1}$ gives $c_1 \in M(x)$, we have the proof.

(ii) It is a $y_\ell - y_{k-1}$ path. Since $c_k \in M(y_{k-1})$, this path is ended with color c_0. That is to say c_0 is also available for xy_{k-1} (not only c_{k-1}). Hence, we color xy_{k-1} with c_0 instead of c_{k-1}, the proof follows by a similar recoloring process.

(iii) It is a $y_k - y_i$ path, $i \notin \{k-1, k\}$.

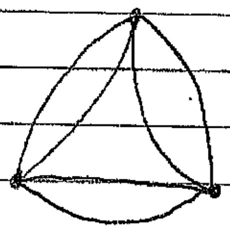
Then, either c_k or c_0 will be available for xy_i and the proof follows by recoloring process. ∎

(•) Based on the same proof technique, we also have a stronger result of Vizing's Theorem.

Theorem 67' (Vizing, 1964)

If G is a multigraph with multiplicity η, then $\chi'(G) \le \Delta(G) + \eta$.

(•) The following graph has $\Delta(G) = 4$ and $\eta = 2$.



Definition (Class 1 and Class 2)

A graph (simple) is of Class 1 if $\chi'(G) = \Delta(G)$ and of Class 2 if $\chi'(G) = \Delta(G) + 1$.

✓ Theorem 77 (König, 1916) A bipartite graph is of Class 1.

Proof. (1st)

By induction on $\|G\|$. Let $xy \in E(G)$ and $G-xy$ can be edge-colored with $\Delta(G)$ colors. Now, since $\deg_{G-xy}(x) < \Delta(G)$ and $\deg_{G-xy}(y) < \Delta(G)$, a color is missing at x and also a color is missing at y. Let them be α and β respectively. Clearly, $\alpha \neq \beta$, and β occurs around x and α occurs around y. Now, we adapt the idea in proving Vizing's Theorem, let P be a longest α-β path from x: $\overset{\beta}{-}\cdot\overset{\alpha}{-}\cdot\overset{\beta}{-}\cdot \cdots$

First, if P is an x-y path and the last edge has color α, then P is a path of even length. Hence, $P \cup \{xy\}$ is an odd cycle. $\rightarrow\leftarrow$ G is bipartite.

Hence, x and y are in different components induced by the set of edges colored α and β. Now, we recolor all the edges of P by interchanging α and β. This gives a coloring in which β is missing at x and also at y. By coloring xy with β, we obtain a Δ-edge-coloring of G. ∎

2nd proof

Lemma Let G be a bipartite graph. Then, there exists a $\Delta(G)$-regular bipartite graph $\tilde{G} \geq G$. (Exercise)

By Lemma $\tilde{G}$ is a $\Delta(G)$-regular bipartite graph and thus $\tilde{G}$ can be decomposed into $\Delta(G)$ perfect matchings by König's Theorem. This implies that $\chi'(\tilde{G}) = \Delta(G)$. Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) \leq \Delta(G)$. Hence, we conclude the proof.

(○) A graph G is said to be overfull if $\|G\| > \lfloor \frac{|G|}{2} \rfloor \cdot \Delta(G)$.

(○○) If G is overfull, then G is of Class 2.

(○) If G is overfull, then $|G|$ is odd.

Theorem 73

Petersen graph is of Class 2.

Proof. If G is the Petersen graph and $\chi'(G) = 3$, then G can be decomposed into 3 1-factors: F_1, F_2 and F_3 (3 color classes).

Now, consider the set of 5 link-edges e_1, e_2, e_3, e_4 and e_5, see Figure 45.

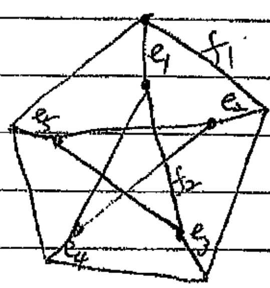


Figure 45 Petersen graph

At least one of F_1, F_2 and F_3 will contain at least two link-edges by Pigeon-hole principle, let it be F_1. Clearly, F_1 can not contain all the 5 link-edges. For otherwise, two C_5's is the union of F_2 and F_3 which is impossible. So, there are three cases to consider.

(i) $|F_1 \cap \{e_1, e_2, \cdots, e_5\}| = 4$

Let e_1 be the edge not in F_1. But, now all the edges of $G - e_1$ not in $\{e_2, e_3, e_4, e_5\}$ are incident to an edge of $\{e_2, e_3, e_4, e_5\}$. So, no other edge can be chosen for F_1.

(ii) $|F_1 \cap \{e_1, e_2, \cdots, e_5\}| = 3$

Let e_1 and e_2 be the edges not in F_1. Then, other than link-edges, we can choose at most one more edge f_1. The case e_1 and e_3 are not in F_1 has similar conclusion (only f_2 is available).

(iii) $|F_1 \cap \{e_1, e_2, \cdots, e_5\}| = 2$

This case comes out that we can find two more edges which not link-edges. ∎

(*) The proof of Theorem 69 implies that the Petersen graph contains no Hamilton cycles.

Proof. If G contains a Hamilton cycle C, then $\chi'(G)=3$ by coloring the cycle with two colors and $G-C$ (1-factor) with another color. ∎

Theorem 74

A 3-regular planar graph G is of Class 1.

Proof. Let G be embedded in S_0. Then, by 4-color Theorem, G is 4-face-colorable (or 4-map-colorable). Let the 4 colors used be obtained from the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$. Since each edge is in the boundary of two adjacent faces, let the edge be colored by $(a_1,b_1)\oplus(a_2,b_2)$ where (a_1,b_1) and (a_2,b_2) are the colors of those two adjacent faces. As a conclusion, we obtain a 3-edge-coloring of G, since $(0,0)$ will not be used. The coloring is proper since three adjacent faces will receive three different colors, see Figure 46. ∎

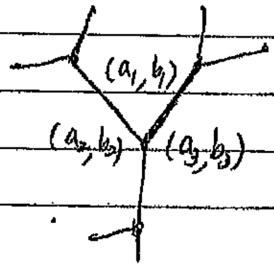


Figure 46.

Theorem 75 (Equitable edge-coloring)

If G has a k-edge-coloring f, then G has an equitable edge-coloring, i.e., for any two $i, j \in \{1, 2, \cdots, k\}$, $\big||f^{-1}(i)| - |f^{-1}(j)|\big| \leq 1$.

Proof. If there exist i and j such that $|f^{-1}(i)| - |f^{-1}(j)| \geq 2$, then we consider the graph H induced by the set of edges colored i and j. Then, H is a subgraph of G such that each component of H is either a path or an even cycle. Since i occurs more times than j, there exists an i-j path: $\circ \overset{i}{-} \circ \overset{j}{-} \circ \overset{i}{-} \circ \overset{j}{-} \circ \cdots \circ \overset{i}{-} \circ$ whose end edges are colored i. Now, by switching the colors on this path, we obtain a new edge coloring of G such that i occurs one less time and j occurs one more. It turns out that we can obtain an k-edge-coloring s.t. $\big||f^{-1}(i)| - |f^{-1}(j)|\big| \leq 1$. As a consequence, we are able to adjust all of them and obtain an equitable k-edge-coloring. ■

(•) This theorem is also not difficult to prove, but very useful.

(◦) Without using 4CT, the proof of Theorem 70 is very difficult.

(*) It was conjecture that if G is planar and $\Delta(G)$ is large enough, say, ____.

Theorem 76

The complete graph K_n is of Class 2 if and only if K_n is overfull or equivalently n is odd.

Proof. First, we claim that for each $m \geq 1$, K_{2m} is of Class 1. It suffices to give a $(2m-1)$-edge-coloring of K_{2m}. For convenience, let $V(K_{2m}) = \mathbb{Z}_{2m} = \{0, 1, 2, \cdots, 2m-1\}$. For each color $i \in \{1, 2, \cdots, 2m-1\}$, let the set of edges colored i be

$F_i = \{(0,i), (i+1, i-1), (i+2, i-2), \cdots, (i+m-1, i-m+1)\} \pmod{2m-1}$.

See Figure 46 for an example of $m=5$ and $i=3$.

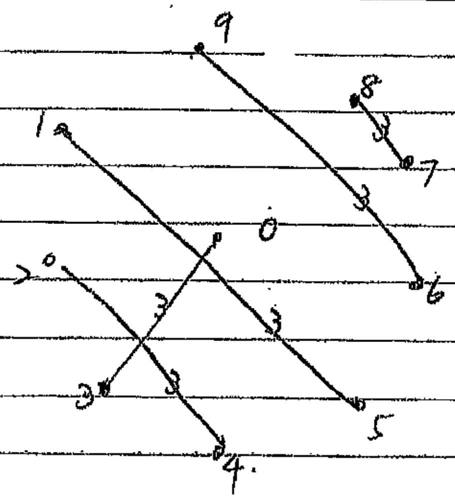


Figure 46. $\chi'(K_{10}) = 9$.

Since $\Delta(K_{2m}) = 2m-1$, $\chi'(K_{2m}) = 2m-1$.

Now, by deleting 0 in K_{2m}, we obtain a $(2m-1)$-edge-coloring of K_{2m-1}. On the other hand, it is not difficult to check that K_{2m-1} is overfull for $m \geq 2$, this concludes that $\chi'(K_{2m-1}) > \Delta(K_{2m-1}) = 2m-2$. ∎

(∘) This theorem is not difficult to prove, but it is very useful in the construction of "Combinatorial Designs".

(∘∘) Equivalently, K_{2m} can be decomposed into $2m-1$ 1-factors, which is also know as a 1-factorization of K_{2m}.

(∘∘∘) If G is an r-regular graph and $\chi'(G) = r$, then G has a 1-factorization.

(***) It was conjectured that if G is r-regular and $r \geq \frac{|G|}{2}$, then G has a 1-factorization or equivalently $\chi'(G) = r$.

Theorem 26′ (D. Hoffman et al.)

A complete multipartite graph G is of Class 2 if and only if G is overfull.

(o) <u>Total coloring</u>

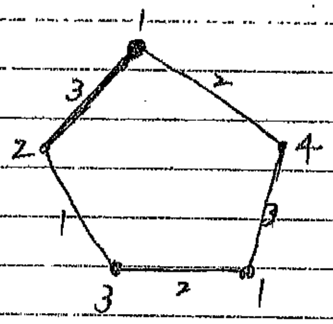
A k-total coloring of a graph G is a mapping $\psi : V(G) \cup E(G) \longrightarrow \{1, 2, \cdots, k\}$ such that

(i) adjacent vertices receive distinct images,

(ii) incident edges receive distinct images, and

(iii) each vertex has a distinct image wrt the images of its incident edges.

e.g.

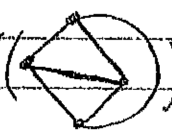


A 4-total coloring of C_5.

(o) $\chi''(G) = \min\{k \mid G \text{ has a } k\text{-total coloring}\}$.

(Total chromatic number)

<u>Theorem 7.9</u>

$\chi''($  $) = 5$ (?)

$\chi''(K_{2n+1}) = \chi''(K_{2n}) = 2n+1$.

<u>Proof.</u> $\chi''(K_{2n+1})$ can be obtained by using $\chi'(K_{2n+1}) = 2n+1$.

No. '12-17

TCC Conjecture $\chi''(G) \le \Delta(G)+2$.

Note that $\chi''(G) \ge \Delta(G)+1$. As to the total coloring of K_{2n}, we claim that $2n$ colors are not enough.

Observe that each color class has at most one vertex and $n-1$ edges. So, $2n$ color classes will contain at most $2n$ vertices and $2n(n-1)$ edges. Hence, there are $2n^2$ elements (vertices and edges) in total. But, K_{2n} has $2n + \frac{2n(2n-1)}{2}$ elements to color, which is $2n^2+n$. Clearly, $2n$ color is not enough. Since K_{2n+1} is $(2n+1)$-total colorable, K_{2n} is also $(2n+1)$-total colorable. The proof follows. ∎

(○) Based on TCC Conjecture, a graph G is called Type 1 if $\chi''(G) = \Delta(G)+1$ and Type 2 otherwise.

Theorem 28. $K_{m,n}$ is of Type 1 if and only if $m \ne n$.

Proof ($\Rightarrow$) If $m=n$, then there are $2n+n^2$ elements to color. Since each color class contains at most n elements, $\Delta(G)+1 = n+1$ colors are not enough, $n(n+1) < 2n+n^2$. Hence, $\chi''(K_{n,n}) \ge n+2$, and thus $K_{n,n}$ is not of Type 1.

($\Leftarrow$)

On the other direction, let $m = n+k$. Now, $\Delta(K_{m,n}) = n+k$. By the edge-coloring of $K_{m,n}$, we have an $n \times (n+k)$ Latin rectangle based on $\{1, 2, \cdots, n+k\}$, see Figure 47. Since $k \geq 1$, we may extend this rectangle to $(n+1) \times (n+k)$ and the last row can be used to color the vertices of A. Finally, color all vertices of B by one extra color, we have $\chi''(K_{m,n}) \leq n+k+1 = \Delta(K_{m,n})+1$. ∎

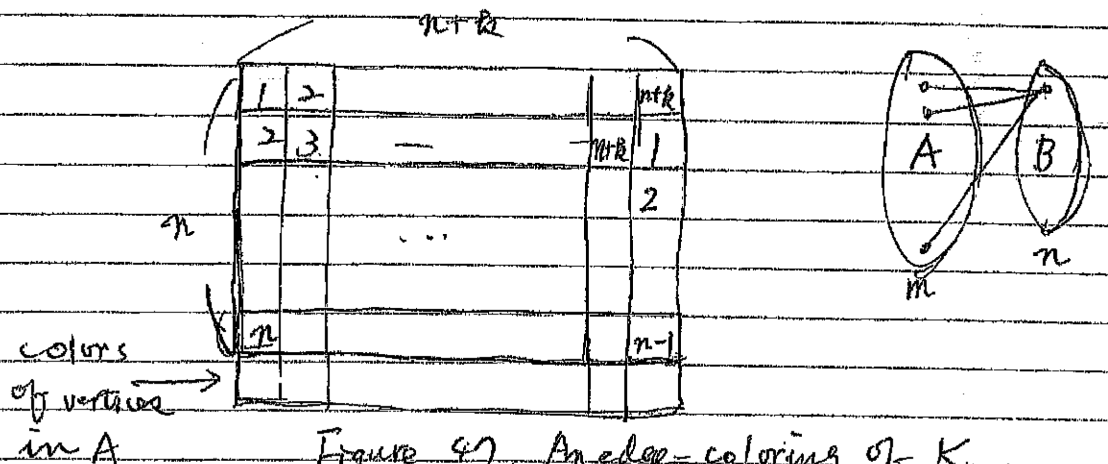
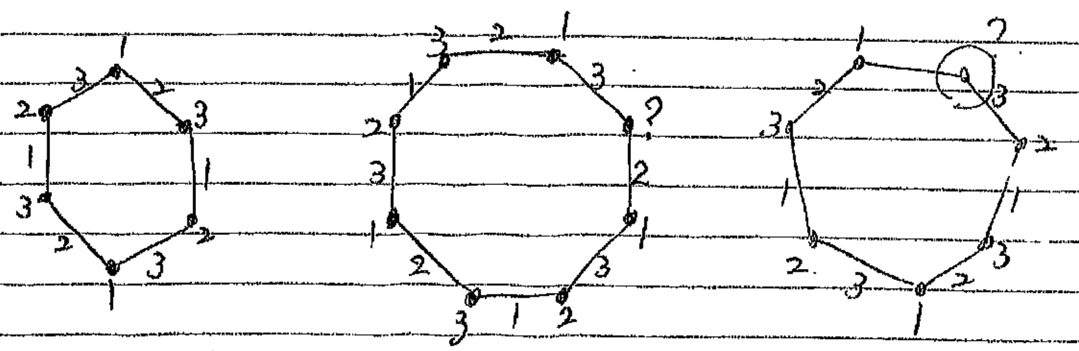


Figure 47, An edge-coloring of $K_{m,n}$.

(0) A cycle C_n is of Type 1 if and only if $n \equiv 0 \pmod 6$.



If we use three colors, then starting from one vertex and one edge, all the colors of the others are forced!

(∘) The deficiency of a graph G, $\mathrm{def}(G)$, is defined as $\sum_{v \in V(G)} (\Delta(G) - \deg_G(v))$.

(∘∘) G is <u>conformable</u> if G has a vertex coloring $\varphi : V(G) \to \{1, 2, \dots, \Delta(G)+1\}$ such that $\mathrm{def}(G) \geq |\{ i \mid |G| - |\varphi^{-1}(i)| \equiv 1 \pmod 2 \}|$.

<u>Theorem 79</u>

Let S_i be a star with i edges. Then, $K_{2n} - S_1 - S_{2n-3}$ ($= G(1, 2n-3)$) is of Type 2.

<u>Proof.</u> Assume that $G(1, 2n-3)$ is of Type 1, i.e., there exists a total coloring ψ of $G(1, 2n-3)$ using $\Delta(G)+1 = 2n-1$ colors. Let uv be the edge subdivided by w, see Figure 48. First, if $\psi(u) = \psi(v)$ ($\neq 0$), then let $\psi(w) = 1$, $\psi(uw) = 2$ and $\psi(wv) = 3$. Let r_j the number of vertices v in which j occurs in either v or an edge incident to v. Hence, $\sum_{j=0}^{2n-2} r_j = |K_{2n}| + 2\|G(1, 2n-3)\|$. (?) Now, 0 occurs in at most $2n-2$ vertices. Hence, $\sum_{j=0}^{2n-2} r_j \leq (2n-2) + 3 \cdot (2n) + (2n-5) \cdot (2n-1) = 4n^2 - 4n + 3 < |K_{2n}| + 2\|G(1, 2n-3)\|$. $\to\in$

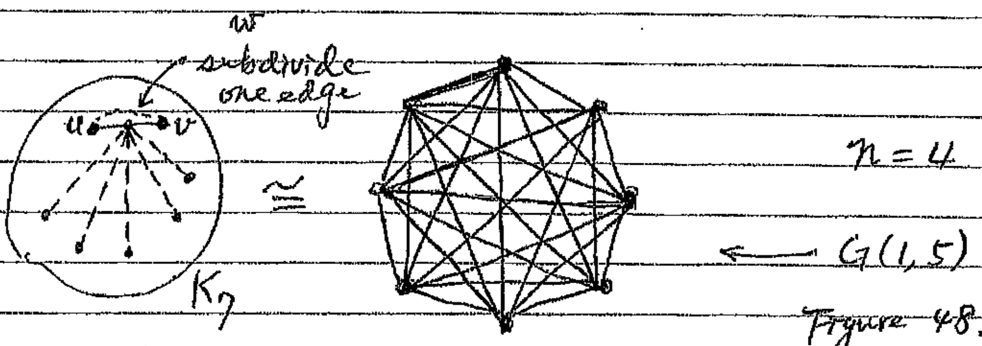


Figure 48. $G(1, 5)$.

On the other hand, if $\psi(u) \neq \psi(v)$, a similar argument shows that $2n-1$ colors are not enough. Hence, $G(1, 2n-3)$ is of Type 2. ■

(*) If G is of Type 1, then G is conformable.

Proof. If there exists a color i such that $|G| - |\psi^{-1}_{V(G)}(i)| \equiv 1 \pmod 2$, then i occurs in at most $|G|-1$ vertices. Since every color occurs around a major vertex (Δ-degree), $\Delta(G)+1$ colors are not enough if $\mathrm{def}(G) < |\{ i \mid |G| - |\psi^{-1}_{V(G)}(i)| \equiv 1 \pmod 2 \}|$. So, if G is not conformable, then G is not of Type 1. ■

Conjecture (Chetwynd and Hilton, 1988)

Let G be a simple graph with $\Delta(G) \geq \left[\frac{|G|+1}{2}\right]$. Then, G is of Type 2 if and only if there exists a non-conformable subgraph H of G such that $\Delta(H) = \Delta(G)$ (and G is not the Chen and Fu graph).
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(∘∘∘) The (original) conjecture was disproved by using Theorem 7.5.

(∘) The graph $G(1, 2n-3)$ is known as Chen and Fu graph.