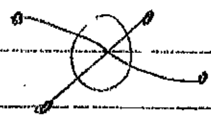
(*) Topological Graph Theory studies the "drawing" of a graph on a surface.

(**) A proper drawing on a surface of a graph G with p vertices and q edges follows the rules:

(1). There are p points on the surface which corresponds to the set of vertices in G; and

(2). There are q curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

(3). Two edges have at most one crossing.

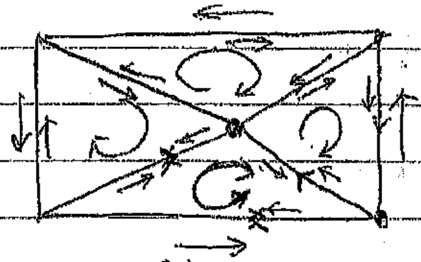


2-manifold : A connected topological space in which every point has a neighborhood homeomorphic to the open unit disk defined on R^2.

Bounded subspace : A subspace M of R^3 is bounded if $\exists\ K \in R^+$ such that $M \subseteq \{(x,y,z) \mid x^2+y^2+z^2 \leq K\}$.

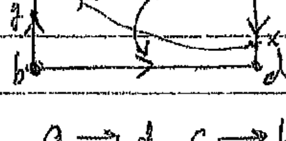
Closed : M is closed if its boundary ∂M coincides with M.

Orientable : M is orientable if for every simple closed curve C on M, a clockwise sense of rotation is preserved once around C, Otherwise, M is non-orientable.



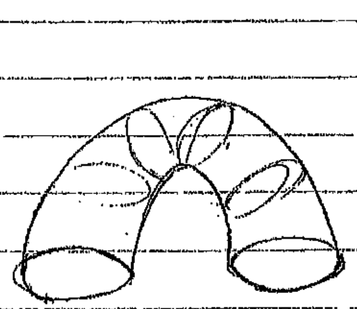
Orientable Surface : A surface S_k is a compact "orientable" 2-manifold that may be thought of as a sphere on which has been placed (inserted) a number k of "handles" (holes),

Non-orientable Surface : A surface obtained by adding k cross-caps to a sphere (S_0) is a non-orientable surface N_k.

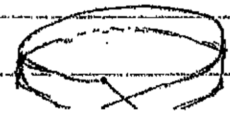


Möbius band

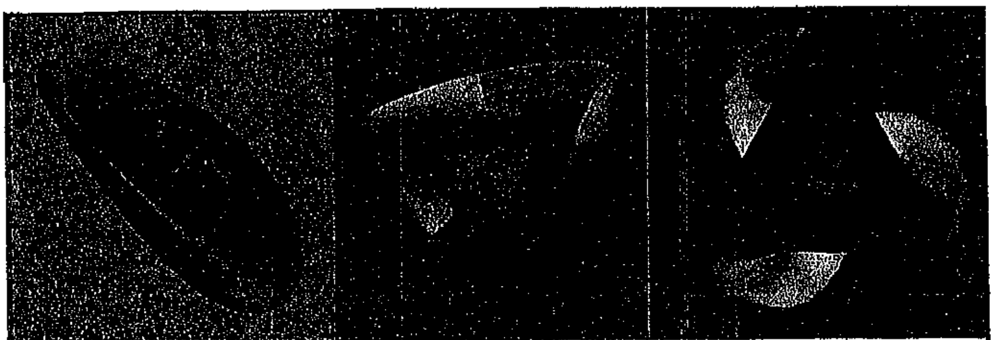
$a \to d$, $c \to b$, $x \to y$



Handle



(*) Adding a crosscap : Attach the boundary of a Möbius band to a cycle on S_0.



The Crosscap | The Roman Surface | Boy's Surface

<u>Definition</u> (Embeddable)

A (p,q)-graph G is said to be embeddable on a surface if it is possible to draw G <u>properly</u> on the surface.
(drawing without crossings)

<u>Definition</u> (Planar graph)

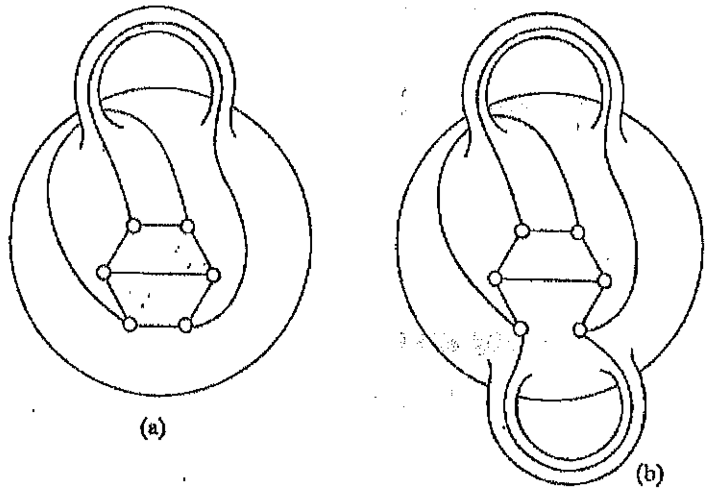
A graph is planar if it can be embedded in the plane, equivalently, embedded on the sphere.

<u>Definition</u> (2-cell embedding)

Date

An embedding in which for each region all curves in that region can be continuously deformed or contracted in that region to a single point. ... A 2-cell is topologically homeomorphic to R^2. An embedding of G on a surface is a 2-cell embedding of G if all the regions so determined are 2-cells.

The following figure shows embeddings of $K_{3,3}$ on S_1 and S_2 respectively. S_k is a surface obtained by attaching k handles.



(a) (b)

Figure 3.1. Embeddings of $K_{3,3}$ on surfaces S_1, S_2.

(•) S_0 : Sphere

(•) $N_0 \simeq S_0$, N_h : Attach h crosscaps to $N_0(S_0)$.

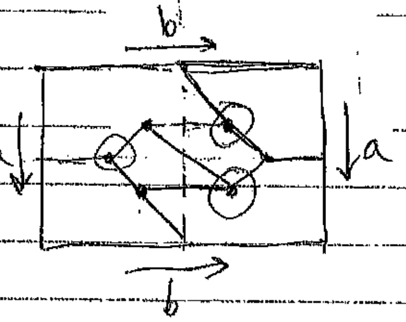
Definition (Genus)

The number of handles (resp. crosscaps) (on a surface) is referred to as genus of the orientable surface (resp. non-orientable surface). We use $\gamma(G)$ (respectively $\tilde{\gamma}(G)$) to denote the smallest genus of all orientable surfaces (resp. non-orientable surfaces) on which G can be embedded.

(*) If G is a planar graph, then $\gamma(G)$ (so is $\tilde{\gamma}(G)$) is equal to zero. But, G can be embedded on a surface with genus larger than "0".

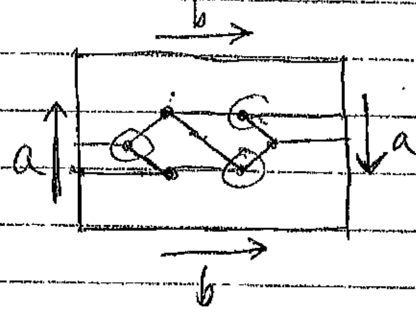
(**) Given a graph G, determining $\gamma(G)$ is a difficult problem.

Orientable embedding of $K_{3,3}$ (S_1)



$bab^{-1}a^{-1}$

Non-orientable embedding of $K_{3,3}$ (Möbius band)



$bab^{-1}a$

Theorem 50 (Euler)

Let G be a connected planar graph with p vertices, q edges and f faces (regions). Then $p-q+f=2$.

Proof. By induction on q. Since G is connected, G has at least $p-1$ edges. (?) If G has $p-1$ edges and G is connected, then G is a tree which contains no cycles. This implies that $f=1$ and thus $p-(p-1)+1=2$. The assertion is true for "minimal" graphs. Let the hypothesis is true for $\|G\| = k \geq p-1$. Now, consider G with $k+1$ edges. Clearly, G contains a cycle. Let e be a cycle edge. Since G is a connected planar graph (with f' faces), $G-e$ is also a connected planar graph. Moreover, $\|G-e\| = k$ and $G-e$ has k edges and $f'-1$ faces. By induction $p-k+f'-1=2$ and thus

$p-(k+1)+f'=2$. This concludes the proof. ∎

Theorem 51 If G is a planar graph with largest size, then $\|G\| = 3|G| - 6$.

Proof. By observation, if G has maximum size, then each region of G is a triangle. Since each edge of G is in the boundary of exact two regions, $3 \cdot f = 2 \cdot q$ where f is the number of regions and q is the size of G, i.e., $q = \|G\|$. Now, by Euler's formula $p - q + f = 2$ equivalently.

$$|G| - \|G\| + \frac{2}{3}\|G\| = 2$$

$$\Rightarrow 3 \cdot |G| - 6 = \|G\|. \quad (G \text{ is a maximal planar graph!}) \blacksquare$$

Corollary If G is a planar graph, then $\|G\| \le 3|G| - 6$.

(*) K_5 is not planar.

Corollary In any planar graph, there exists at least one vertex of degree smaller than 6.

Corollary The degree sum of a planar graph is at most $6 \cdot |G| - 12$.

This corollary is very useful.

(0) We can give a more accurate estimation of the above corollary

(*) If G is a planar graph with girth $g(G) \ge 4$, then $q \le 2p - 4$.

$4f \le 2q$, $q \ge 2f$, $p - q + f = 2 \le p - q + \frac{q}{2}$. $\Rightarrow \frac{q}{2} \le p - 2 \Rightarrow q \le 2p - 4$.

$\Rightarrow K_{3,3}$ is not planar.

Theorem 52.

Let G be a maximal planar graph (triangulated) of order p, and let p_i denote the number of vertices of degree i in G for $i=3,4,\cdots,\Delta(G)=d$. Then

$$3p_3+2p_4+p_5=p_7+2p_8+\cdots+(d-6)p_d+12.$$

Proof. Since $p=\sum_{i=3}^{d}p_i$ and $2q=\sum_{i=3}^{d}ip_i$, we have

$$\sum_{i=3}^{d}ip_i=2(3p-6)=6\sum_{i=3}^{d}p_i-12.$$

This implies the conclusion. ∎

Theorem 53

There are exactly five regular polyhydra.

Proof. Notice that a regular polyhydron is a polyhydron whose faces (regions) are bounded by congruent (全等) regular polygons and whose polyhydral angles are congruent (相等).

First, we convert a polyhydron into a regular planar graph. (See Figure 32.) Let the number of vertices, edges and faces be

By Theorem 50, $p-q+f=2$. Hence,

$$-8 = 4q-4p-4f = 2q+2q-4p-4f$$

$$= \underline{\sum_{i\geq 3} i f_i} + \sum_{i\geq 3} i p_i - 4\sum_{i\geq 3} p_i - \underline{4\sum_{i\geq 3} f_i} \quad (f_i : \# \text{ of } i\text{-face})$$

$$= \sum_{i\geq 3}(i-4)f_i + \sum_{i\geq 3}(i-4)p_i.$$

Since the polyhydron is regular, all degrees and face sizes are the same, let them be k and h respectively. Therefore,

$$-8 = (h-4)f_h + (k-4)p_k.$$

By the fact that every planar graph contains a vertex of degree less than six, we only have nine cases to consider: $3 \leq h \leq 5$ and $3 \leq k \leq 5$.

From direct checking, only 5 cases are possible, namely,

(1) $f_3 = p_3 = 4$ (Tetrahedron) 四面体
(2) $f_3 = 8$ and $p_4 = 6$ (Octahedron) 八面体
(3) $f_3 = 20$ and $p_5 = 12$ (Icosahedron) 二十面体
(4) $f_4 = 6$ and $p_3 = 8$ (Cube) 六面体
(5) $f_5 = 12$ and $p_3 = 20$ (Dodecahedron) 十二面体。 ▨

See Figure 32 for regular polyhydra.

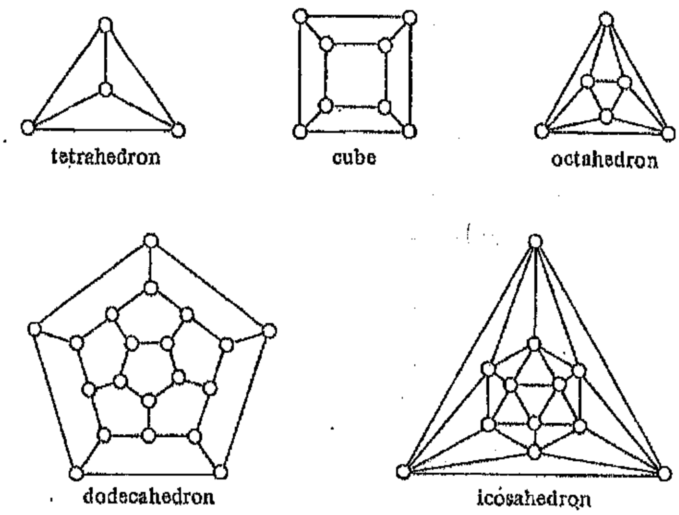


Figure 32 *The graphs of the regular polyhedra*

Theorem 54 (Fáry (1948), Wagner (1936))

A planar graph G can be embedded in the plane so that each edge is a straight line segment.

Proof. The proof is by induction on the order of G. It suffices to prove the case when G is a connected maximal planar graph.

Clearly, it is true for small orders. Assume the hypothesis is true for order k and let G be a connected maximal planar graph of order k+1.

Since G is maximal, $3 \leq \delta(G) \leq 5$.

Case 1. $\delta(G) = 3$

Let $v_0 \in V(G)$ such that $\deg_G(v_0) = 3$ and v is adjacent to v_1, v_2 and v_3. Since G is maximal, $\langle \{v_1, v_2, v_3\} \rangle_G \cong K_3$. This implies that $G - v_0$ is also a maximal planar graph. By induction $G - v_0$ has a straight line segment embedding. Now, put v_0 back to the graph $G - v_0$ such that v_0 is inside the region bounded by $\langle \{v_1, v_2, v_3\} \rangle$ and connect v_0 to the three vertices by straight line segment. This concludes the proof of this case.

Case 2. $\delta(G) = 4$

The proof follows by a similar process by letting $N(v_0) = \{v_1, v_2, v_3, v_4\}$. Now, $G - v_0 + v_1v_3$ is a maximal planar graph and thus it has a straight line segment embedding. The proof follows by placing v_0 back to $G - v_0 + v_1v_3 - v_1v_3$. By considering the drawing of the embedding (Figure 33(a)), we are able to put segment.

Case 3. $\delta(G) = 5$.

Again, we use the same technique and the drawing can be seen in Figure 33(b). ∎

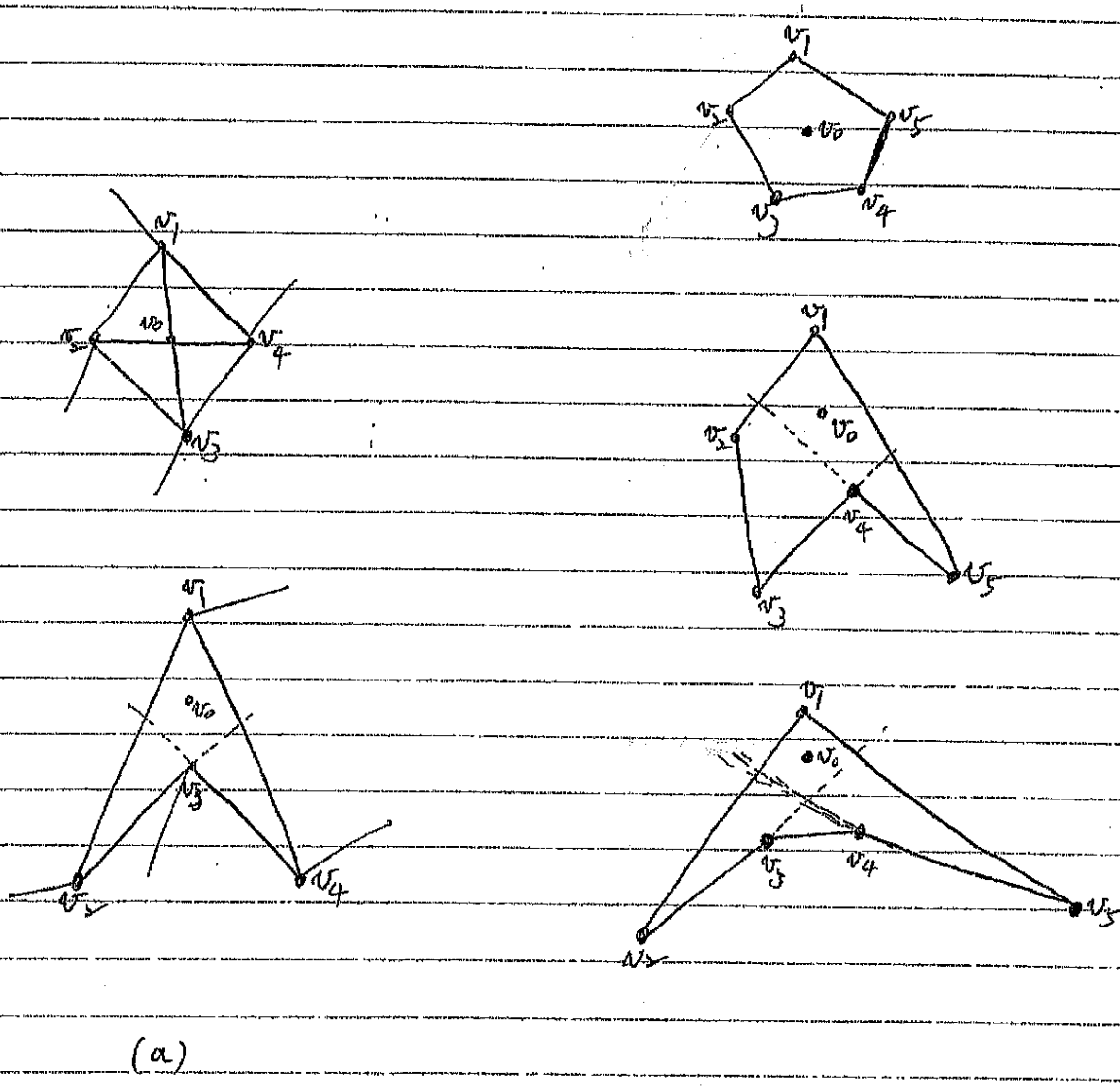
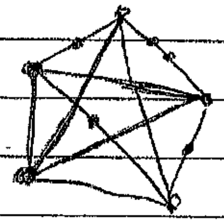


Figure 33 Location of v_0.

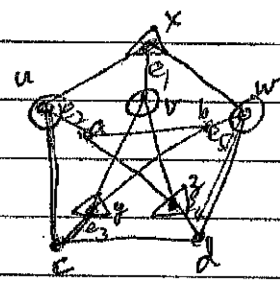
Characterization of planar graphs

(*) K_5 and $K_{3,3}$ are not planar graphs so are their homeomorphic graphs.



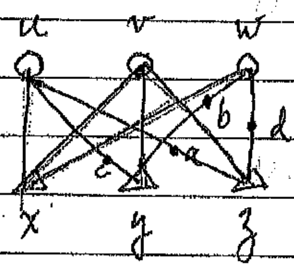
Subdivision of edges

(*) A graph which contains K_5, $K_{3,3}$ as minors.



$((((G/e_1)/e_2)/e_3)/e_4)/e_5$

$G/\{e_1, e_2, e_3, e_4, e_5\} \Rightarrow K_5$



delete ab, cd

Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proof. (Refer to books of Graph Theory)

(◎) The following theorem considers pseudographs, i.e., loops and multiedges are allowed.

Theorem 55 (Euler-Poincaré Theorem) ← Ex. 3-3 (5 points).

Let G be a (p,q)-pseudograph which has a 2-cell embedding on S_n. Then, $p-q+f = 2-2n$ where f is the number of faces in the embedding.

Proof. By induction on n and it's true when $n=0$ (by Euler's planar graph formula). Assume that the assertion is true when $n=k\geq 0$ and G is a (p,q)-pseudograph which has a 2-cell embedding on S_{k+1}. Since $k+1\geq 1$, there exists a handle in the embedding, see Figure 34(a). It suffices to consider the embedding such that there exists at least one edge which passes through the handle (on the surface). Note that if we can pull back an edge without passing the handle, then pull it back, see Figure 34(b). Now, we apply the idea of cut and past to obtain a 2-cell embedding of $\tilde{G}$ on S_k.

By using a circle around the handle, we can cut the handle through the circle and obtain $\tilde{G}$, see Figure 34(c). As a consequence the graph $\tilde{G}$ is embedded in S_k. If there are t edges passing through the handle. Then $|\tilde{G}| = p+2t$, $\|\tilde{G}\| = q+3t$, and the embedding in S_k has $f+t+2$ faces. Hence,

$$(p+2t) - (q+3t) + (f+t+2) = 2-2k.$$

This implies that $p-q+f = 2-2(k+1)$. ∎

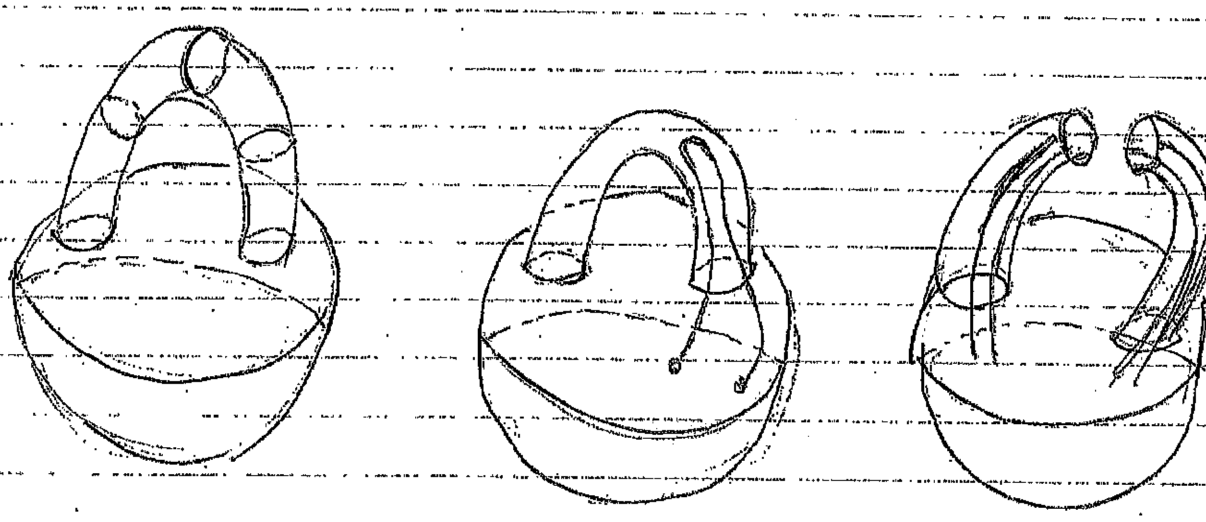
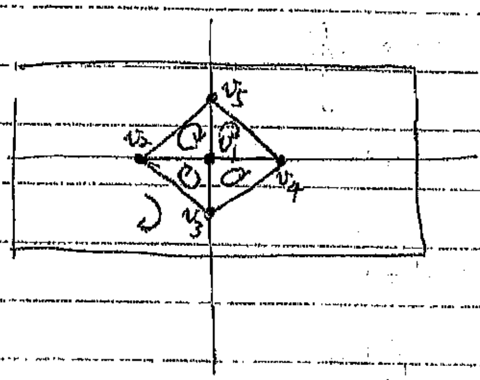


Figure 34 (a) Figure 34 (a), pull edges back Figure 34 (c)

For reference

Date $\quad$ 10-16

(∘) How to find a 2-cell embedding on S_k?



The above figure provides an embedding of K_5 on S_1, their regions are $((v_1, v_2, v_5))$, $((v_1, v_3, v_2))$, $((v_1, v_4, v_3))$, $((v_1, v_5, v_4))$, and

$((v_2, v_3, v_5, v_2, v_4, v_5, v_3, v_4))$.

(∘) Each arc of D_5 occurs exactly once in a region (face).

(∘∘) By considering each vertex, we observe that there are five permutations to determine this embedding.

$\pi_1 = (2, 3, 4, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ (Defined on v_1.)

$\pi_2 = (3, 1, 5, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{pmatrix}$ (Defined on v_2.)

$\pi_3 = (4, 1, 2, 5)$

$\pi_4 = (3, 2, 5, 1)$

$\pi_5 = (1, 4, 3, 2)$

On the other hand, if we are given five permutations one from each vertex, then we have a 2-cell embedding.

For example, $\pi_1 = (3,2,4,5)$, $\pi_2 = (3,1,5,4)$, $\pi_3 = (4,1,2,5)$, $\pi_4 = (3,2,5,1)$, and $\pi_5 = (1,4,3,2)$.

Now, if we start from the arc $(v_1 v_2)$, then the permutation of π_2 will give the next vertex of the oriented 2-cell containing $(1,2)$,

$$v_1 - v_2 - v_{\pi_2(1)} \Rightarrow v_1 - v_2 - v_5 - v_{\pi_5(2)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_{\pi_1(5)}$$

$$\Rightarrow v_1 - v_2 - v_5 - v_1 - v_3 \Rightarrow \cdots \Rightarrow (v_1, v_2, v_5, v_1, v_3, v_2, v_1, v_4, v_3).$$

$(v_{\pi_3(4)} = v_1$ and $v_{\pi_1(3)} = v_2)$

(*) Therefore, given a (p;q)-graph G, we can define p permutations for p vertices v_i such that each permutation is a cycle using the indices of the vertices in $N_G(v_i)$. Then, a 2-cell embedding will be obtained.

Theorem 56 (The (Heffter) Rotational Embedding Scheme)

Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \cdots, v_p\}$. For each 2-cell embedding of G on a surface, there exists a unique p-tuple $(\pi_1, \pi_2, \cdots, \pi_p)$ where for $i = 1, 2, \cdots, p$, π_i

is a cyclic permutation of $V(i)$ that describes the subscripts of the vertices in $N_G(v_i)$ in counterclockwise order about v_i. Conversely, for each p-tuple $(\pi_1, \pi_2, \cdots, \pi_p)$, there exists a 2-cell embedding of G on some surface such that for $i = 1, 2, \cdots, p$, the subscripts of the vertices adjacent to v_i are in counterclockwise order about v_i are given by π_i. Moreover, the set $\{\pi_1, \pi_2, \cdots, \pi_p\}$ induces a mapping π such that $\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$ for each pair of adjacent vertices v_i and v_j, $1 \leq i \neq j \leq p$.

Proof. The scheme was first observed and used by Dyck (1888) and Heffter (1891). A formalized version was obtained later in 1960 by Edmonds.

The main idea has been mentioned before the statement of Theorem 52. The genus of surface can be obtained after the number of faces (regions) has been determined. □

Theorem 57 $\gamma(K_{2m,2n}) = (m-1)(n-1)$, $m \le n$.

Proof. For convenience, let $K_{2m,2n} = (A_o, A_e)$ where $A_o = \{a_1, a_3, \cdots, a_{4m-1}\}$ and $A_e = \{a_2, a_4, \cdots, a_{4n}\}$. Note that m may not be equal to n. See Figure 35 for the case $2m=6$ and $2n=8$.

Now, let
$$\begin{cases} \pi_1 = \pi_5 = \cdots = \pi_{4m-3} = (2\ 4\ 6\ \cdots\ 4n); \\ \pi_3 = \pi_7 = \cdots = \pi_{4m-1} = (4n\ 4n-2\ \cdots\ 6\ 4\ 2); \\ \pi_2 = \pi_6 = \cdots = \pi_{4n-2} = (1\ 3\ 5\ \cdots\ 4m-1);\ \text{and} \\ \pi_4 = \pi_8 = \cdots = \pi_{4n} = (4m-1, 4m-3\ \cdots,\ 5\ 3\ 1). \end{cases}$$

We may check that this embedding does have $2mn$ regions each of them is bounded by a 4-cycle. Hence, by Theorem 51,

$$2\gamma(G) = 2-p+q-r = 2-(2m+2n)+4mn-2mn = 2mn-2m-2n+2.$$

Hence, $\gamma(G) = (m-1)(n-1)$. ∎

$(\pi_1, \pi_2, \cdots, \pi_{12}, \pi_{14}, \pi_{16})$

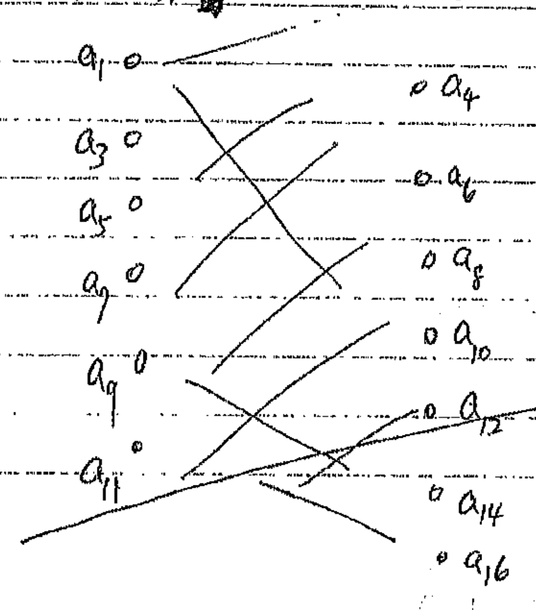


Figure 35, $K_{6,8}$

Date / /

(••) There are $\prod_{i=1}^{p} (deg_G(v_i) - 1)!$ p-tuples of $(\pi_1, \pi_2, \cdots, \pi_p)$.

(•••) The 2-cell embedding with the largest number of faces gives the genus of G.

(*) The 2-cell embedding with the smallest number of faces gives the "maximum" genus of G, denoted by $\gamma_M(G)$.

(**) Finding $\gamma(G)$ is a very difficult problem in general.

(**) Finding $\gamma_M(G)$ is comparatively easier.

Theorem 58. Let cr(G) denote the crossing number of G. Then, $cr(K_5) = cr(K_{3,3}) = 1$ and $cr(K_6) = 3$.

Proof. Since $\gamma(K_5) = \gamma(K_{3,3}) = 1$, the proof follows by a drawing with "1" crossing number. Now, we consider K_6. By Figure 35, $cr(K_6) \leq 3$. Let $cr(K_6) = k$. Then, we may convert the crossings into new vertices. Hence, we have a planar graph G (obtained from above): $|G| = 6+k$ and $\|G\| = 15+2k$. By the fact $15+2k \leq 3(6+k) - 6$, we have $k \geq 3$. ▨

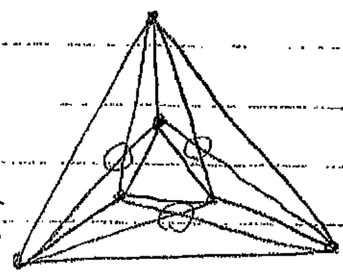


Figure 35

Theorem 59 $cr(K_9) = 36$.

Proof. For the upper bound, it suffices to give a drawing which has exactly 36 crossings. But, it is very technical to show the lower bound. Here, we provide a drawing for $cr(K_9) \le 36$, see Figure 36.

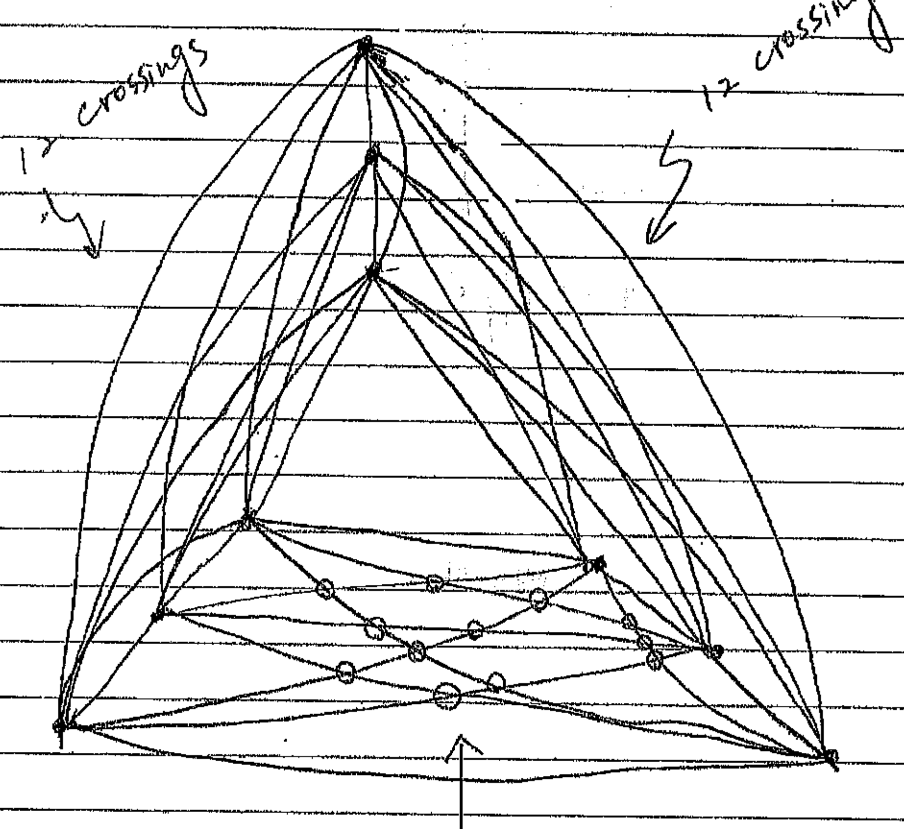


Figure 36, $cr(K_9)$

Conjecture $cr(K_p) = \frac{1}{4}\lfloor\frac{p}{2}\rfloor\lfloor\frac{p-1}{2}\rfloor\lfloor\frac{p-2}{2}\rfloor\lfloor\frac{p-3}{2}\rfloor$.
(True for $1 \le p \le 10$.)

Ex. 3-4, Find $cr(K_8)$ and $cr(K_{4,4})$.