

Lecture 9

Nov. 27

NO.

- (*) An edge-colored graph is a graph G such that each edge is assigned a color. For convenience, we use C_G to denote the color set.
- (*) If $|C_G| = k$, then G is k -edge-colored.
- (*) If adjacent edges of G are assigned with ^{k} different colors, then we say G has a proper k -edge-coloring.
- (*) The minimum number k such that G has a proper k -edge-coloring is called the chromatic index of G , denoted by $\chi'(G)$.

Theorem If G is a simple graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.
(Vizing)

(Note) The study of edge-coloring (may not be proper) is an important topic in Graph Theory.

Defn. (Rainbow subgraph)

If G is edge-colored, then a rainbow subgraph (or hetero-chromatic) is a subgraph of G such that all its edges are assigned with different colors.

Defn. (Mono-chromatic subgraph)

All edges in the subgraph are of the same color.

(*) In a proper coloring, a mono-chromatic subgraph induces a "matching".

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Theorem 39 $ex(n; K_{\alpha, t}) \leq \frac{1}{2}(\alpha-1) \cdot n^{2-\frac{1}{t}} + \frac{1}{2}(\alpha-1)n$, i.e.,

$$ex(n; K_{\alpha, t}) \leq \frac{1}{2} z(n, n; \alpha, t).$$

Proof. Let G be an extremal graph such that $\|G\| = ex(n; K_{\alpha, t})$.
(1st)

Define a bipartite graph $H = (A, B)$ based on G . Let $V(G) = \{v_1, v_2, \dots, v_n\}$.

Let $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$ and $a_i \sim_H b_j$ if

and only if $v_i \sim_G v_j$, see Figure 29 for an example. Now,

clearly, $\|H\| = 2\|G\|$ and $a_i \not\sim_H b_i$ for $i = 1, 2, \dots, n$. Moreover,

if $G \not\cong K_{\alpha, t}$, then $H \not\cong K_{\alpha, t}$. (?) This concludes that
(bipartite).

$$\|G\| = \frac{1}{2}\|H\| \leq \frac{1}{2} z(n, n; \alpha, t).$$

1st

(2nd proof) Two-way counting

Consider the number of stars $K_{1, t}$. Since $G \not\cong K_{\alpha, t}$,

every set of t vertices of $V(G)$ has at most $\alpha-1$ centers

of stars whose pendant vertices are these vertices. So, there are

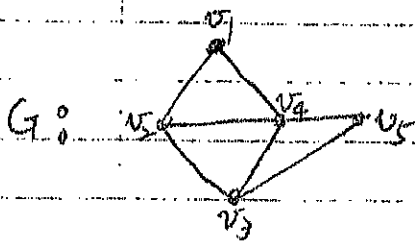
at most $(\alpha-1) \binom{n}{t}$ t -stars. The number of stars $K_{1, t}$ can be

obtained by $\sum_{i=1}^n \binom{d_i}{t}$ where $d_i = \deg_G(v_i)$. Now, let $m = \|G\|$.

$\sum_{i=1}^n d_i = 2m$. By a similar technique as Theorem 35, we conclude the proof

$$n \cdot \binom{2m/n}{t} \leq (\alpha-1) \binom{n}{t}. \quad (\text{in Lecture 8})$$

$$\left(\binom{7}{3} + \binom{5}{3} \right) \geq 2 \cdot \binom{6}{3}.$$



H:

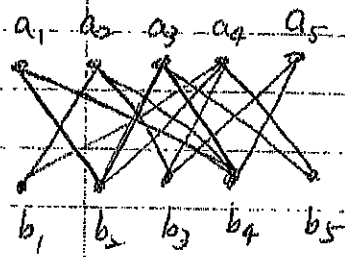


Figure 29. Bipartite version

Ramsey Theory

(*) The Ramsey number $R(s, t)$ is the smallest value "n" for which either a graph G of order n contains K_s or $\bar{G} \geq K_t$.

(*) Edge-coloring version of Ramsey number.

The Ramsey number $R(s, t)$ is the smallest value "n" for which in any 2-edge-colored K_n (red and blue), either there exists a red K_s or a blue K_t . (A red K_s is a complete graph of order s such that all its edges are colored red.)

(*) $R(3, 3) = 6$ (Do you know this fact?)

Theorem 40 The following statements are true.

(1) $R(a, 2) = a$ and $R(2, t) = t$,

(2) $R(a, t) = R(t, a)$,

(3) For $a \geq 2, t \geq 2$, $R(a, t) \leq R(a, t-1) + R(a-1, t)$; and

(4) $R(a, t) \leq \binom{a+t-2}{a-1} = \binom{a+t-2}{t-1}$.

如果等号成立, 则不含 K_a 在 G , 及不含 K_t 在 \bar{G} 的图(图)
 其数为 $R(a, t-1) + R(a-1, t) - 1$.
 当 $R(a, t-1)$ 及 $R(a-1, t)$ 皆为偶数时不成立(?)

and $R(a, t) \leq R(a, t-1) + R(a-1, t) - 1$ if both $R(a, t-1)$ and $R(a-1, t)$ are even.

Proof. (1) and (2) are easy to see.

Claim of (3).

Let $n = R(a, t-1) + R(a-1, t)$. Then, in K_n , each vertex is of degree $R(a, t-1) + R(a-1, t) - 1$. Therefore, if K_n is 2-edge-colored by red and blue, then the edges incident to a fixed vertex $x \in V(K_n)$ are either red edges or blue edges. By Pigeon-hole principle, either there are $R(a, t-1)$ blue edges or $R(a-1, t)$ red edges. If the first case holds, then in $\langle N_{K_n}(x) \rangle_{K_n}$ (a complete graph of order $R(a, t-1)$), either there exists a red K_a or blue K_{t-1} . Hence, we have a red K_a or a blue K_t in K_n . The other case can be obtained by a similar argument.

Claim of (4) By inductive argument. (Or induction)

$$R(n, t) \leq R(n, t-1) + R(n-1, t)$$

$$\begin{aligned} &\leq \binom{n+t-1-2}{n-1} + \binom{n-1+t-2}{t-1} = \binom{n+t-3}{n-1} + \binom{n+t-3}{n-2} \\ &= \binom{n+t-3+1}{n-1} = \binom{n+t-2}{n-1}. \end{aligned}$$

Theorem 41 (Erdős and Szekeres, 1935)

$$\text{For each } n \geq 2, R(n) \leq \frac{2^{2n-2}}{n^{1/2}}. \quad (R(n) =_{\text{def}} R(n, n))$$

Proof. $R(n, n) \leq \binom{2n-2}{n-1}$. We claim $\binom{2n-2}{n-1} \leq \frac{2^{2n-2}}{n^{1/2}}$ by induction on n . First, if $n=2$, $2 \leq \frac{4}{\sqrt{2}}$, the assertion is true.

Assume that the assertion is true for $n=k$, then $\binom{2k-2}{k-1} \leq \frac{2^{2k-2}}{k^{1/2}}$.

$$\begin{aligned} \text{Now, we calculate } \binom{2k}{k} &= \frac{(2k)!}{k!k!} = \frac{2k \cdot (2k-1) \cdot (2k-2)!}{k^2 (k-1)! \cdot (k-1)!} \\ &= \frac{2k(2k-1)}{k^2} \cdot \binom{2k-2}{k-1} \leq \frac{4k-2k}{k^2} \cdot \frac{2^{2k-2}}{k^{1/2}} = \frac{(4k-2)}{4k} \cdot \frac{2^{2k}}{k^{1/2}}. \end{aligned} \quad (1)$$

Since $\binom{2k}{k+1}^{1/2} \leq \frac{4k \cdot k^{1/2}}{4k-2}$, we conclude that $\binom{2k}{k} \leq \frac{2^{2k}}{(k+1)^{1/2}}$.

(*) The result has been there for almost 50 years before

the improvement due to Thomason in 1988:

$$R(n) \leq 2^{2/n}.$$

(**) The original proof by Ramsey shows that

$$R(k) \leq 2^{\binom{n-1}{k-1}} = \frac{2^{\binom{n-1}{k-1}}}{2}. \quad (1930)$$

Theorem 42 $R(k) \geq \lceil 2^{\lfloor \frac{k}{2} \rfloor} \rceil$. ($k \geq 3$)

Proof. (probabilistic method) (和率方法)

Consider a random red-blue coloring of the edges of K_n .

For a fixed set T of k vertices, let A_T be the event that

$\langle T \rangle_{K_n}$ is monochromatic. Hence $P(A_T) = \left(\frac{1}{2}\right)^{\binom{k}{2}} \cdot 2 = 2^{1-\binom{k}{2}}$
red or blue

Since there are $\binom{n}{k}$ possible sets for T , the probability that

at least one of A_T occurs is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$. Now, if

$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then no event A_T occurs is of positive

probability, i.e., there exists a coloring of edges such that

no monochromatic K_k occurs. Therefore, for such n , $R(k) > n$.

Let $n = \lfloor 2^{\lfloor \frac{k}{2} \rfloor} \rfloor$. (It suffices to show that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.)

$$\begin{aligned} \binom{n}{k} 2^{1-\binom{k}{2}} &< \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} & (1 - \binom{k}{2} = 1 - \frac{k^2}{2} + \frac{k}{2}) \\ &\leq \frac{\left(\frac{k}{2}\right)^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} & \leq \frac{2^{1+\frac{k}{2}}}{k!} < 1 \quad (k \geq 3). \end{aligned}$$

Hence, $R(k) \geq \lfloor 2^{\lfloor \frac{k}{2} \rfloor} \rfloor$. ▀

(*) Combining Theorems obtained above

$$2^{\frac{n}{2}} \leq R(n) \leq 2^{2n-3} \text{ for } n \geq 2.$$

(**) Open problem: $R(n) = 2^{(c+o(1))n}$, (c may be equal to 1).

Theorem 43 - Known results of $R(n, t)$.

$t \backslash n$	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	36-41	49-61	59-84	73-115
5			(43-48)	58-87	80-110	101-216	133-316
6				102-165	115-298	134-198	183-780
7					205-540	217-1031	252-1713
8						282-1870	329-3583
9							565-6589

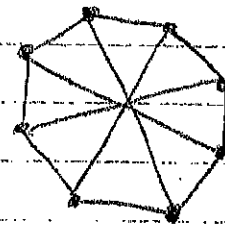
$$R(n, t) = R(t, n)$$

(*) The results of lower bounds are obtained by "a special edge-coloring" with two colors. Corresponding to the coloring we have G and \bar{G} of order (prescribed).

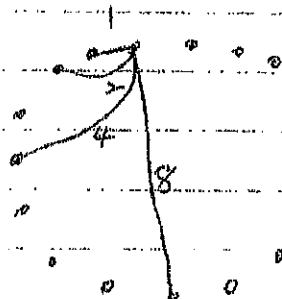
$R(3, 3) :$
(> 5)



$R(3, 4) :$
(> 8)



$R(4, 4)$
(> 17)



$G(17; \{1, 2, 4, 8\})$

problem Find as many vertices (n) as possible such that a

graph G of order n satisfying $G \not\cong K_5$ and $\bar{G} \not\cong K_5$.

(Try 43!)

Exercise (Bonus) Find a better lower bound for $R(5)$ and a better upper bound

(*) Ramsey number : Not limited to graphs! There are

versions on Algebra, Geometry and Number Theory.

Theorem 44 $R(p_1, p_2, \dots, p_t) \leq R(p_1-1, p_2, \dots, p_t) + R(p_1, p_2-1, p_3, \dots, p_t) + \dots + R(p_1, p_2, \dots, p_{t-1}) - t + 2.$

Proof. By a similar argument as the proof $R(a, t) \leq R(a-1, t) + R(a, t-1) - 1.$

(i) $R(3, 3, 3) \leq 6 + 6 + 6 - 3 + 2 = 17$ (Theorem 4.1).

(ii) There exists a 3-edge-coloring of K_{16} such that no mono-chromatic triangles occur.

Theorem 45 $R(\underbrace{3, 3, \dots, 3}_{k\text{-tuples}}) = \text{def } R_k(3) \leq \lfloor e \cdot k! \rfloor + 1.$

Proof. Since $R(3, 3) = 6$, $R(3, 3, 3) = 17$, the assertion is true

for $k=2$ and 3 . Assume that it holds for $k-1$ when $k \geq 3$.

Hence, $R_{k-1}(3) \leq \lfloor e(k-1)! \rfloor + 1$. By Theorem 4.4,

$$\begin{aligned} R_k(3) &\leq k(\lfloor e(k-1)! \rfloor + 1) - k + 2 && R_k(3) = R(\overbrace{3, 3, \dots, 3}^k) \\ & && \leq k(R(3, 3, \dots, 3, 2)) - k + 2 \\ & && = k R_{k-1}(3) - k + 2 \\ & && \leq k(\lfloor e(k-1)! \rfloor + 2) \leq \lfloor e \cdot k! \rfloor + 1 \end{aligned}$$

Now, $k \lfloor e(k-1)! \rfloor = k \lfloor (k-1)! \cdot (1 + \frac{1}{1} + \frac{1}{2!} + \dots + \frac{1}{(k-1)!} + \frac{1}{k!} + \dots) \rfloor =$

$= k \lfloor M + \frac{1}{k} + \frac{1}{(k+1)k} + \frac{1}{(k+2)(k+1)k} + \dots \rfloor = \dots$

(?) $\equiv \lfloor (k!) \cdot e \rfloor - 1.$

(*) Instead of $R(a, t)$, we use $R(H_1, H_2)$ to denote the smallest integer n such that any 2-edge-coloring of K_n , ~~is either~~
 (red, blue)
 there exists a red H_1 or a blue H_2 .

Theorem 4b $R(C_4, C_4) \stackrel{\text{def}}{=} R(C_4) \leq 8$.

Proof. Consider a graph $G \not\geq C_4$ and $|G| = 8$. By Theorem 35-1

$$|E(G)| \leq \frac{1}{2} f(8, 8; 2, 2) \leq \frac{n \cdot (1 + \sqrt{4n-3})}{4} = 2(1 + \sqrt{29}) < 14. \text{ That is,}$$

is
 if a graph G of order 8 and size 14, then $G \geq C_4$. Now,

in a 2-edge-colored K_8 , either red or blue edges induce
 such a graph, the proof follows. ■

Theorem 4c $R_k(C_4) \leq k^2 + k + 2$.

Proof. Let $n = k^2 + k + 2$ and consider a k -edge-colored K_n .

$$\text{Since } ex(n; C_4) \leq \frac{n}{4} (1 + \sqrt{4n-3}) = \frac{k^2 + k + 2}{4} \cdot (1 + \sqrt{4k^2 + 4k + 5})$$

$$\leq \frac{k^2 + k + 2}{2} \cdot \frac{1 + \sqrt{4k^2 + 4k + 5}}{2}$$

Now, compare $(k^2 + k + 1)$ and $\frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2}$. By direct calculation

we have $k^2 + k + 1 > \frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2}$. This implies that

$$k \cdot ex(n; C_4) \leq \frac{k^2 + k + 2}{2} \cdot \frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2} < \frac{(k^2 + k + 2)(k^2 + k + 1)}{2} = \binom{n}{2}. \quad \blacksquare$$

(找到至少一个颜色, 令其包含同色 C_4 !)

Def. (*) $R(H_1, H_2) = \min \{n \mid |G|=n, G \geq H_1, \text{ or } \bar{G} \geq H_2\}$.

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Theorem 48

For $l \geq 1$ and $p \geq 2$, $R(lK_2, K_p) = 2l + p - 2$.

Proof. Let O_n denote the graph of order n which contains no edges.
(Stable set)

Let H be a graph of order $2l + p - 3$ such that $H = O_{p-2} \cup K_{2l-1}$

(see Figure 30).

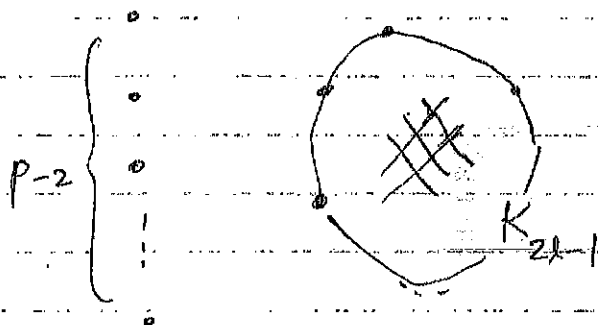


Figure 30.

So, it is not difficult to see that H does not contain a matching of size l . As a consequence, $\bar{H} = O_{2l-1} \vee_{(join)} K_{p-2}$ does not contain a K_p . Hence, $R(lK_2, K_p) \geq 2l + p - 2$.

On the other direction of inequality, let $n = 2l + p - 2$ and assume that a graph G of order n does not contain a matching of size l . Let $s < l$ be the maximum number of independent edges in G .

Now, consider \bar{G} . Since in G , the set of vertices not incident to these s edges induces a graph of order $n-s$ which has no edges, \bar{G} contains a complete graph of order $n-2s \geq n-2l-1 = p$. This concludes the proof. ■

Appendix (For your reference in writing exercises!)

Date

No

Use max-flow min-cut theorem to prove Hall's Theorem.

Proof. Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, $m \leq n$.

Claim: There exists a matching in $G = (X, Y)$ saturates X if and only if for each subset $A \subseteq X$, $|P(A)| \geq |A|$.

(Note) We shall prove the theorem for both (\Rightarrow) and (\Leftarrow) by using network argument.

First, we construct a network N by (1) adding s and t such that (s, x_i) and (y_j, t) are arcs in N , $1 \leq i \leq m$ and $1 \leq j \leq n$; (2) orienting $x_i y_j$ with $(x_i, y_j) \in E(G)$, and (3) $c(s, x_i) = c(y_j, t) = 1$ and $c(x_i, y_j) = M > |X| = m$. (See Figure below.)

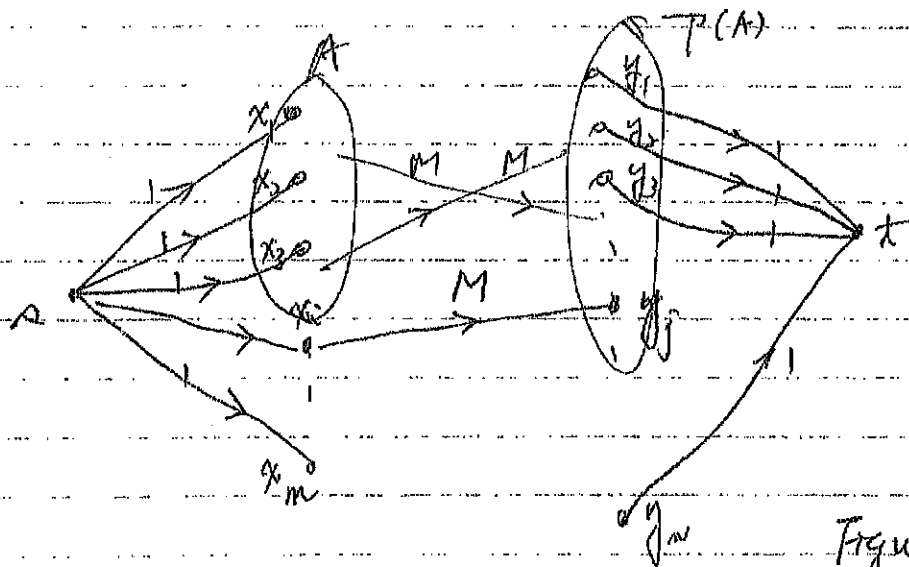


Figure 1.

Now, it is easy to see that a flow of value $|X| = m$ will provide a matching saturates X since each $y_j \in Y$ take at most one flow value through the flow.

(\Rightarrow) Suppose that there exists a subset A of X such that

$|A| > |P(A)|$. Let $S = \{s\} \cup A \cup T(A)$. Then, $\langle S, \bar{S} \rangle$ is a cut

with capacity $\frac{(|X| - |A|) + |P(A)|}{(s \rightarrow \bar{S}) \quad (S \rightarrow t)}$ since there are no arcs from the

vertices of A to the vertices of $Y \setminus T(A)$. This capacity of

cut is less than $|X|$, Hence, there exists no flow with value

$|X|$ and thus no matching saturates X .

(\Leftarrow) Assume that $|P(A)| \geq |A|$ for each $A \subseteq X$. It suffices to

claim that all cuts have capacity at least $|X|$. Let $S =$

$\{s\} \cup A \cup B$, see Figure 2. ($\underline{A} \subseteq X$ and $\underline{B} \subseteq Y$.)

They can be empty.

(B is not necessarily be $T(A)$.)

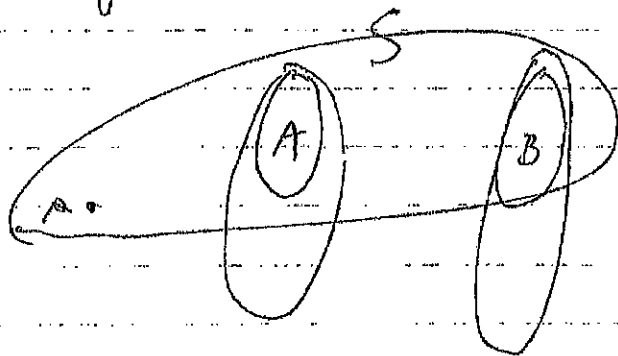


Figure 2.

Now, if there exists an arc from the vertices of A into $Y \setminus B$,

then $c(S, \bar{S}) \geq M > |X|$. On the other hand, if $P(A) \subseteq B$,

then $c(S, \bar{S}) = \underbrace{(|X| - |A|)}_{n \rightarrow \bar{S}} + |B| \geq |X| - |A| + |P(A)| \geq |X|$. Since

$c(S, \bar{S}) = |X|$ in the case $S = \{s\}$, we obtain a min-cut with

capacity $|X|$ and thus there exists a flow with maximum value

$|X|$. The proof follows. ■

Use max-flow min-cut theorem to prove Menger's Theorem.

Proof. First, we prove a directed version of Menger's Theorem.

(*) If s and t are distinct vertices of a digraph D such that $s \neq t$, then the maximum number of internally disjoint $u-v$ directed paths in D equals the minimum number of vertices in a $u-v$ separating set in D .

(**) For the undirected version, we replace each edge uv by a pair of arcs (u,v) and (v,u) .

Proof of (*).

Let \tilde{D} be the digraph obtained as follows:

(1) $\forall x \in V(D) \setminus \{s, t\}$, split x into two vertices x' and x'' ,

also let $(x', x'') \in A(\tilde{D})$;

(2) $\forall (x, y) \in A(D)$ such that $\{x, y\} \cap \{s, t\} = \emptyset$, replace (x, y)

with (x'', y') ;

(3) For $\begin{matrix} x \neq t, \\ (s, x) \in A(D), (x, s) \in A(D), \end{matrix}$ replace them with

(s, x') and (x'', s) respectively, and

(4) Replace (t, x) with (t, x') and (x, t) with (x'', t) if $x \neq s$.

As a consequence, we have a network defined on D by assigning each arc a capacity "1".

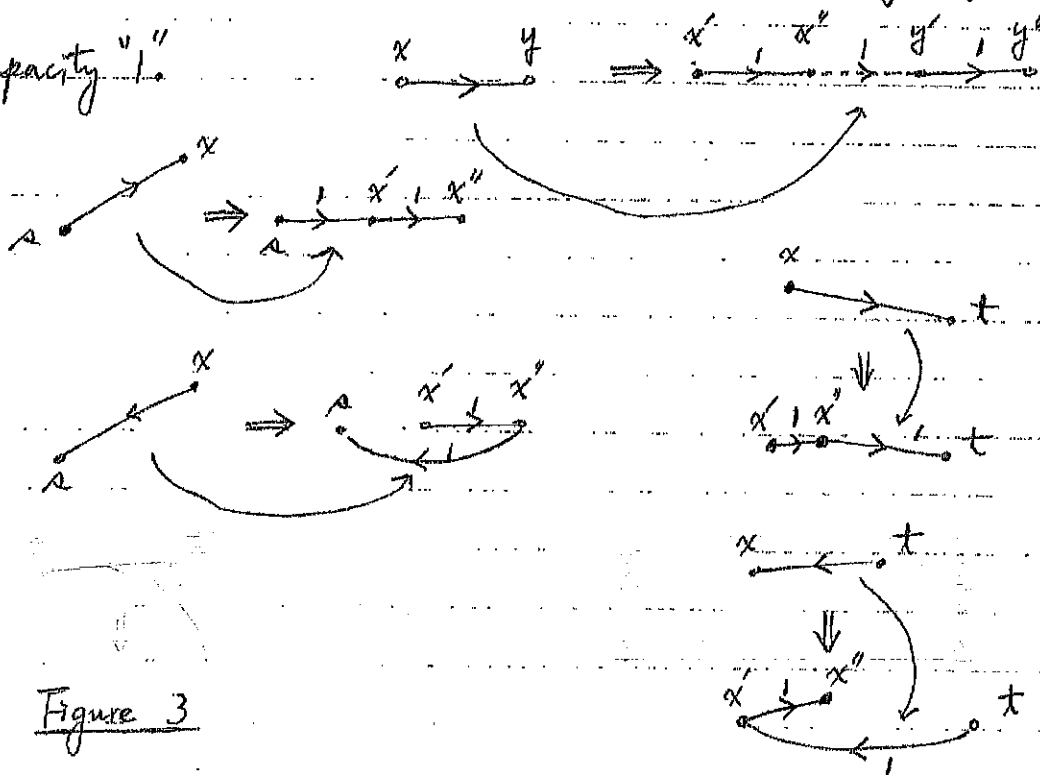


Figure 3

Let m be the maximum number of internally disjoint $s-t$ paths in D and n be the minimum number of vertices in an $s-t$ separating set in D .

Let A be an $u-v$ separating set of arcs in D and $|A| = n'$.

First, we observe that if (S, \bar{S}) is a cut, then $cap(S, \bar{S}) \leq n'$.

Here, S contains s if $(s, x') \in A$, S contains x if $(x', x'') \in A$ and $(x'', t) \in A$.

Moreover, $cap(S, \bar{S}) \geq n$. As a matter of fact, the min-cut is of capacity n and thus, N has a maximum flow, n . By the way,

The construction of network shows that a flow value 1 will give

a path from s to t . Since each arc has capacity one, these
(directed)

paths are internally disjoint. The proof follows. □