(○) A 1-factor F in a graph G is a 1-regular spanning subgraph of G. ($V(F)=V(G)$)

(○) An r-factor ($r \geq 1$) in a graph G is an r-regular spanning subgraph of G.

(*) The study of the existence of r-factors is called "Factor Theory".

(○) A 1-factorization of a graph G is a decomposition of G into 1-factors. Clearly, G must be a graph which is of even order and regular.

Latin squares

(*) Both K_{2n} and $K_{n,n}$ have 1-factorizations respectively.

(**) A cubic "planar" graph G has a 1-factorization, i.e.,
(可以畫在平面上使得所有边都不会互相跨越。)
G has three edge-disjoint 1-factors.

(**) To determine whether a cubic graph has a 1-factorization or not is a difficult problem.

(○) Petersen graph does not have a 1-factorization.

(•) A 1-factor of a graph G is a 1-regular spanning subgraph of G.

Theorem 30 (Tutte's 1-factor theorem)

A nontrivial graph G has a 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G-S$, $o(G-S) \le |S|$. (⊛ ← Tutte's condition)

Proof. ($\Rightarrow$) Assume that F is a 1-factor of G and there exists a proper subset W of $V(G)$ such that $o(G-W) > |W|$. Since an odd component H has an odd number of vertices, one of the vertices in H incident to F must be joining a vertex of W. But, we have more odd components than $|W|$. One of the vertices in W will be incident to at least two edges in F, a contradiction.

($\Leftarrow$) Since $o(G-\phi) \le 0$, G contains only even components. Hence, $|G|$ is even. Furthermore, if $|S|$ is odd (even), $o(G-S)$ must be odd (even). So, $|S|$ and $o(G-S)$ are of the same parity.

We shall prove the sufficiency by induction on $|G| = n$.

Clearly, if $n=2$, then $G \cong K_2$. Assume for all graphs H of even order less than n that if $o(H-W) \le |W|$ for every proper subset of $V(H)$, then H has a 1-factor. Let G be a graph of order n and $o(G-S) \le |S|$ for each proper subset S of $V(G)$.
(Tutte's condition holds)

We claim that G has a 1-factor.

Case 1. $\forall\ S \subsetneq V(G)$, $|S| \ge 2$ and $o(G-S) < |S|$. —— $(*')$
(扣一点不会有这种情形。)

The fact of parity shows that $o(G-S) \le |S|-2$ for all S.

Let $e = uv$ be an edge of G and consider $G' = G-\{u,v\}$. By the fact that: for each proper subset T of $V(G')$ ..., $o(G'-T) = o(G-\{u,v\}-T) = o(G-(\underbrace{\{u,v\} \cup T}_{S})) \le |T|+2-2 = |T|$ and induction hypothesis, $G-\{u,v\}$ has a 1-factor, so is G. (If $o(G-\{u,v\}-T) > |T| = |T \cup \{u,v\}|-2$, then $o(G-(\underbrace{\{u,v\} \cup T}_{S})) \ge |\underbrace{T \cup \{u,v\}}_{S}|$, a contradiction to $(*')$.)

Case 2. $\exists\ R \subseteq V(G)$, s.t. $o(G-R) = |R|$ where $1 \le |R| < n$.

Among all such R's, let S be a set of maximum cardinality $|S| = h$. Now, let $G_1, G_2, \cdots, G_h$ denote the odd components of $G-S$.

Next page

Note that those h odd components are the only components in $G-S$. For otherwise, let G_0 be an even component of $G-S$ and $v_0 \in V(G_0)$. Then, $o(G-S\cup\{v_0\}) \geq h+1 = |S\cup\{v_0\}|$. In fact, $o(G-S\cup\{v_0\}) = |S\cup\{v_0\}|$ by the assumption. Now, we have a larger "K" for S, a contradiction.

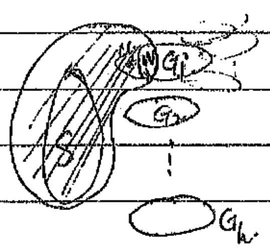
For $i=1,2,\cdots,h$, let S_i be the set of vertices in S which are adjacent with vertices in G_i. None of S_i's will be empty. For otherwise, G_i is an odd component of G and it is not possible. (G has only even components.)

Now, for $1 \leq k \leq h$, consider the union T of "any" k sets in $\{S_1, S_2, \cdots, S_h\}$. Suppose that $|T| < k$. Since $o(G-T)$ is at least k, $o(G-T) \geq k > |T|$ which violates the assumption $o(G-S) \leq |S|$. So, $\{S_1, S_2, \cdots, S_h\}$ has an SDR $(v_1, v_2, \cdots, v_h)$ where $v_i \in S_i$. Moreover, in G_i, let $u_i \sim_G v_i$.

For showing that G has a 1-factor, it's left to show that for each $i=1,2,\cdots,h$, G_i-u_i has a 1-factor.

Therefore, let W be a proper subset of $V(G_i-u_i)$ and we claim $o(G_i-u_i-W)\le|W|$. (This will imply the existence by induction.)

Suppose not. Let $o(G_i-u_i-W)>|W|$. Again, since $o(G_i-u_i-W)$ and $|W|$ are of the same parity, we have $o(G_i-u_i-W)\ge|W|+2$. Now, combining with S,



$$\begin{aligned} o(G-u_i-W-S) &= \underset{\ge\ |S|}{o(G-S)} + \underset{\ge\ |W|+2}{o(G_i-u_i-W)} - 1 \\ &\ge |S| + |W|+2-1 \\ &= |S|+|W|+1 = |\{u_i\}\cup W\cup S|. \end{aligned}$$

Hence, we conclude that $o(G-(\{u_i\}\cup W\cup S)) = |\{u_0\}\cup W\cup S|$.

Since $\{u_0\}\cup W\cup S$ is larger than S (in size), this contradicts to the choice of S. As a consequence, we have the fact: G_i-u_i contains a 1-factor and thus G has a 1-factor. ■

✓ Theorem 31 (Petersen's) Every 2-edge-connected (bridgeless) cubic graph G has a 1-factor F and $G-F$ is a 2-factor.

(is necessary!)

⇐ has no 1-factor!

Proof. Let $S \subseteq V(G)$ and consider an odd component C in $G-S$.
(Notice that if $o(G-S) = 0$, then $0 \le |S|$.) (done!) Since G is cubic,
the number of edges between S and C must be odd. (Otherwise,
(C中每一點的degree都是3, 所以 $3\cdot|C|$ 是奇數) S C → 奇數
the degree sum of $V(C)$ in $G-S$ is odd.) By the assumption that
G is 2-edge-connected, there are at least three edges in $\langle S, C\rangle$. (至少三边)
This implies that the total edges between S and $G-S$ is
at least $3\cdot o(G-S)$. By the fact that G is cubic, such edges
are at most $3\cdot|S|$ (由S連出去). Hence, $3\cdot o(G-S) \le 3\cdot|S|$. By Tutte's 1-factor
theorem, G has a 1-factor F and $G-F$ is clearly a 2-factor. ■

Ex. 2-6. Prove that Petersen graph can not be decomposed into three 1-factors and also does not contain a Hamilton cycle. (5 points)

Theorem 32 (Petersen's 2-factor theorem)

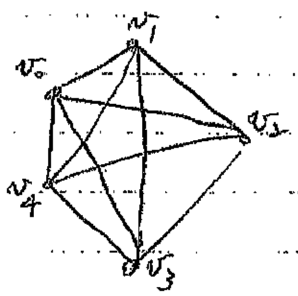
Let k be an even integer. Then, a k-regular graph contains $\frac{k}{2}$ edge-disjoint 2-factors.

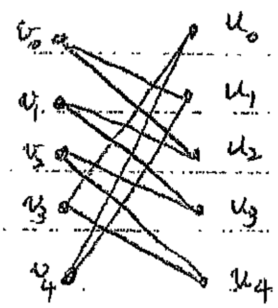
Proof. It suffices to consider a connected k-regular graph G.
Let $k = 2h$. By Euler's circuit theorem, G has an eulerian
circuit Z: $(v_0, v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_t, v_0)$.

Now, we defined a bipartite graph $\tilde{G} = (A, B)$, such that $|A| = |B| = |G|$, (See Figure 26.) and $v_i \underset{\tilde{G}}{\sim} u_j$ if v_i, v_j are two consecutive vertices in Z. Since G is $2k$-regular, $\tilde{G}$ is k-regular. By König's Theorem, $\tilde{G}$ contains k edge-disjoint perfect matchings. It is not difficult to see that a perfect matching in $\tilde{G}$ gives a 2-factor in G. This concludes the proof. ■

(*) Unfortunately we are not able to control the type of 2-factors we are going to obtain. (Bonus: $G(n;\{1,2\})$ contains any possible 2-factor with n vertices and a Hamilton cycle.)

(**) A perfect matching in $\tilde{G}$ can be represented as a permutation.

G: 

$\tilde{G}$: 

$((v_0, v_1, v_2, v_3, v_4, v_0, v_2, v_4, v_1, v_3))$

is an eulerian circuit of G.

<u>Ex. 3-1</u> For $n \geq 5$, <u>$G(n;\{1,2\})$</u> contains a Hamilton cycle and an arbitrarily given 2-factor. (5 points)

Lecture 8 – continued

For reference

- A graph F (or a class $\mathcal{F}$) is said to be forbidden in a class of graphs $\mathcal{G}$ if for each $G \in \mathcal{G}$, $G \not\supseteq F$ (or $G \not\supseteq F$ for each $F \in \mathcal{F}$).
- $ex(n; F) = \max\{\|G\| \mid G$ is a graph of order n such that $G \not\supseteq F\}$. $ex(n; \mathcal{F})$ can be defined accordingly.
- The graph G of order n with $\|G\| = ex(n; F)$ is called an extremal graph of order n with forbidden graph F.
- The class of bipartite graphs with partite sets of sizes m and n respectively is denoted by $\mathcal{G}_2(m,n)$.
- The extremal size of graphs in $\mathcal{G}_2(m,n)$ which do not contain $K_{s,t}$ is denoted by $z(m,n; s,t)$. (The notation is in honor of Zarankiewicz.)
- Notice that $ex(n; K_{s,t})$ is different from $z(m,n; s,t)$.
- $z(n,n; s,t) \geq 2\, ex(n; K_{s,t})$. (?)
- $T_r(n) =_{def} K_{\lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor, \dots, \lceil \frac{n}{r} \rceil}$ and $\|T_r(n)\| = t_r(n)$.

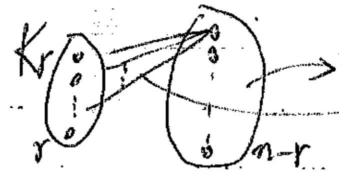
Date / /　　No.

Theorem 37. (Turán, 1941)

$ex(n; K_{r+1}) = t_r(n)$ and $T_r(n)$ is the unique extremal graph.

Proof. (1st) By induction on n. (To show $ex(n; K_{r+1}) = t_r(n)$.)

Since $T_r(n)$ does not contain K_{r+1}, $ex(n; K_{r+1}) \geq t_r(n)$ ($= \|T_r(n)\|$). We claim $ex(n; K_{r+1}) \leq t_r(n)$. Let G be a graph such that $G \not\supseteq K_{r+1}$ and G is of maximum size. Then, $G \supseteq K_r$. For otherwise, we may add more edges to G. Let $W \subseteq V(G)$ and $\langle W \rangle_G \cong K_r$. Let $U = V(G) \setminus W$.



Now, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + \|\langle U \rangle_G\|$. The term $(r-1)(n-r)$ comes from the fact that each vertex of U is incident to at most $r-1$ vertices of W. By induction hypothesis, $\|\langle U \rangle_G\| \leq t_r(n-r)$. Hence, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + t_r(n-r) = t_r(n)$. This is a direct consequence of adding one vertex of W to one partite set (not incident to one partite set) of $T_r(n-r)$ and $\lfloor \frac{n-r}{r} \rfloor + 1 = \lfloor \frac{n}{r} \rfloor$ ($\lceil \frac{n-r}{r} \rceil + 1 = \lceil \frac{n}{r} \rceil$). (?) 共有 r 个 partite set

Next, we claim the uniqueness. The proof is also by induction on n ($|G|$). Let $y \in V(G)$ such that $\deg_G(y) = \delta(G)$.

Clearly, $G-y$ does not contain K_{r+1} and thus $\|G-y\| = \|G\| - \delta(G) \geq t_r(n-1)$ by the proof of the first part. By induction, $T_r(n-1)$ is the unique graph which is isomorphic to $G-y$. This implies that in $G-y$ the smallest partite set is of size $\lfloor \frac{n-1}{r} \rfloor$. Since $T_r(n-1)$ contains a K_r from r-partite sets, y is incident to at most $r-1$ partite sets of $T_r(n-1)$. Therefore, y can be recognized as a vertex in one of the partite sets, and thus the number of edges between y and $G-y$ is $(n-1) - \lfloor \frac{n-1}{r} \rfloor = n - \lceil \frac{n}{r} \rceil$. This implies that $G \cong T_r(n)$. ▨

(2nd proof) (Zykov) Only $ex(n; K_{r+1}) = t_r(n)$.

Let $v_1 \in V(G)$ such that $\deg(v_1) = \Delta(G)$ and let $W = N(v_1)$. Let $G_1 = G - \langle N(v_1) \rangle_G + T_{r-1}(\Delta(G))$, and $U_1 = V(G_1) \setminus (W \cup \{v_1\})$. See Figure 26.

G_1 :

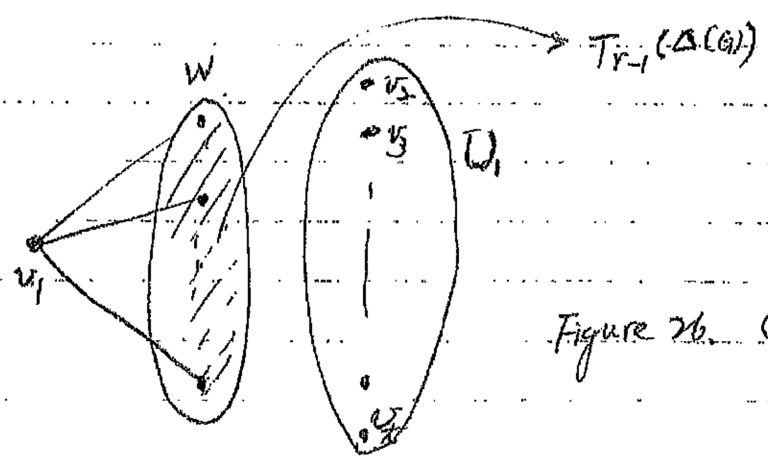


Figure 26. G_1

Date / / No.

If U_1 is an empty set, then we stop and evaluate $\|G_1\|$.

Otherwise, if $U_1 \neq \emptyset$, let $v_2 \in U_1$. Now, delete all edges E_2 in G_1 which are incident to v_2 and add v_2u for each $u \in W$ to $G_1 - E_2$ with $U_2 = V(G) \setminus (W \cup \{v_1, v_2\})$. The new graph is defined as G_2. Since $T_{r-1}(\Delta(G))$ defined on W does not contain K_r, G_2 does not contain K_{r+1}. By continuing this process (until U_t is empty), we shall obtain a complete r-partite graph H such that $\|H\| \geq \cdots \geq \|G_2\| \geq \|G_1\| \geq \|G\|$. (Notice that $\{v_1, v_2, \cdots, v_t\}$ is a new partite set.)

(3rd proof)

We can replace all the vertices of U_1 at the same time by deleting all the edges incident to U_1 and add $\langle W, U_1 \rangle$ to obtain a complete r-partite graph. ∎

(4th proof)

Theorem 3 (Erdős, 1970)

Let $G \not\supseteq K_{r+1}$. Then, there exists an H satisfying (1) H is an r-partite graph, (2) $V(H) = V(G)$, and (3) $\forall x \in V(G)$, $\deg_G(x) \leq \deg_H(x)$.

Moreover, if G is not a complete r-partite graph, then there exists a vertex $z \in V(G)$, s.t. $deg_G(z) < deg_H(z)$.

Proof. By induction on r for the whole statement, and $r=1$ is true. Let the assertion be true for $r' < r$.

Let $x \in V(G)$ s.t. $deg_G(x) = \Delta(G)$, $N(x) = W$ and $\langle W \rangle_G = G_0$. Clearly, $G_0 \not\supseteq K_r$. By induction, there exists an $(r-1)$-partite graph H_0, s.t., $V(H_0) = W$, $\forall\, y \in W$, $deg_{G_0}(y) \le deg_{H_0}(y)$, moreover, if G_0 is not a complete $(r-1)$-partite graph, then there exists a $y' \in W$, s.t. $deg_{G_0}(y') < deg_{H_0}(y')$.

Now, let $H = H_0 \underset{(join)}{\vee} (V \setminus W)$, see Figure 27. So, H is an r-partite graph. For $z \in V \setminus W$, $deg_G(z) \le \Delta(G) = |W| = deg_H(z)$, and if $z \in W$, $deg_G(z) \le deg_{G_0}(z) + n - |W| \le deg_{H_0}(z) + n - |W| = deg_H(z)$. This concludes the first part. For the second part,

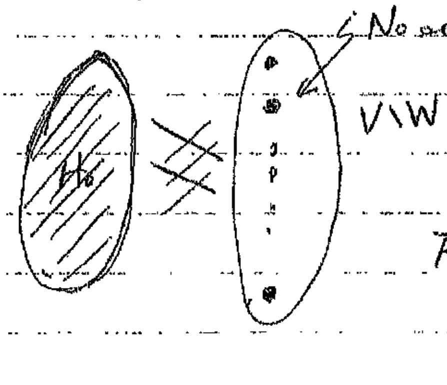


Figure 27

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Assume that $\deg_G(x) = \deg_H(x)$ for all $x \in V(G)$. Hence, $\|G\| = \|H\|$ and thus $\|G_0\| = \|H_0\|$. (For otherwise, $\|H\| > \|G\|$.) Moreover, $\deg_{G_0}(x) = \deg_{H_0}(x)$ for each $x \in W$. Suppose not. Let $\deg_{G_0}(x') < \deg_{H_0}(x')$ for some vertex $x' \in W$. This implies that $\deg_G(x') < \deg_H(x') = \deg_{H_0}(x) + n - |W|$, a contradiction. As a consequence, G_0 (H_0) is a complete $(r-1)$-partite graph and G is a complete r-partite graph as well. ∎

(∘∘) Try to estimate $z(m,n;s,t)$

Theorem 35 (Important Lemma)

Let $2 \le s \le m$, $2 \le t \le n$, $0 \le r \le m$, $z = km + r$ and $z = my$, $y \in \mathbb{R}^+$.

Let G be a bipartite graph, $G \in \mathcal{G}_2(m,n)$ and $G \not\supseteq K_{s,t}$. Then,

$$m \cdot \binom{y}{t} \le (m-r)\binom{k}{t} + r\binom{k+1}{t} \le (s-1)\binom{n}{t}.$$

(Remark. A function $f: \mathbb{R} \to \mathbb{R}$ is convex if $tf(x) + (1-t)f(y) \ge f(xt + y\cdot(1-t))$, $0 \le t \le 1$.)

Proof. Let $G = (A,B)$ where $|A| = m$ and $|B| = n$. Define a graph $H = (A, \binom{B}{t})$. $\binom{B}{t}$ is the collection of all t-subsets of B.

Date / / No.

And $x \sim_H T$ if and only if $x \sim_G y$ for each $y \in T$. Figure 28 is an example $|A| = 5, |B| = 6$ and $t = 3$.

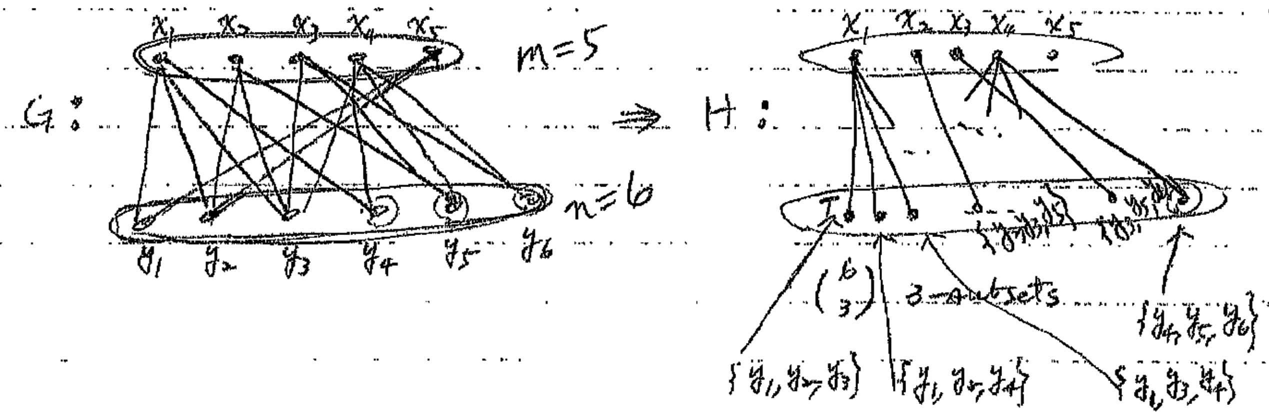

Figure 28 H induced by G

$\|H\| = \binom{4}{3} + \binom{3}{3} + \binom{3}{3} + \binom{4}{3} =$

Hence, we have

(1) $\|H\| = \sum_{x \in A} \binom{\deg_G(x)}{t}$, (For example, in Figure 28, $\|H\| = 10$.)

(2) $\forall\ T \in \binom{B}{t}$, $\deg_H(T) \le s-1$, $(G \not\supseteq K_{s,t})$

(3) $\|H\| \le (s-1)\binom{n}{t}$. (From (2).)

average degree

Now, since $z = km + r = my = \sum_{x \in A} \deg_G(x)$,

(*) $m\binom{y}{t} \le (m-r)\binom{k}{t} + r\binom{k+1}{t} \le \sum_{x \in A} \binom{\deg_G(x)}{t} \le (s-1)\binom{n}{t}$. ■

$\left(f(x) = \binom{x}{t} \right)$

(*) comes from the property of combination number. For example, $z = 16, k = 3, m = 5, r = 1$, and $t = 2$. Then, $\overset{16}{5 \cdot \binom{3.2}{2}} \le \overset{16}{4\binom{3}{2} + \binom{4}{2}} \le \overset{16}{\binom{a}{2} + \binom{b}{2} + \cdots + \binom{e}{2}}$.

Date / / No.

<u>Theorem 36</u> $z(m,n;s,t) \leq (s-1)^{\frac{1}{t}}(n-t+1)\cdot m^{1-\frac{1}{t}} + (t-1)m$.

Proof. By Theorem 35, $m\binom{y}{t} \leq (s-1)\binom{n}{t}$, $\frac{\binom{y}{t}}{\binom{n}{t}} \leq \frac{s-1}{m}$.

Hence, $\frac{y(y-1)\cdots(y-t+1)}{n(n-1)\cdots(n-t+1)} \leq \frac{s-1}{m}$.

By the fact $\frac{y-i}{n-i} \geq \frac{y-t+1}{n-t+1}$ for each $0 \leq i \leq t-1$,

we have $\left(\frac{y-t+1}{n-t+1}\right)^t \leq \frac{s-1}{m}$, i.e., $(y-t+1)^t \leq (s-1)\cdot(n-t+1)^t \cdot m^{-1}$.

This implies that $y-t+1 \leq (s-1)^{\frac{1}{t}}\cdot(n-t+1)\cdot m^{-\frac{1}{t}}$ and

$$y \leq (s-1)^{\frac{1}{t}}\cdot(n-t+1)\cdot m^{-\frac{1}{t}} + (t-1).$$

Hence, $z = my \leq (s-1)^{\frac{1}{t}}(n-t+1)\cdot m^{1-\frac{1}{t}} + (t-1)m$. ∎

<u>Theorem 37</u> $z(n,n;2,2) \leq \frac{n}{2}\left[1+(4n-3)^{\frac{1}{2}}\right]$.

Proof. By Theorem 35, $n\cdot\binom{y}{2} \leq \binom{n}{2}$.

Hence, $n\cdot y(y-1) \leq n(n-1)$ and we have $y^2 - y - (n-1) \leq 0$.

A direct calculation shows that $y \leq \frac{1+\sqrt{4n-3}}{2}$. This implies

that $z = m\cdot y = n\cdot y \leq \frac{n(1+\sqrt{4n-3})}{2}$. ∎

<u>Theorem 38</u> If $n = q^2+q+1$ and q is a prime power,

then $z(n,n;2,2) = \frac{n}{2}\left[1+(4n-3)^{\frac{1}{2}}\right]$. (Proof. By the existence of a projective plane of order q.)