Lecture 7, Oct. 30

Review

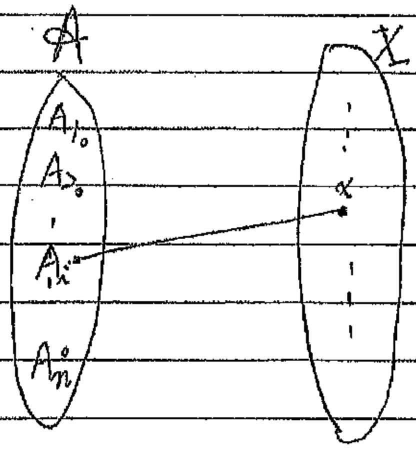
System of Distinct Representatives, SDR

Definition Let $\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$ be a collection of subsets of a given set X. Then, an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ is called an SDR of $\mathcal{A}$ if $a_i \in A_i$, $i = 1, 2, \cdots, n$ and all elements a_i's are distinct.

Hall's Theorem (1935)

$\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$ has an SDR if and only if for each $1 \le k \le n$, the union of any k subsets in $\mathcal{A}$ contains at least k distinct elements, i.e., $\left|\bigcup_{j=1}^{k} A_j\right| \ge k$. (Hall's condition)

We can use a bipartite graph to depict the above idea.



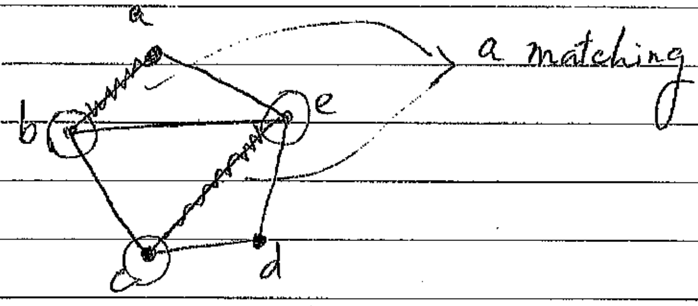
x is incident to A_i iff $x \in A_i$.

Figure 24 (Marriage problem)

(•) Vertex cover

A set of vertices S in a graph G is called a vertex cover of G if for each edge e in G, e is incident to a vertex of S.

G:



$\{b, c, e\}$ is a vertex cover of G.

(•) Matching

A set of independent (vertex disjoint) edges of G is called a matching of G.

(*) We are looking for vertex cover with minimum cardinality and (maximum size) matching of G, $\alpha(G)$ and $\alpha_1(G)$, respectively.

Theorem 27: A bipartite graph $G=(A,B)$ contains a matching saturates A if and only if for every $S \subseteq A$, $\Gamma(S) = \bigcup_{x \in S} N_G(x)$ contains at least $|S|$ elements of B, i.e., $|\Gamma(S)| \geq |S|$.

Proof. (1st) ($\Longrightarrow$) By the existence of a matching saturates A.

($\Longleftarrow$) By Theorem 24, it suffices to prove that there are $|A|$ vertex-disjoint A–B paths (and thus a matching saturates A). Suppose not.

See 7-2' Then, there exists a subset A_1 of A and a subset B_1 of B such that there is no edge between $A \setminus A_1$ and $B \setminus B_1$, see Figure 24, and $|A_1| + |B_1| < |A|$. (The number of A–B paths is less than $|A|$.)

Hence, there are no edges between $A \setminus A_1$ and $B \setminus B_1$, equivalently $\Gamma(A \setminus A_1) \subseteq B_1$. Then, $|\Gamma(A \setminus A_1)| \leq |B_1| < |A| - |A_1| = |A \setminus A_1|$. ■

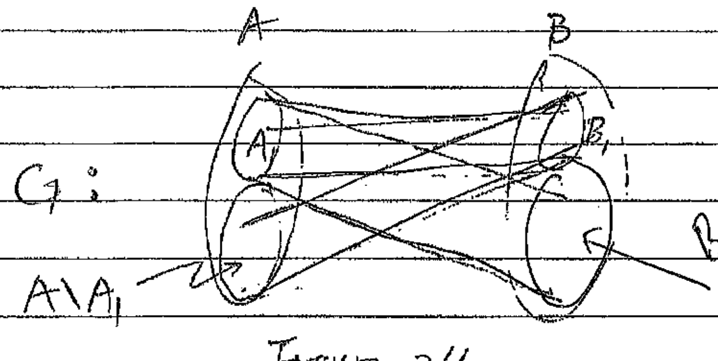


Figure 24.

(König's Thm, 1931) $A_1 \cup B_1$ is a vertex cover with minimum cardinality. (所以 $A\setminus A_1$, $B\setminus B_1$ 之间没有边!)

$\sigma(G) = \alpha_1(G)$.

(*) 這樣的 A, 與 B, 存在是因為以下的定理。

Theorem (König, 1931)

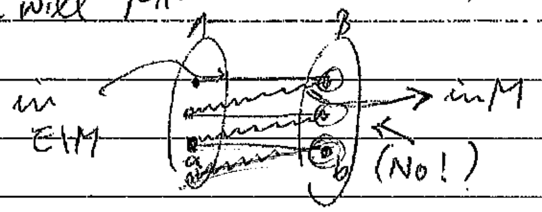
The maximum cardinality of a matching in $G=(A,B)$ is equal to the minimum cardinality of a vertex cover of its edges.

Proof. Let U be a vertex cover of G with minimum cardinality, $\sigma(G)$ and M be a matching with maximum cardinality $\alpha_1(G)$.

Clearly, $\alpha_1(G) \le \sigma(G)$ since we have to choose a (distinct) vertex to cover an (distinct) edge in M. On the other hand, we claim that U can be obtained by taking one vertex from each edge of M, i.e. U covers $E(G)$. The way to construct U is as follows: (from each edge of M)

① if some alternating path ends in B, choose that end vertex;

② Otherwise, its end in A. Since there exist no augmenting paths, the claim will follow. (10 min. later)



ends at a or ends at b. (augmenting path)

(○) A path in $G=(A,B)$ which starts in A at an unmatched vertex and then contains, alternatively, edges from $E(G)\setminus M$ and from M, is an alternating path with respect to M.

(*) 集合版本的形式，如 2nd proof.

2nd proof. By induction on $|A|$. Clearly, it's true for $|A|=1$.

First, if for each $S \subsetneq A$, $|\Gamma(S)| \geq |S|+1$, then let $a_1 \in A$ and $a_1b_1 \in E(G)$ where $b_1 \in N_G(a_1)$. Now, consider the bipartite graph $G' = (A_1\setminus\{a_1\}, B_1\setminus\{b_1\})$. Since for each $S' \subseteq A_1\setminus\{a_1\}$, $\Gamma(S') \geq |S'|$, there exists a matching saturates $A_1\setminus\{a_1\}$. Combining with a_1b_1, we have a matching needed.

Second, if there exists a proper subset S of A such that $|\Gamma(S)| = |S|$. By induction, we have a matching M_S saturates S. Now, consider $(A-S, B-T)$ where T is the set of vertices used in M_S. If there exists an $S' \subseteq A-S$ such that $|\Gamma(S')| < |S'|$, then $|\Gamma(S\cup S')| < |S\cup S'|$ a contradiction. Hence, the Hall's condition holds for the graph $(A-S, B-T)$. By induction, we have a matching saturates $A-S$. As a consequence, G has a matching saturates A.

3rd proof. (Rado)

Let $\underset{(A,B)}{\overset{\parallel}{G}}$ be a minimal (size) graph satisfying the condition.

(如果去掉邊可以滿足 Hall's Condition, 就不是 Minimal.)

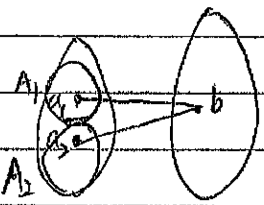
It suffices to claim that G contains $|A|$ independent edges.
(matching of size $|A|$.)

Suppose not. There exist two vertices a_1 and a_2 in A and b in B such that a_1b and a_2b are edges of G. Since both $G-a_1b$ and $G-a_2b$ violate Hall's condition, there exist two subsets A_1 and A_2 of A such that $|\Gamma(A_1)|=|A_1|$, $|\Gamma(A_2)|=|A_2|$ and a_i is the only vertex of A_i which is adjacent to b.
$(i=1,2)$

Hence, $|\Gamma(A_1)\cap\Gamma(A_2)| \geq |\Gamma(A_1-a_1)\cap\Gamma(A_2-a_2)|+1$
(a_1, a_2 除了 b 之外还有其它的共同邻居.)

$\geq |\Gamma(A_1\cap A_2)|+1 \geq |A_1\cap A_2|+1.$
(?) ($\forall x\in\Gamma(A_1\cap A_2)$, $x\in\Gamma(A_1-a_1)$ and $x\in\Gamma(A_2-a_2)$.)

On the other hand, $|\Gamma(A_1\cup A_2)| = |\Gamma(A_1)\cup\Gamma(A_2)|$



$= |\Gamma(A_1)|+|\Gamma(A_2)|-|\Gamma(A_1\cap A_2)|$

$\leq |A_1|+|A_2|-|A_1\cap A_2|-1$

$= |A_1\cup A_2|-1.$ $(\rightarrow\leftarrow)$. ∎

(0) Can you find another proof?

Ex. 2-5. Find at least three different ways to prove Hall's Theorem. (5 points)

(1) 以下為 Hall's Theorem 的應用。

Theorem 28 (König)

Every r-regular bipartite graph contains r edge-disjoint perfect matchings.

Proof. By induction on r. Clearly, it is true for $r=1$. Let $r \geq 2$. Let G be the r-regular bipartite graph where $G=(A,B)$. Then $|A|=|B|$. So, it suffices to find a matching saturates A. Now, for any subset S of A, $\Gamma(S) = \bigcup_{x \in S} N_G(x)$. If $|S|=k$, then S is incident to $k \cdot r$ edges. Since each vertex of B is of degree r, it takes at least k vertices of B to join with those $r \cdot k$ edges. This implies that $|\Gamma(S)| \geq |S|$. So, by Hall's Theorem, a matching saturates A can be obtained. Following the same process, we conclude the proof. ■

Theorem 29

Let $G=(A,B)$ be a bipartite graph such that for each $S \subseteq A$, $|\Gamma(S)| \geq |S|-d$, $d < |A|$. Then, G contains a matching with $|A|-d$ edges.

Proof. Clearly if $d=0$, then we have a matching with $|A|$ edges.
Now, let $d>0$ and $B'=B\cup D$ where $D=\{y_1,y_2,\cdots,y_d\}$ and $D\cap B=\phi$.
Let $G'=(A,B')$ such that $E(G')=E(G)\cup\{y_ia_j \mid i=1,2,\cdots,d;\ j=1,2,\cdots,|A|\}$.
(Join each vertex in D to every vertex of A.)

Now, for each $S\subseteq A$, $|\Gamma(S)|\geq S$ (in G'). Hence G' has a matching saturates A. This implies that G has a matching of size at least $|A|-d$. ∎

Remark. The following results can be obtained by applying Hall's Theorem.

1. A Latin rectangle can be extended to a Latin square.
2. An $n\times n$ matrix $A=(a_{ij})$ (non-negative) is said to be doubly stochastic if $\sum_{i=1}^{n} a_{ij}=1$ for every j and $\sum_{j=1}^{n} a_{ij}=1$ for every i. Then, there exist $\lambda_k\geq 0$, $\sum_{k=1}^{m}\lambda_k=1$, and permutation matrices $P_1,P_2,\cdots,P_m$ such that $A=\sum_{k=1}^{m}\lambda_k P_k$. (Ex. 2-3)
3. More ...