

Graph Theory Lecture 6 Oct. 23, 2014

6-0

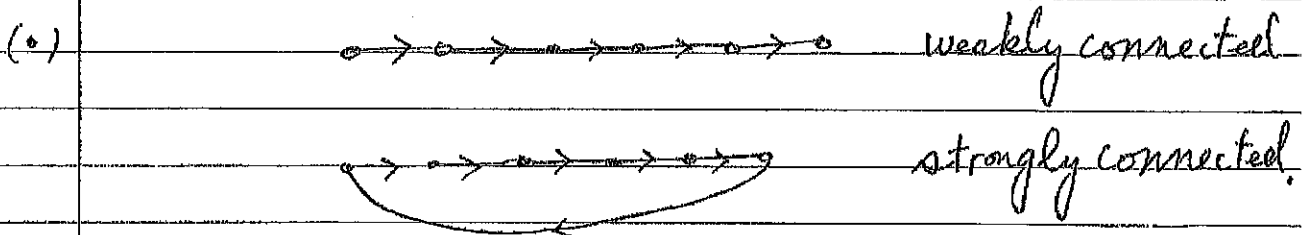
Review

(*) A directed graph D (digraph) is an ordered pair (V, A) where V is the set of vertices and A is a set of ordered pairs (u, v) where $u, v \in V$, (u, v) is called an arc, denoted also by \vec{uv} .

(*) Clearly, (u, v) is different from (v, u) .

Definition (Connectedness)

A digraph graph $D = (V, A)$ is strongly connected if for any two vertices u and v , there exists a directed path (dipath) from u to v . D is weakly connected if for any two vertices u and v , there exists either a dipath from u to v or a dipath from v to u .



(*) Tournaments are good examples to learn digraphs.
 T_n : Give an orientation to each edge of K_n .

Networks play an important role in applications.

Definition (Network)

A network N is a digraph D with two distinguished vertices s and t ; called the source and sink of N , respectively, and a non-negative integer-valued function c on $E(D)$. The digraph is the underlying digraph of N and the function c is the capacity function on N . For convenience, $c(\vec{a}) = c((x, y)) = c(x, y)$ for each arc $\vec{a} = (x, y)$, is the capacity of \vec{a} .

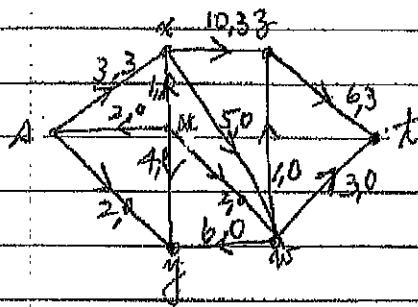


Figure 20. A network with capacity

$$N^+(x) = \{y \mid y \in V(D) \text{ and } (x, y) \in A(D)\}, \quad \deg^+(x) = |N^+(x)|$$

(outdegree of x)

$$N^-(x) = \{y \mid y \in V(D) \text{ and } (y, x) \in A(D)\}, \quad \deg^-(x) = |N^-(x)|$$

(indegree of x)

(*) A flow in a network N , f , is a function on $A(D)$, s.t.

$$(f: A(D) \rightarrow \mathbb{R}^+ \cup \{0\})$$

① $0 \leq f(\vec{a}) \leq c(\vec{a})$ for every $\vec{a} \in E(D)$, and
(capacity bound)

② $\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x)$ for every $x \in V(D) \setminus \{s, t\}$.
(Conservation law)

• The net flow into x is equal to $\sum_{y \in N^-(x)} f(y, x) - \sum_{y \in N^+(x)} f(x, y)$
which is zero except $x \in \{s, t\}$.

• $(V_1, V_2) = \{(x, y) \in E(D) \mid x \in V_1, \text{ and } y \in V_2\}$ (digraph version!)
↳ (Only arcs from V_1 to V_2)

(oo) A cut in N is $(X, V(D) \setminus X)$ such that $s \in X$ and $t \in V(D) \setminus X$.

Definition Let $K = (X, X')$ be a cut in N . Then, the capacity

of K , $\text{cap } K = c(X, X') = \sum_{(x, y) \in K} c(x, y)$.

e.g. In Figure 20, let $X = \{s, x, u, y\}$, then $c(X, X') = 17$.

(*) Definition The value of a flow f in N is defined as the

net value flow out the source and therefore, the net value

flow into the sink. (Denoted by $\text{val } f$)

Theorem 23 Let f be a flow in a network N and K

$= (X, X')$ be a cut in N . Then, $\text{val } f \leq \text{cap } K$.

Proof. Note that $\text{val } f = f(A, \{A\}') - f(\{A\}', A)$ and

$$f(x, \{x\}') - f(\{x\}', x) = 0, \quad \forall x \in X \setminus \{A\}. \quad (A \in X, A \notin X')$$

This implies that $\sum_{x \in X} [f(x, \{x\}') - f(\{x\}', x)] = \text{val } f.$

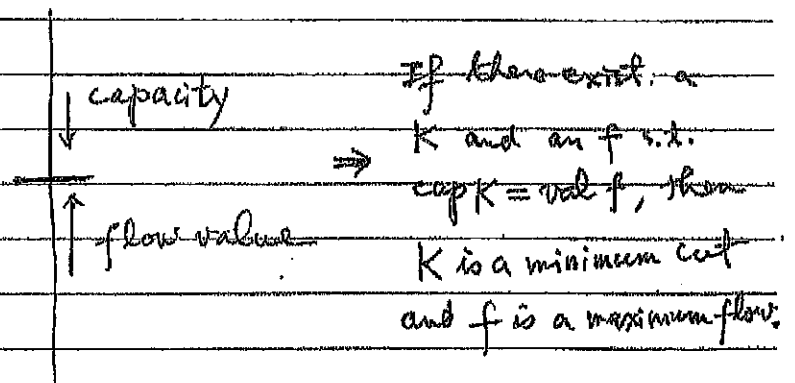
$$\parallel \\ f(X, X') - f(X', X) \quad (\text{Calculation}).$$

Now, $f(X, X') \leq \text{cap}(X, X')$ and $f(X', X) \geq 0$. Hence,

$$\text{val } f \leq \text{cap } K.$$

• A minimum cut is a cut K in N such that for every cut K' in N , $\text{cap } K \leq \text{cap } K'$.

• A maximum flow is a flow f in N such that for every flow f' in N , $\text{val } f \geq \text{val } f'$.



min-max problem

Theorem 24 (Ford and Fulkerson, 1956-1962)

N defined on D with source s and sink t

In any network, the value of a maximum flow equals the

the capacity of a minimum cut.

Proof. Clearly, if there exist no cuts K such that its capacity of

the cut is $\text{val } f$, then f does not exist. (Theorem 21) So, it

is sufficient to claim that if the value of a maximum flow f is v

then there exists a cut K such that $\text{cap } K = \text{val } f = v$.

(*) Let f be a maximum flow of N .

↳ Define a subset $S \subseteq V(D)$ recursively as follows. Let $s \in S$.

If $x \in S$, and $c(x, y) > f(x, y)$ or $f(y, x) > 0$, then let $y \in S$.

We shall prove that (S, S') is a cut with capacity v .

First, we claim $t \notin S$. Suppose not, i.e., $t \in S$. Hence, we can

find a sequence of vertices in N such that $s = x_0, x_1, \dots, x_l = t$.

Moreover, if we let $\varepsilon_i = \max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\}$,

$i = 0, 1, \dots, l-1$, then $\varepsilon_i > 0$. Let $\varepsilon = \min\{\varepsilon_i\}$. Now, let

$f^*(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \varepsilon$ if $c(x_i, x_{i+1}) - f(x_i, x_{i+1}) = \varepsilon_i > 0$ and

$f^*(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \epsilon$ if $f(x_{i+1}, x_i) = \epsilon_i > 0$. As a consequence, f^* is a flow from s into t with value $\text{val } f^* = v + \epsilon$, a contradiction.

(See Figure 21) By the definition of a flow,

$$\text{val } f = v = \sum_{x \in S, y \in S'} f(x, y) - \sum_{x \in S', y \in S} f(x, y). \quad (1)$$

Again, by the definition of S , if $x \in S$ and $y \in S'$, then

$c(x, y) = f(x, y)$ and $f(y, x) = 0$. This implies that (1) = $\sum_{x \in S, y \in S'} c(x, y)$,

the proof follows. $v = \text{cap}(S, S')$ \square

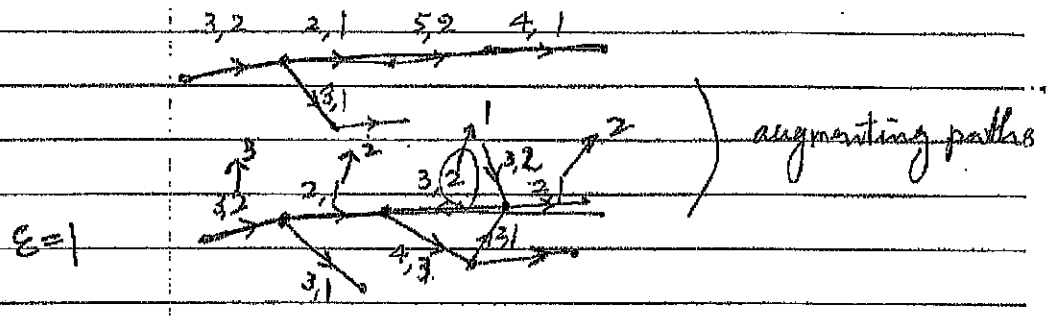


Figure 21 augmenting flow f^*

Theorem 25 (Menger, 1927)

Let s and t be two nonadjacent vertices of a graph G . Then, the minimal number of vertices separating s from t is equal to the maximal number of vertex-disjoint $s-t$ paths. (Internally)

Proof (1st)

Let the number of vertices separating s and t be k . Then, it is easy to see that there are at most k independent paths connecting s and t , (vertex-disjoint)

Also, if $k=1$, then we have a path joining s and t .

Now, suppose the assertion is not true, i.e., we can find less than k independent $s-t$ paths for certain k . Now, take the minimal

k in which we have a counterexample. Then, among all such examples, let G be the one with minimum size (number of edges)

First, we notice that s and t have no common neighbors. For otherwise, let sx and xt be edges of G . Then, $G-x$ will be a counterexample for " $k-1$ " (smaller than k).

Let W be a separating set of s and t and $|W|=k$. Suppose, neither $N_G(s)=W$ nor $N_G(t)=W$. (Figure 2.2-1)

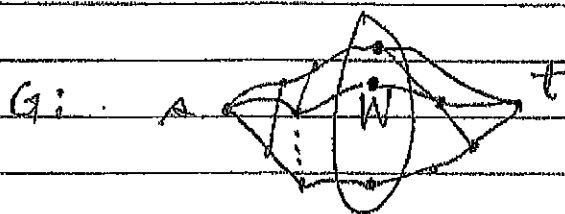


Figure 2.2-1.

Let G_2 be obtained by deleting all the vertices to the left of G in Figure 22-1 and adding a replacing s' with edges joining to W , see Figure 22-2. Now, G_2 has fewer edges than G and thus there are k independent s' - t paths. Hence, we have k W - t independent paths. With the same technique, we derive k s - W independent paths (by changing s to t).

(新圖不再是反例)

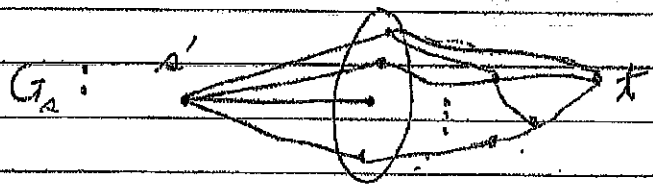


Figure 22-2.

② So, as a conclusion, either s or t must have their neighbors

W . Let $N_p(s) = W$ and $\langle s, x_1, x_2, \dots, x_k, t \rangle$ be a shortest s - t path.

Then, $k \geq 2$. Consider $G - x_1, x_2$.

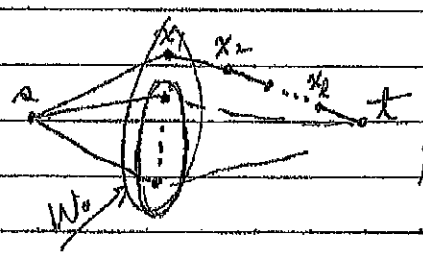


Figure 22-3.

In $G - x_1 x_2$, there exists an $s-t$ separating set W_0 of size $k-1$. Then, both $W_1 = W_0 \cup \{x_1\}$ and $W_2 = W_0 \cup \{x_2\}$ are $s-t$ separating sets of G . $\textcircled{4}$

By the fact that P is a shortest $s-t$ path, s is not adjacent to x_2 and t is not adjacent to x_1 . This implies that $N_G(s) = W_1$ since t is not adjacent to a vertex of the separating set W_1 .

Similarly, $N_G(t) = W_2$. Hence, $N_G(s) \cap N_G(t) = W_0$, a contradiction. $\textcircled{5}$

$$|W_0| = k-1 \geq 1$$

(s and t have common neighbors.)
 $|W_0| = k-1 > 0$

← Ex. 2-4. (请参考其它图论课本.)

Find a second proof of Menger's Theorem (5 points)
 (不同于上述的证明)

(*) The "Edge" version of Menger's theorem can be stated as follows:

Let s and t be two vertices of G . Then, the minimal number of edges separating s from t is equal to the maximal number of edge-disjoint $s-t$ paths. (We can prove this part by using Theorem 2-4. Replace each edge xy of G by (x,y) and (y,x) and assign capacity "1" to each arcs.)

Theorem 26 (Set version of Menger's Theorem)

If S and T are arbitrary subsets of $V(G)$, then the maximal number of vertex-disjoint (including endvertices) $S-T$ paths is $\min\{|W| \mid W \subseteq V(G) \text{ and } G-W \text{ has no } S-T \text{ paths}\}$.

Proof. By adding to new vertices s and t as in Figure 23, we have a new graph \tilde{G} . Now, by Menger's Theorem, the maximal number of $S-T$ paths is the same as that of $s-t$ paths in \tilde{G} . Hence, we have the proof. ■

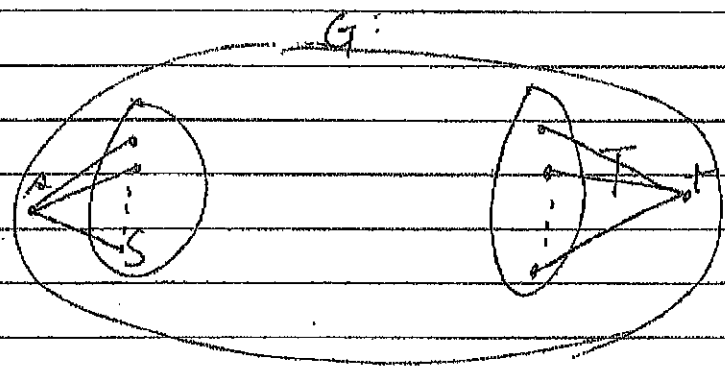
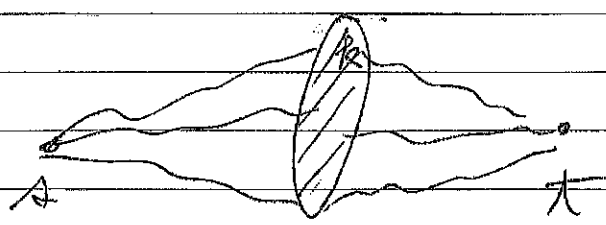


Figure 23, graph \tilde{G}

講義中 Menger's Theorem 的證明概念說明

NO.

1. 如果 ^{A, t 的} 最小分隔集 (Separating Set) 的邊數是 k ; A 的獨立路徑數為 h , 則 $h \leq k$. 顯然當 A, t 的獨立路徑是 h 時, 就至少要拿走 h 個邊才能分隔 A 與 t .
2. 所以主要的證明內容是當最小分隔集有 k 個邊時, 一定可以找到 k 條獨立 $A-t$ 的路徑.



3. 假設當 $k \geq 2$ 時, 這個敘述不成立. ($k=1$ 時必成立)

令 G 為所有反例中具有最少的邊數, 而且 G 的最小分隔集有 k 個邊, 但是能夠找到的 $A-t$ 獨立路徑數小於 k .
(最小的 k)

Claim: 這樣的 G 具有以下的性質.

① $N_G(A) \cap N_G(t) = \emptyset$.

如果 $x \in N_G(A) \cap N_G(t)$, 則 $G-x$ 為 " $k-1$ " 狀況下的反例, 此為不可能.

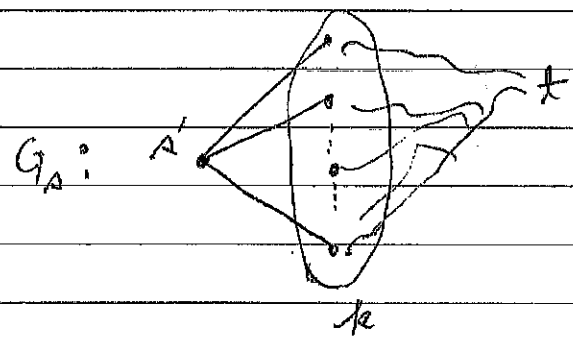
② 令 W 为 s, t 分离集而且 $|W| = k$.

Claim: Either $N_G(s) = W$ or $N_G(t) = W$.

(不妨 W 与 s 相邻就是 W 与 t 相邻.)

假如 $N_G(s) \neq W$ 且 $N_G(t) \neq W$.

令 G_s 为下图 (把 $s-W$ paths 用边取代)



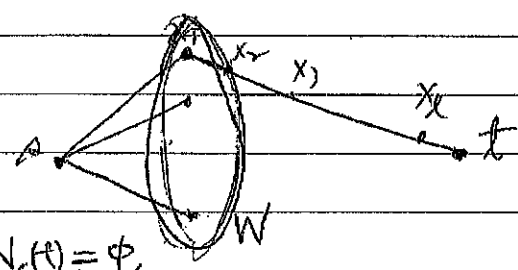
显然 $\|G_s\| < \|G\|$, 因此 G_s 不再是反例; G_s 中可以找

到 k 条独立路径; 再把 G_s 换成 G_t , 可以找到 k 条

$s-t$ 独立路径, 于是 G 就可以找到 k 条独立路径. $\rightarrow \leftarrow$

③ 因此, $N_G(s) = W$ or $N_G(t) = W$. Let $N_G(s) = W$. 令 $P = \langle s, x_1, x_2, \dots, t \rangle$

为一条 $s-t$ 的
最短路径.



(因为 $N_G(s) \cap N_G(t) = \emptyset$,
 $k \geq 2$.)

④ 現在考慮圖 $G - x_1 x_2$ 。 ($G - x_1 x_2$ 不再是反橋。)

因此, $G - x_1 x_2$ 的分隔集會變小; 令 W_0 為 $G - x_1 x_2$ 的一個 k 分隔集, $|W_0| = k - 1$ 。這也表示 $W_0 \cup \{x_1\}$, $W_0 \cup \{x_2\}$ 皆為大小是 k 的 k 分隔集 (G 中)。

由於 p 是最短路徑, $x_1 x_2 \in E(G)$ 而 $x_1 x_2 \notin E(G)$ 。

又因為 $x_1 \in E(G)$, $N_G(x_1) = W_1$, 同理 $N_G(x_2) = W_2$ 。

於是 $N_G(x_1) \cap N_G(x_2) = W_0 \neq \emptyset \rightarrow \leftarrow |W_0| = k - 1 > 0$ 。

