Graph Theory Lecture 5 (18-22)

- A maximal connected subgraph without a cutvertex is called a "block".

  ($\Downarrow$ increase the # of components)

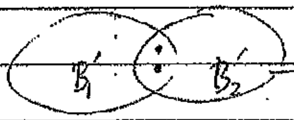
(**) Every block of a graph G is either a maximal 2-connected subgraph or a bridge (with its endvertices), or an isolated vertex. (See Figure 16.)

(其它書的定義可能不同)

- A block graph of G is a bipartite graph $bc(G) = (A, B)$ where A is the set of cutvertices of G and B is the set of blocks, and $a \in A$ is incident to $B_i \in B$ if $a \in V(B_i)$.

Theorem 18. The block graph of a connected graph is a tree.

Proof. Observe that each block is an induced subgraph and any two blocks have at most one cutvertex in common.



$\to$ A larger graph (2-connected)

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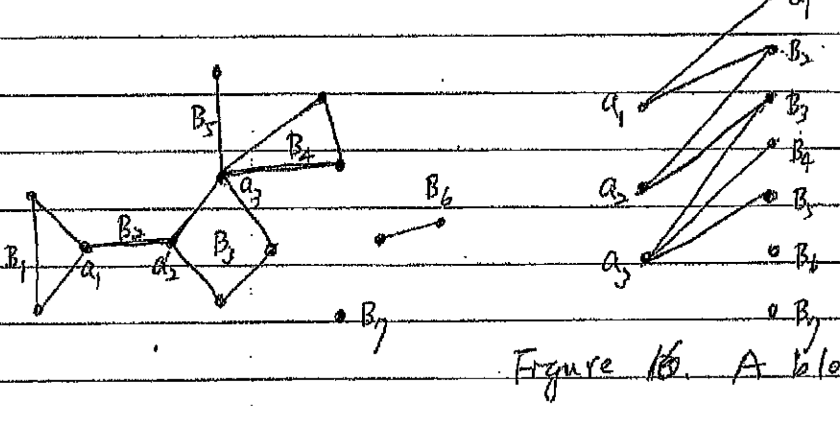


Figure 16. A block graph

Since G is connected, if G has only one block and thus contains no cutvertex, $bc(G)$ is a single vertex. The proof is trivial. Assume that G contains more than one block. Then, each vertex is either in a block or a cutvertex itself. Now, consider a cutvertex v, and a block B_i. Let $u \in B_i \setminus \{v\}$. ($B_i \setminus \{v\}$ is non-empty since G is connected.) Then, we have a path P connecting v and u. Clearly, v is going to connect to a block which contains some vertices of P. If this block B_i, then vB_i is an edge of $bc(G)$, done. Otherwise, the path will contain a cutvertex following the block and travels to another block and finally to B_i. The other two cases, cutvertex to cutvertex and block to block can be verified similarly.

Now, for the acyclic part, a cycle in $bc(G)$ will produce a cycle in G which passes all cutvertices involved. But, in that case, none of these cutvertices are cutvertices anymore, a contradiction. This completes the proof. ∎

Theorem 19 A graph G is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed. (Ear construction)

Proof ($\Leftarrow$) From the construction, it is clear that G contains no cutvertices. Hence, G is 2-connected.

($\Rightarrow$) Assume that G is 2-connected and H is a maximum (size) subgraph following the construction. This is possible, since G contains a cycle. In fact, H is an induced subgraph, since for any two vertices x and y in $V(H)$ and $xy \in E(G) \setminus E(H)$, we have an H-path and a larger subgraph will be obtained.

Now, assume that $H \neq G$. $\exists\, v \in V(G) \setminus V(H)$; $w \in V(H)$ and $vw \in E(G) \setminus E(H)$. Since G is 2-connected, $G-w$ contains a v–H path P. (See Figure 17) This implies that $\langle w, v, P \rangle$ is an H-path. Hence, a larger subgraph is obtained, a contradiction. ∎

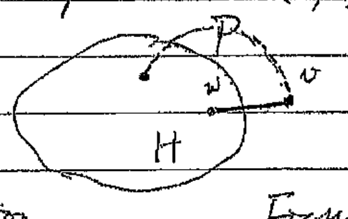


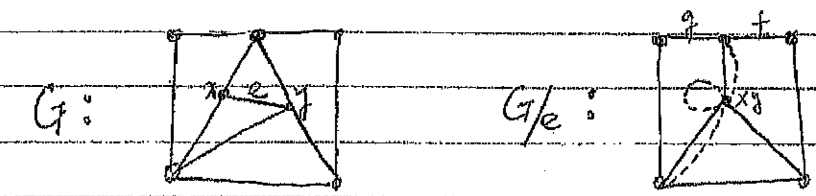
Figure 17.
An extra H-path

Definition (Graph minors)

A graph M is called a minor of G if M can be obtained from G by contracting edges, deleting vertices and edges.

Review (Edge-contraction)

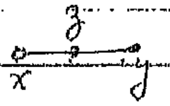
Given an edge $\underset{=}{xy}^{e}$ of a graph G, the graph G/e is obtained from G by contracting e; that is to identify the vertices x and y and deleting resulting loops and duplicate edges.



Example K_4 is a minor of the above G.
(Contracting e, f and g.)

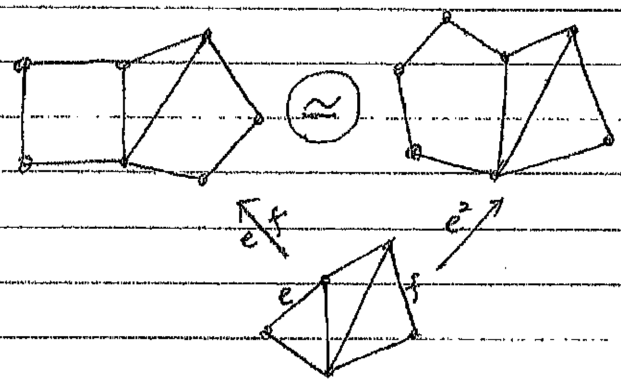
Definition (Subdivision)

A subdivision of an edge xy is obtained by adding a new vertex z such that we have edges xz and zy.



<u>Definition</u> (Homeomorphic Graphs)

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

<u>Remark</u>: Two cycles are homeomorphic.

(*) The reverse of subdivision can be "<u>considered</u>" as <u>contraction</u>.

P_4 $\qquad\qquad$ $P_4/e \rightsquigarrow P_3$

Theorem 20 If G is 3-connected and $|G| \geq 4$, then G has an edge such that G/e is again 3-connected.

Proof. Suppose not. Then, for each $xy \in E(G)$, the graph $G/_{xy}$ contains a separator S with $|S| \leq 2$. ($G-S$ is disconnected.) Since $\kappa(G) \geq 3$, the contracted vertex $v_{xy} \in S$ and $|S| = 2$, i.e., $\exists\, z \in V(G)$, s.t. $S = \{v_{xy}, z\}$. Therefore $T = \{x, y, z\}$ is a separator set of G. Since no proper subset of T can separate G, each vertex of T is incident to every component of $G-S$. (See Figure 18)

Among all edges of G, we choose an edge xy and its corresponding vertex z such that the component C has minimum size. $(*)$ Let $v \in V(C)$ and $vz \in E(G)$. By assumption, $G/_{vz}$ is again not 3-connected and there exists a corresponding vertex w such that $\{v, z, w\}$ separates G.

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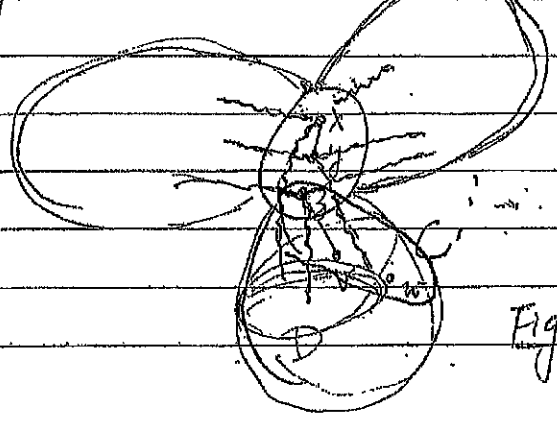

Figure 18 3-connected graph

Moreover, each vertex of $\{v,z,w\}$ is incident to every component of $G-\{v,z,w\}$. Since $xy \in E(G)$, $G-\{v,z,w\}$ has a component D s.t. $D \cap \{x,y\} = \phi$. By the fact $v \in V(C)$, the neighbor of v in D is also in C. Hence, $D \cap C \neq \phi$. This implies that D is a proper subset of C, i.e., $|D| < |C|$, a contradiction to the choice of C. ($|C|$ is minimum.) ∎

Theorem 21. (Tutte, 1961)

A graph G is 3-connected if and only if there exists a sequence $G_0, G_1, \cdots, G_n$ of graphs satisfying:

(a) $G_0 = K_4$ and $G_n = G$; and

(b) G_{i+1} has an edge xy (with degree of x, y at least 3) such that $G_i = G_{i+1}/xy$, $1 \le i < n$.

Proof. ($\Rightarrow$) By Theorem 20, we start with G as G_n and end at $K_4 = G_0$.

($\Leftarrow$) Let $G_0, G_1, \cdots, G_n$ be a sequence of graphs satisfying (a) and (b). It suffices to show that if $G_i = G_{i+1}/xy$ is 3-connected, then G_{i+1} is also 3-connected, for all $1 \le i < n$.

Suppose not. Let S be a separator with $|S| \le 2$. Also, let C_1 and C_2 be two components of $G_{i+1} - S$. Since $xy \in E(G_{i+1})$, let $\{x, y\} \cap V(C_1) = \phi$. (Figure 19) Now, if $\{x, y\} \subseteq C_2$, then $G_i - S$ is disconnected, a contradiction. Hence, at most one of $\{x, y\}$ is in C_2, either x or y, but not both. Furthermore, if $v \notin \{x, y\}$ and $v \in V(C_2)$, then $G_i - S$ is also disconnected, a contradiction to the fact G_i is 3-connected. Hence, C_2 contains exactly one vertex of degree at most 2, $\rightarrow\leftarrow$. ∎

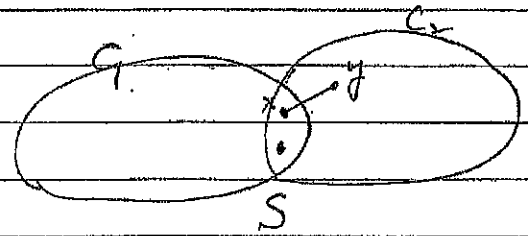

Figure 19, G_{i+1}

(Review)
Theorem 22 Every non-trivial graph G contains at least two vertices which are not cutvertices.

Proof. Review that if v is a cutvertex of G, then the number of components of G, $c(G)$ is smaller than that of $c(G-v)$.

Now, consider u and v such that $d(u,v) = diam(G)$. We show both u and v are not cutvertices. Suppose not. Let u be a cutvertex, then $G-u$ is disconnected. Let w be a vertex which is in a component different from v belongs. Since u is a cutvertex and v, w are in different components, all $v-w$ paths must pass through u. This implies that $d(v,w) > d(v,u) = diam(G)$. Hence, u can not be a cutvertex. Similarly v is not a cutvertex either. ∎

(*) Determine whether a graph G contains a minor H is considered as the most important problem in the study of graph structure.

Ex. 2-3. Prove that in an n-connected graph G, any n distinct vertices are contained in a cycle of G.
($n=3$ for 2 points and $n \in \mathbb{N}$ for 3 points.)