Theorem 8

Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$ (provided $\delta(G) \geq 2$).

Proof. Let $\langle v_0, v_1, \cdots, v_\ell \rangle$ be a longest path. Then, $N_G(v_\ell) \subseteq \{v_0, v_1, \cdots, v_{\ell-1}\}$. For otherwise, we have a longer path. Since v_ℓ has at least $\delta(G)$ neighbors $\ell \geq \deg_G(v_\ell) \geq \delta(G)$. This concludes the first part. Now, let i be the smallest index in $\{0, 1, 2, \cdots, \ell-1\}$ such that $v_i v_\ell \in E(G)$. Hence, $(v_i, v_{i+1}, \cdots, v_\ell)$ is a cycle of length at least $\delta(G)+1$. ∎

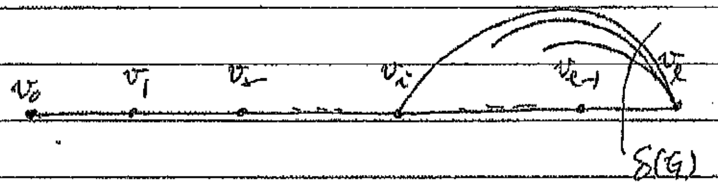


Figure 1. $\ell - i \geq \delta(G)$.

(*) Bonus Every connected graph G contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$. Moreover, if $\delta(G) \geq \frac{|G|}{2}$, then G contains a Hamilton cycle.

(o) A Hamilton cycle of G is a cycle of G of length $|G|$.

(A cycle passes every vertex of G exactly once.)

<u>Theorem 8.5</u> Let $\mathcal{C}$ be the collection of cycles. Then, $ex(n;\mathcal{C}) \le n-1$. The equality holds for trees of order n.

<u>Proof.</u> By induction on n and it's true for $n=1$ and 2. Let the assertion be true for $n=k\ge 2$. Consider a graph G of order $k+1$ which forbids $\mathcal{C}$. $(ex(k;\mathcal{C})\le k-1)$

Since $\delta(G)\ge 2$ will imply that G contains a cycle, there exists a vertex $v\in V(G)$ satisfying $deg_G(v)=1$. Clearly, $G-v$ also forbids $\mathcal{C}$ and thus $\|G\|-1=\|G-v\|$. By hypothesis $ex(k;\mathcal{C})\le k-1$. $\|G-v\|\le ex(k;\mathcal{C})\le k-1$ and thus $\|G\|\le k$. This concludes the proof. ∎

(•) A graph which forbids cycles is an acyclic graph.

(•) An <u>acyclic</u> <u>connected</u> graph is called a tree.

(•) An acyclic graph is called a forest.

(*) Trees (order n) are extremal graphs (order n) to satisfy $ex(n;\mathcal{C})=n-1$.

(Ref. to Theorem 6 in Lecture 2.)

<u>Proof.</u> By induction on n.

$G = (V, E)$. 3-3

(Review)

$\Delta(G)$: maximum degree

$\delta(G)$: minimum degree

$d(G)$: average degree $\quad d(G) = \dfrac{\sum_{v \in V(G)} \deg_G(v)}{|G|} = \dfrac{Vol(G)}{|G|}$

(•) G is connected if for any two vertices u and v of G, there exists a path P connecting u and v, denoted by $u \underset{P}{\leadsto} v$.

(•) The distance of two vertices u and v in G is the length of a shortest path P, such that $u \underset{P}{\leadsto} v$, denoted by $\mathrm{dist}_G(u,v)$. (If no such P exists, then $\mathrm{dist}(u,v) = +\infty$.)

$\rightarrow$ Metric $\rightarrow$ 3-3'

(•) Let G be a graph. The eccentricity of a vertex v, $ecc(v) =_{def} \max.\{\mathrm{dist}(v,u) \mid u \in V(G)\}$. (離心率)

(•) The diameter of G, denoted by $\mathrm{diam}(G)$, is equal to $\max\limits_{v \in V(G)} ecc(v)$, and $\min\limits_{v \in V(G)} ecc(v)$ is the radius of G.

(×) A graph with diameter k is called a diameter k graph. (The well-known class of graphs is diameter 2 graphs.)

(·) A Metric space is a pair (M,d) where $d: M\times M \longrightarrow \mathbb{R}$ such that

(1) $d(x,x)=0 \quad \forall\, x\in M$;

(2) $d(x,y)>0$ if $x,y\in M$ and $x\neq y$;

(3) $d(x,y)=d(y,x) \quad \forall\, x,y\in M$; and

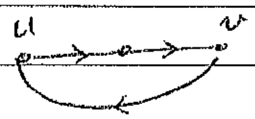
(4) $d(x,y)\le d(x,z)+d(z,y)$. (d is called a metric defined on M.)

(·) A Metric space can be defined on a simple graph G.

Let $d(u,v) =_{def} \text{dist}(u,v)$ where $u,v\in V(G)$.

(※) 如果 G 是有向圖，距離的定義會修改，如下：

$$\text{dist}(u,v) =_{def} \underset{\text{最少的 arcs}}{u\to \cdots \to v}.$$



$$\begin{cases} \text{dist}(u,v)=2 \\ \text{dist}(v,u)=1 \end{cases}$$

$\Uparrow$ 不符合 Metric 的定義。

<u>Theorem 9</u> For each graph G, $rad(G) \le diam(G) \le 2\,rad(G)$.

Proof. It suffices to consider the second inequality. Let u and v be two vertices in G such that $d(u,v) = diam(G)$. Let w be a vertex in the center of G, i.e, $ecc(w) = rad(G)$. By the fact that "d" is a metric, $d(u,w) + d(w,v) \ge d(u,v)$. This implies that

$ecc(w) + ecc(w) = 2\,rad(G) \ge d(u,w) + d(w,v) \ge d(u,v) = diam(G)$. ∎

(Note. The eccentricity of $w \in V(G)$ is $\max\{d(x,w) \mid x \in V(G)\}$.)

<u>Ex. 1-5</u> For positive integers $a \le b \le 2a$, construct a ~~2-connected~~ graph G such diam$(G) = b = 11$ and rad$(G) = a = 7$ (2 points) and a, b in general (3 points).

<u>Theorem 10</u> A graph of minimum degree δ and girth g (shortest cycle) has at least $n_0(\delta, g)$ vertices where

$$n_0(\delta,g) = \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta-1)^i, & \text{if } g =_{def} 2r+1; \text{ and} \\ 2 \cdot \sum_{i=0}^{r-1} (\delta-1)^i, & \text{if } g = 2r. \end{cases}$$

Proof.

Case 1, $g = 2r+1$, $r \ge 1$.

Let v_0 be a fixed vertex in G, see Figure 8.

Connectivity

(∘) A separating set or vertex cut of a graph G is a set $S \subseteq V(G)$ such that $c(G-S) > c(G)$.
# of components in G.

(∘) The connectivity of a graph G, denoted by $\kappa(G)$, is

$\kappa(G) =_{def} \min\{|S| \mid S \subseteq V(G)$ such that $G-S$ is disconnected or has only one vertex.

(*) If G is disconnected, then $\kappa(G) = 0$.

(∘) A graph G is k-connected if $\kappa(G) \geq k$.

(∘) The edge-connectivity of a (connected) graph G, denoted by $\kappa'(G)$, is the minimum size of an edge set F such that $G-F$ is disconnected.

(**) $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

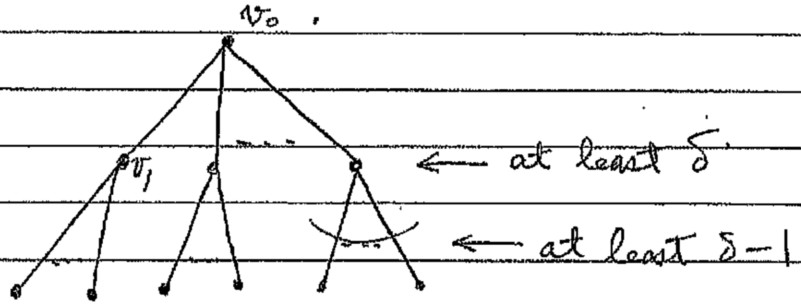


Figure 9

Then, there are at least δ neighbors of v_0, and for each neighbor say v_1, v_1 has at least $\delta-1$ neighbors. Since $g=2r+1$, G contains at least $1+\delta+\delta(\delta-1)+\cdots+\delta(\delta-1)^{r-1}$ vertices. This concludes the proof of the Case 1.

Case 2. $g=2r$

In this case, we start with an edge u_0v_0, see Figure 10. By a similar argument, G contains at least $2\cdot[(\delta-1)+(\delta-1)^2+\cdots+(\delta-1)^{r-1}]$ vertices. ∎

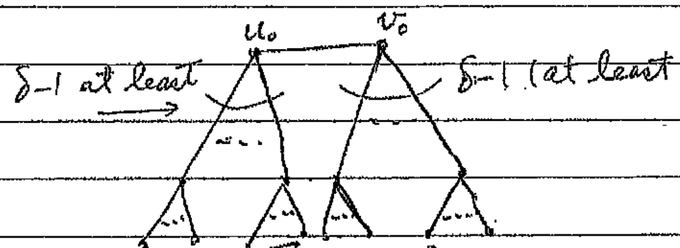


Figure 10.

Theorem 11

If $\delta(G) \geq 3$, then $g(G) < 2\log_2 |G|$. ($g(G)$: girth of G).

Proof. Note that if $\delta_1 \geq \delta_2 \geq 3$, then $n_0(\delta_1, g) \geq n_0(\delta_2, g)$.

It suffices to consider $n_0(3,g)$. By Theorem 10,

$|G| \geq n_0(3,g) = 2^r + 2^r - 2 > 2^r$ ($g = 2r$) and

$|G| \geq n_0(3,g) = 1 + 3 \cdot \frac{2^r - 1}{2-1} = \frac{3}{\sqrt{2}} 2^{\frac{g}{2}} - 2 > 2^{\frac{g}{2}}$ ($g = 2r+1$).

This implies that $r < \log_2 |G|$ and thus $g < 2\log_2 |G|$. ∎

[(d,g)-graph, see 3-5]

Theorem 12 A (d,g)-cage is a d-regular graph with girth g
(Harder one)

and minimum number of vertices. Prove that A (d,g)-cage is

2-connected. ($g \geq 3$)

Proof. First, we claim that if $g_1 > g_2$, then A (d, g_1)-cage (G_1)

contains more vertices than the order of a (d, g_2)-cage (G_2).

Suppose not. Let G_1 and G_2 be two cages respectively and $|G_1| \leq |G_2|$.

It suffices to consider the case $g_1 = g_2 + 1$. Let $\|G_1\| = f(d; g_1)$

and $\|G_2\| = f(d, g_2)$.

↓ 3-6

Review

(•) The girth of a graph G, $g(G)$, is the size of a smallest cycle in G. If G contains no cycle, then $g(G) =_{def} +\infty$.

(•) The perimeter of a graph G, $pm(G)$, is the largest size of a cycle in G. Clearly, $pm(G) \le |G|$ and the equality holds when G has a Hamilton cycle (hamiltonian cycle).

(•) A (d,g)-graph is a d-regular graph with $g(G) = g$.

(•) A (d,g)-cage is a (d,g)-graph with minimum order.

(*) To determine whether a graph contains a cycle of length $3 \le k \le |G|$ is very difficult in the sense of algorithms.

結構最漂亮的圖：(連通性強, 直徑短)

Examples

1. Petersen graph is a $(3,5)$-cage. ($k=2$, $g=3$ in Theorem 8.) $n_0(3,5) = 1 + 3\cdot(1+2) = 10$
2. K_4 is a $(3,3)$-cage. $n_0(3,3) = 1+3 = 4$.
3. Q_3:  is a $(3,4)$-cage, $n_0(3,4) = 2\cdot(1+2) = 6$.

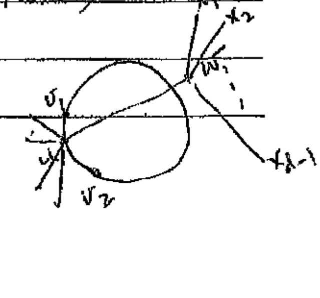
(a) d is even

Let C be a cycle (in G_1) of length g_1 and $uv_1, uv_2 \in E(C)$, moreover $N_{G_1}(u) = \{v_1, v_2, \cdots, v_d\}$. Let $E' = \{v_1v_2, v_3v_4, \cdots, v_{d-1}v_d\}$. Now, consider $G' = G_1 - u + E'$ and denote the component contains v_1 by G_1'. Clearly, G' is a simple graph and G_1' contains a cycle of length g_2: $C - u + v_1v_2$.

Further, if C' is a cycle of G_1' and $E(C') \cap E' = \phi$, then C' is a
(確定 G_1' 中每一個 cycle 長度至少為 g_2.)
cycle of G_1 and thus of length at least g_1. On the other hand, if $E(C') \cap E' \neq \phi$, then let v_iv_j be one of the edges. Let P be a $\langle v_i, \cdots, v_j \rangle$ path on C' satisfying $E(P) \cap E' = \phi$. So, $P + \{uv_i, uv_j\}$ is a cycle of G_1. This implies that $|E(C')| \geq g_1 - 1 = g_2$. This concludes that G_1' is a d-regular graph with girth at least g_2 and $|G_2| \leq |G_1'| = f(d; g_1) - 1 \leq |G_1|$, hence $|G_2| = f(d, g_2) < f(d, g_1) = |G_1|$.

(b) d is odd

Let C be a cycle (in G_1) of length g_1 and $uv_1, uv_2 \in E(C)$. Let $N_{G_1}(u) = \{v_1, v_2, \cdots, v_{d-1}, w\}$. Clearly, $w \notin V(C)$. For otherwise, we have a cycle of length less than g_1. (See Figure 11.)



Now, let $N_{G_1}(w) = \{u, x_1, x_2, \cdots, x_{d-1}\}$ and G_1' be the component (contains v_1) of $G - \{u, w\} + \{v_{2i-1}v_{2i},\ x_{2i-1}x_{2i} \mid 1 \le i \le (d-1)/2\}$.
(一次扣掉两点)

Again, G_1' is simple and G_1' is a (d,g)-graph with at most $f(d;g_1) - 2$ vertices.

Hence, $f(d;g_2) < f(d,g_1)$.

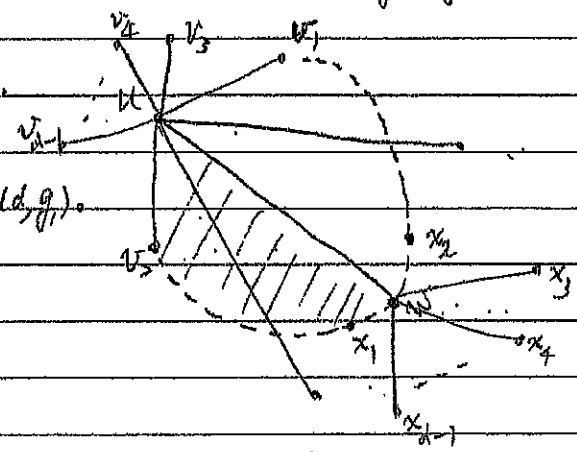


Figure 11, Shorter cycle

<u>Proof of the theorem</u> (A (d,g)-cage is 2-connected.)

Suppose not. Let u be a cut-vertex. Let $C_1, C_2, \cdots, C_w$ be the components of $G-u$, with $|V(C_i)| \le |V(C_{i+1})|$, $i = 1, 2, \cdots, w-1$.

Consider C_1. In C_1, $\forall\ v_1, v_2 \in V(C_1) \cap N_G(u)$, $d(v_1, v_2) \ge g-2$.

(Figure 12) Let C' be an isomorphic copy of C_1 with isomorphism φ. Now, construct a new graph H where $V(H) = V(C') \cup V(C_1)$ and $E(H) = E(C') \cup E(C_1) \cup \{v\varphi(v) \mid \underline{v \in V(C_1) \cap N_G(u)}\}$. By observation, $|H| < |G|$, H is d-regular and H has girth at least $\min\{g, 2g-2\} = g$.

Hence, $|G| > |H| \ge f(d,g)$,

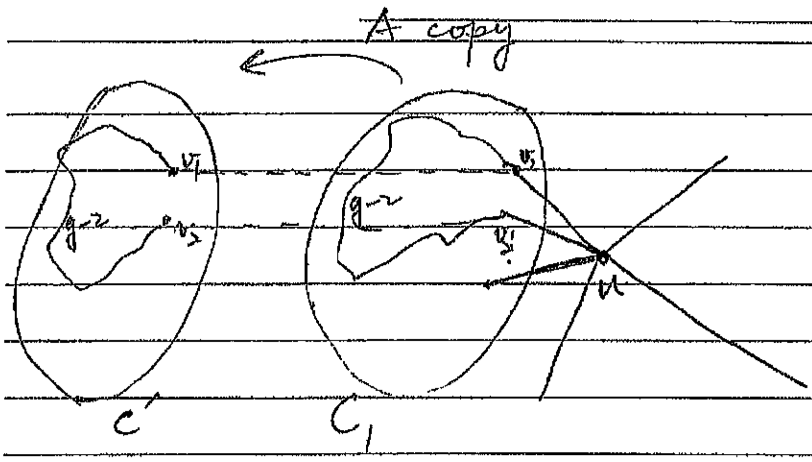


Figure 12, Construction of H.

This implies that G is not a (d,g)-cage, a contradiction. ■

Facts

1. It has been proved that a $(3,g)$-cage is 3-connected.
2. It is conjectured that a (d,g)-cage is d-connected.

Reference

H. L. Fu, K. C. Huang and C. A. Rodger, Connectivity of cages, JGT, Vol. 24, No. 2, 187–191 (1997).