Theorem 3-7 (Lecture 2) Sep. 11, 2024

Theorem 3 (Veblen, 1912)

The edge set of a graph can be partitioned into cycles if and only if every vertex has even degree.

Proof. ($\Rightarrow$) A vertex contained in t cycles has degree $2t$.

($\Leftarrow$) The cycles can be obtained recursively. We start with finding the first cycle. Let $\langle x_0, x_1, \cdots, x_\ell \rangle$ be a path of maximum length ℓ in G. Since $x_0x_1 \in E(G)$, $\deg_G(x_0) \geq 2$.

Key idea!

Let $y (\neq x_1)$ be a neighbor of x_0, i.e., $x_0 y \in E(G)$. Now, $y \in \{x_2, x_3, \cdots, x_\ell\}$. For otherwise, we have a longer path. So, if $y = x_i$, then we have a cycle $C = (x_0, x_1, \cdots, x_i)$. The process continues in $G - E(C)$. (Each vertex is of even degree in the graph $G - E(C)$.) ∎

2-1'

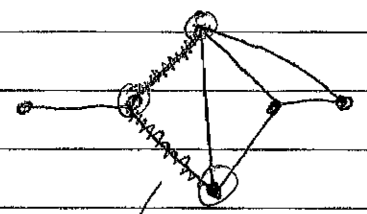
Theorem 4 (Mantel, 1907)

Every graph of order n and size greater than $\lfloor n^2/4 \rfloor$ contains a triangle (C_3 or K_3). ($ex(n; C_3) \leq \lfloor n^2/4 \rfloor$.)

(Original form)

(∘) H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, denoted by $H \leq G$
(See 2'' for general version.)

(∘) H is an induced subgraph of G, denoted by $H \precsim G$, if for each $uv \in E(G)$ and $u, v \in V(H)$, then $uv \in E(H)$. We use $\langle S \rangle_G$ to denote the induced subgraph induced by $S \subseteq V(G)$. Hence, if $H = \langle S \rangle_G$, then $H = \langle V(H) \rangle_G$. $(S = V(H))$



A subgraph, but not an induced subgraph

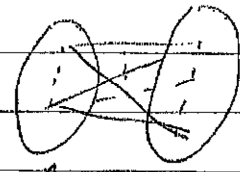
(Review)

(∘) P_m : path with m vertices (order m)

C_ℓ : cycle with ℓ vertices (order ℓ)

K_n : complete graph of order n

K_{n_1,n_2} : complete bipartite graph $((A,B))$



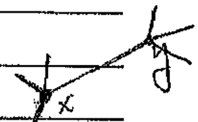
(∘) If we would like to find a graph G of order n which does not contain a subgraph F, then F is called a forbidden graph of G. The maximum size of G is denoted by $ex(n; F)$.

Proof. (1st proof)

Since $K_3 \nsubseteq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \phi$. This implies that $\deg_G(x) + \deg_G(y) \le |G| = n$. (Figure 1) Now, consider

$$\sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) = \sum_{x \in V(G)} (\deg_G(x))^2 \quad \text{(Two-way counting)}.$$

$$\le n \cdot \|G\| = n \cdot e(G)$$



相邻的 $\deg(x)$
边都算一次 $\deg(x)$

By Cauchy's inequality, $(2e(G))^2 = \left(\sum_{x \in V(G)} \deg_G(x)\right)^2 \le n \cdot \sum_{x \in V(G)} (\deg_G(x))^2$

$$\le n^2 \cdot e(G).$$

Hence, $e(G) \le \frac{n^2}{4}$. ∎

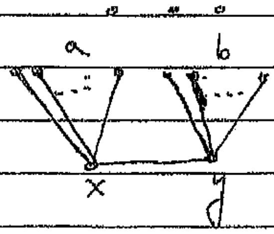


Figure 1. $\deg_G(x) = a+1$, $\deg_G(y) = b+1$

(2nd proof)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2)

Since $K_3 \nsubseteq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies that $\|G\| \le \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G))$.

$\|G\|$ will take a maximum when $\Delta(G) = \lfloor \frac{n}{2} \rfloor$. Hence, we have the proof. ∎

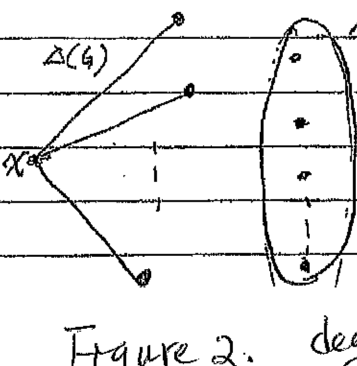


Figure 2. $\deg_G(x) = \Delta(G)$

(*) $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is a graph of size $\lfloor \frac{n^2}{4} \rfloor$ which contains no C_3's. Hence, $ex(n; C_3) \geq \lfloor \frac{n^2}{4} \rfloor$. By Theorem 2, $ex(n; C_3) = \lfloor \frac{n^2}{4} \rfloor$.

(•) The graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is an extremal graph which forbids C_3 (or K_3). (等号成立!)

(***) Find $ex(n; C_4)$. (Or $ex(n; C_k)$ for $k \geq 4$.)
Unsolved

Note. For certain n, $ex(n; C_4)$ is known.

Ex. 1-2. Find $ex(7; C_4)$ (2 points), and $ex(13; C_4)$ (3 points).
Bonus: How about other n's?

(•) Two graphs G_1 and G_2 are isomorphic if there exists a bijection $\varphi : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ iff $\varphi(u)\varphi(v) \in E(G_2)$, equivalently, $u \sim_{G_1} v \Leftrightarrow \varphi(u) \sim_{G_2} \varphi(v)$.

(:) A graph H is a subgraph (general form) of G if H is isomorphic to a subgraph (original form) of G. (長相一樣即可)

(•) φ is called an automorphism of G if $G_1 = G_2 = G$.

(*) Let G be a graph. The set of all automorphisms of G is denoted by $Aut(G)$. ($\langle Aut(G), \circ \rangle$ is a group.)

(*) $Aut(G)$ can be used to characterize the structure of G.

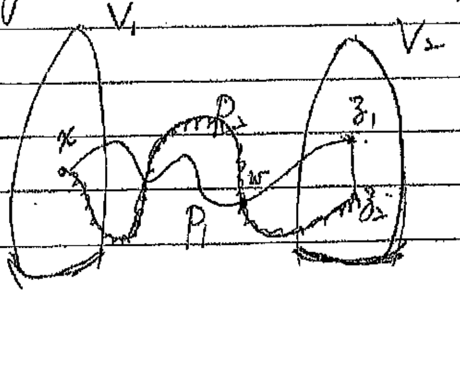
Ex. 1-3. Find $Aut(C_n)$ (2 points) and $Aut(T)$ for $|T| = 5$ (3 points).

Theorem 5 A graph is bipartite if and only if it does not contain an odd cycle.

Proof. ($\Rightarrow$) Let $G=(A,B)$ where A and B are its partite sets. If $(x_0, x_1, \cdots, x_\ell)$ is a cycle of G, then x_0 and x_ℓ must in different partite sets. Hence, the index ℓ must be odd, thus the cycle is of even length.

($\Leftarrow$) W.L.O.G., let G be a connected graph. Let $x \in V(G)$ and $V_1 = \{y \mid y \in V(G) \text{ and } d(x,y) \text{ is even}\}$. Hence, $x \in V_1$. Let $V_2 = V(G) \setminus V_1$. It suffices to claim that both V_1 and V_2 are independent sets. First, consider V_2. Clearly, for each $z \in V_2$, $d(x,z)$ is odd. Suppose that $z_1, z_2 \in V_2$ and $z_1 \sim z_2$ ($z_1 z_2 \in E(G)$). (Figure 3) Let P_1 and P_2 be the two paths such that $P_1 = \langle x, \cdots, z_1 \rangle$ and $P_2 = \langle x, \cdots, z_2 \rangle$, moreover they are the shortest paths connecting x to z_1 and x to z_2 respectively.

Figure 3



Let w be the last vertex in which P_1 and P_2 intersect. Also, let $\|P_1\| = 2s+1$ and $\|P_2\| = 2t+1$. (Note that if $V(P_1) \cap V(P_2) = \{x\}$, then we have an odd cycle (x, P_1, z_1, z_2, P_2) (length $2s+2t+3$).)

Now, if w does exist, then $\langle x, P_1, w\rangle$ and $\langle x, P_2, w\rangle$ are of the same length, let the length be h. (?) So, the cycle $(w, \cdots, z_1, z_2, \cdots)$ is of length $(2s+1-h)+(2t+1-h)+1 = 2s+2t-2h+3$, an odd integer. Thus, an odd cycle exists, a contradiction. Hence, V_2 is an independent set. A similar argument can be applied to show that V_1 is also an independent set. (x is not adjacent to any vertex of $V_1 \setminus \{x\}$.)

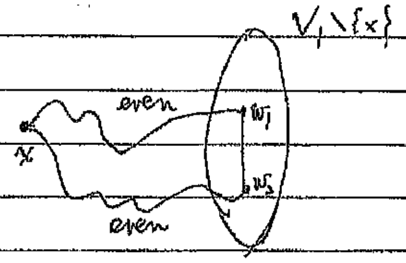


Figure 4.

<u>Open problem</u> How many edges can a bipartite graph of partite sets A, B, $|A|=m$, $|B|=n$, have such that $G \not\supseteq C_4$? $G \not\supseteq C_6$? $\cdots$

(*) The maximum size of $G \not\supseteq C_4$ is denoted by $z(m,n;2,2)$.

Ex. 1-4. Find $z(7,7;2,2)$ (2 points) and $z(91,91;2,2)$ (3 points)
Bonus: How about $z(n,n;2,2)$ for $n \in \mathbb{N}$?

Theorem 6 The following statements are equivalent for a graph G.

(a) G is a tree. (G is connected and acyclic.)

(b) G is connected and every edge of G is a bridge.

(c) G is a maximal acyclic graph. (If x and y are not adjacent, then $G+xy$ contains a cycle.

Proof. (a) $\Rightarrow$ (b)

Let xy be an edge of G and $G-xy$ is connected. Then, there exists a path P connecting x and y in $G-xy$. Clearly, G contains a cycle (x,P,y) in G, a contradiction.

(b) $\Rightarrow$ (c) If G is not acycle, then a cycle edge is not a bridge. Hence, G is acyclic.

If G is not a maximal acyclic graph, then there exists a pair of vertices z_1 and z_2 in G such that $G+z_1z_2$ is also acyclic. Since G is connected, there exists a path connecting z_1 and z_2, say P. This implies that (z_1,P,z_2) is a cycle in $G+z_1z_2$, a contradiction.

(c) $\Rightarrow$ (a) If G is not connected, then there exists a pair of vertices x_1 and x_2 (in different components) such that $G+x_1x_2$ is also acyclic.

x_1 and x_2

Theorem 7 (Euler, 1941)

A nontrivial connected graph (multigraph) has an eulerian circuit (Euler circuit) if and only if each vertex has even degree. Moreover, a connected graph has an eulerian trail from a vertex x to a vertex $y \neq x$ if and only if x and y are the only two vertices of odd degree.

Proof. The second statement follows directly from the first one. We prove the first statement.

($\Rightarrow$) If a circuit passes a vertex x h times, then $\deg_G(x) = 2h$.

($\Leftarrow$) By induction on $\|G\|$. Since $\|G\| \geq 1$, $\delta(G) \geq 2$ and thus G contains a cycle. (G is not a tree!)

Let Z be a circuit in G with the maximum number of edges. If Z is an eulerian circuit, then we are done. Suppose not. Let H be a nontrivial component of $G - E(Z)$. Since G is connected, $V(H) \cap V(Z) \neq \phi$. Let $x \in V(H) \cap V(Z)$. Now, H is a nontrivial connected graph (even graph). Hence, H contains an eulerian circuit Y (by induction). By using x, we can attach

Z and Y together to obtain a larger circuit. (Figure 5) This contradicts to the maximality of $|E(Z)|$. Hence, Z must be an eulerian circuit of G. ▨

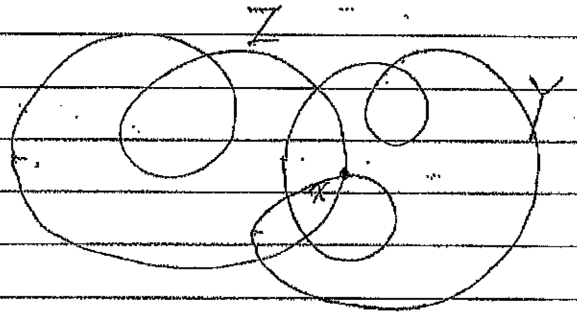
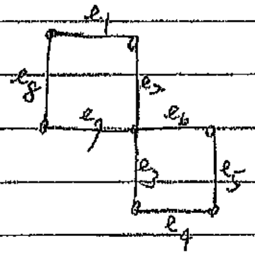


Figure 5. Attaching Z and Y.

(other proofs ?)

Open problem

Find the number of distinct eulerian circuits of an eulerian graph G. (Two circuits are the same if they can be obtained each other by a cyclic shift of edges.)



$\langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \rangle$

$=_{def} \langle e_3, e_4, e_5, e_6, e_7, e_8, e_1, e_2 \rangle$

$\neq_{def} \langle e_1, e_2, e_6, e_5, e_4, e_3, e_7, e_8 \rangle$

(◦) Digraph is a graph whose arcs are "ordered pairs" instead of "edge" in a graph.

Theorem 2.5 (BEST Theorem)

A digraph D has an eulerian (directed) circuit if and only if D is strongly connected and for each vertex $v \in V(D)$, $\deg_D^+(v) = \deg_D^-(v)$. Moreover, if D is an eulerian graph, $s(D)$ is the number of distinct eulerian circuits, then

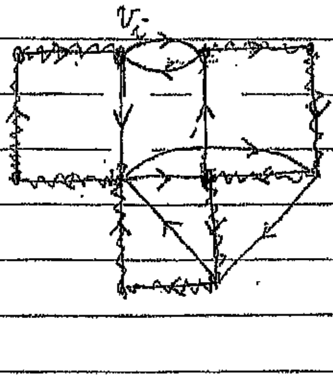
$$s(D) = t_i(D) \cdot \prod_{j=1}^{n} (\deg_D^+(v_j) - 1)! \quad \text{for every } i \in \{1, 2, \dots, n\}$$

where $t_i(D)$ is the number of spanning trees oriented toward v_i.

(Note: The counting part of the theorem was proved by (independently) two groups de Bruijn and van Aardenne-Ehrenfest (Smith and Tutte.)

Proof. The existence part can be obtained by a similar argument as the "multigraph" version.

D:



$s(D) = 1 \cdot 1 = 1$

Figure 6 Spanning tree oriented toward v_i