

Lecture 9

Group Divisible Designs

(Review)

NO. 7
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To start, we use a well-known construction to construct an

$STS(v)$ where $v \equiv 3 \pmod{6}$. Let $v = 6k+3$ and $L = [l_{ij}]$ be an

idempotent commutative Latin square of order $2k+1$. Now, we

are ready to construct the Steiner triple system of order $6k+3$.

(1) Let $X = \mathbb{Z}_3 \times \mathbb{Z}_{2k+1}$.

(2) $\forall i \in \mathbb{Z}_{2k+1}$, let $\{(0, i), (1, i), (2, i)\} \in \mathcal{B}$.

(3) $\forall i < j \in \mathbb{Z}_{2k+1}$, let $\{(0, i), (0, j), (1, l_{ij})\}$, $\{(1, i), (1, j), (2, l_{ij})\}$ and $\{(2, i), (2, j), (0, l_{ij})\}$ be triples in $\mathcal{B}$.

Then, $(X, \mathcal{B})$ is an $STS(6k+3)$.

It is easy to check any two elements of X will occur in a triple

and we have in total $(2k+1) + 3 \cdot \frac{(2k+1)^2 - (2k+1)}{2} = 2k+1 + 6k^2 + 3k$

$$= 6k^2 + 5k + 1 = \frac{(6k+3)(6k+2)}{6}.$$

(*) If $(X, \mathcal{B})$ is an $STS(v)$, then $|\mathcal{B}| = \frac{v(v-1)}{6}$.

(**) In difference method, the part $v \equiv 3 \pmod{6}$ is comparatively more complicate, we can replace it with this construction if we only try to prove the "sufficient" direction.

We can use $\text{MOLS}(n)$ to construct designs with larger blocks.

(***) The existence of an Affine plane of order n where n is a prime power.

Step 1. Construct $n-1$ $\text{MOLS}(n)$, let them be $L^{(1)}, L^{(2)}, \dots, L^{(n-1)}$.
(For convenience, we use $1, 2, \dots, n$ for $\mathbb{Z}_n$)

Step 2. Let $L^{(1)}$ and $L^{(n)}$ be the row-indices and column-indices squares respectively.

$$L^{(1)} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \dots & 1 \\ \hline 2 & 2 & \dots & 2 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline n & n & \dots & n \\ \hline \end{array}$$

$$L^{(n)} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \dots & n \\ \hline 1 & 2 & \dots & n \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 1 & 2 & \dots & n \\ \hline \end{array}$$

Step 3. Let $\bar{X} = (\mathbb{Z}_n \cup \{\infty\}) \times \mathbb{Z}_n = X \cup (\{\infty\} \times \mathbb{Z}_n)$.

Step 4. $\forall i, j \in \mathbb{Z}_n$, let $\bar{B}_{ij} = \{(0, i), (1, j), (2, L^{(1)}(i, j)), (3, L^{(2)}(i, j)), \dots, (\infty, L^{(n-1)}(i, j))\}$ be a block in $\bar{\mathcal{B}}$. (There are n^2 blocks.)

Step 5. Let $\mathcal{B}' = \{\bar{B}_{ij} - (\infty, L^{(n-1)}(i, j)) \mid \bar{B}_{ij} \in \bar{\mathcal{B}}\}$.

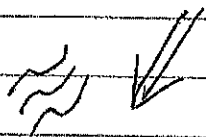
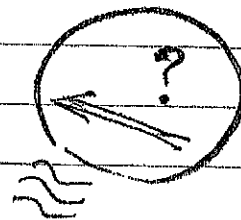
Step 6. Let $\mathcal{B} = \mathcal{B}' \cup \{\{i\} \times \mathbb{Z}_n \mid i \in \mathbb{Z}_n\}$.

Then, we conclude that $(X, \mathcal{B})$ is an Affine plane of order n .

(***) Let $\bar{X} = \{(\infty, \infty)\} \cup \bar{X}$ and $\bar{\mathcal{B}} = \bar{\mathcal{B}} \cup \{(\infty, \infty), \{i\} \times \mathbb{Z}_n\} \mid i \in \mathbb{Z}_n\}$.

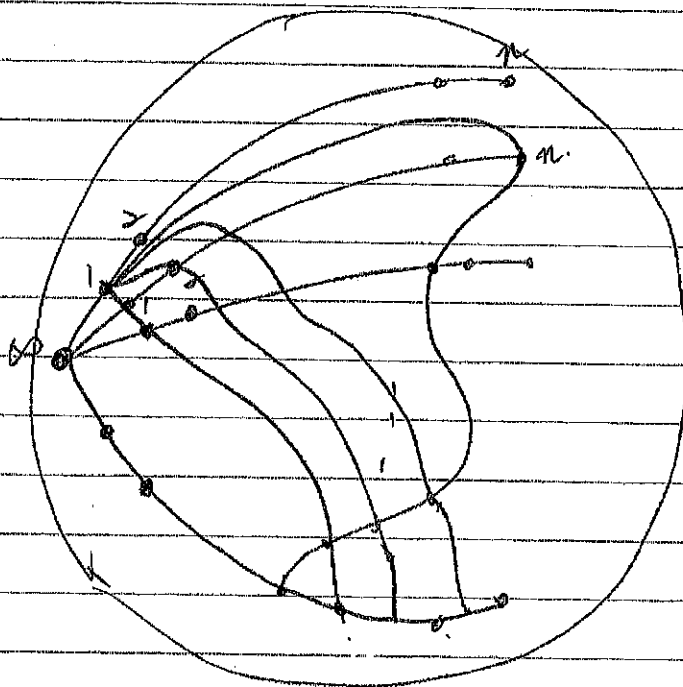
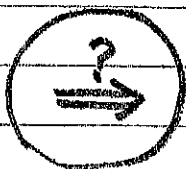
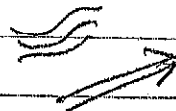
Then $(\bar{X}, \bar{\mathcal{B}})$ is a projective plane of order n .

A complete family of $\text{MOLS}(n)$

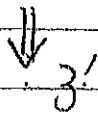
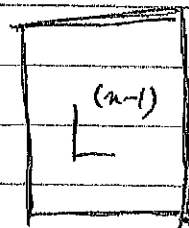
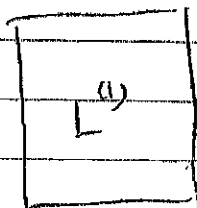
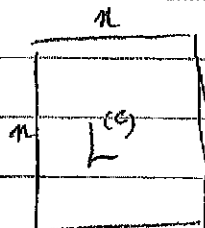
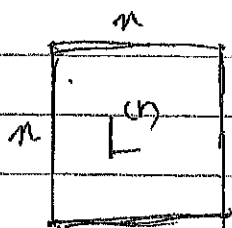


projective plane of order n

Affine plane of order n



projective plane



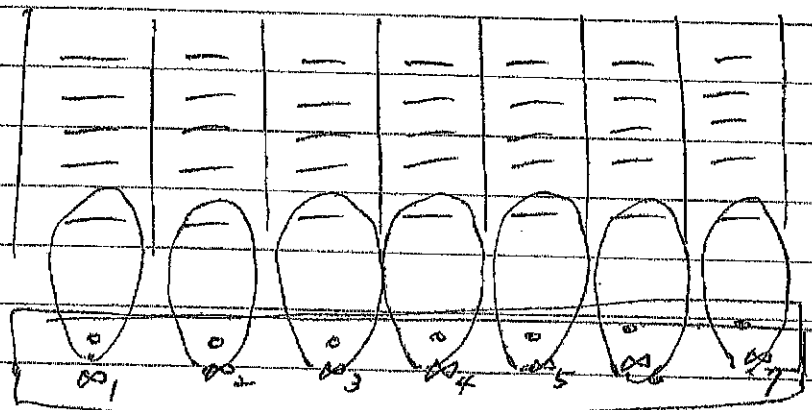
$3'$

Here, we mention some PBD's.

Theorem 1 For each $v \equiv 1 \pmod{3}$, there exists a $2-(v, K, 1)$ -design with $K = \{4, 7\}$ except $v = 10, 19$.

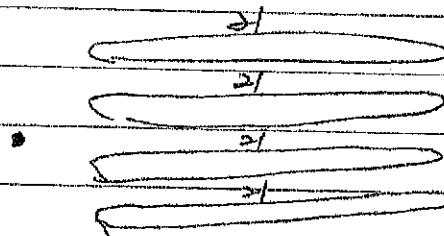
We omit the proof, but we present some examples here.

$v = 22$



By using a Kirkman triple system of order 15, we can attach 7 points in the 'infinity' and obtain the desired PBD.

$v = 85$



First, we have a $2-(85, \{4, 22\}, 1)$ -design by using two MOLS(21). Then, a $2-(85, \{4, 7\}, 1)$ -design will be obtained from $v = 22$ case.

We can also use $MOLS(n)$ to construct PBDs in which K is of size larger than one. For example, we can use an Affine plane of order 5 to construct a PBD $2-(24, \{4, 5\}, 1)$ design $(X, \mathcal{B})$.

The idea comes from deleting an element from X . Then, each block which contains this element becomes a block of size 4, and the other blocks which do not contain this element remain the same.

Hence, we can start with a special type of design, and then either adding or deleting elements (to or from) X to obtain a new design.

(*) Definition (Group Divisible Designs of type n^m)

A design $(X, \mathcal{B})$ is called a group divisible design of type n^m if X can be partitioned in m disjoint subsets $G_1, G_2, \dots, G_m$ such that each $B \in \mathcal{B}$, $|B \cap G_i| \leq 1$, $|B| = k$ and every pair of two elements from different groups occurs together in exactly λ blocks of $\mathcal{B}$.
(called groups)

The design $(X, \mathcal{B})$ is denoted by $GDD(n, m; k; \lambda)$.

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$k=m$, $GDD(n, m; m, \lambda) = \text{def } T(n; m, \lambda)$ Transversal design
(TD)

Existence: $T(n; 3, 1) \equiv LS(n)$.

A GDD $(n, m; k; \lambda)$ can be shorten as a k -GDD of type n and index λ . We shall solve the case $k=3$ and $\lambda=1$ in what follows. First, we need a theorem.

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Theorem 2. For each $v \equiv 5 \pmod{6}$, a $2-(v, \{3, 5\}, 1)$ -design exists.

Moreover, we have such a design with exactly one block of size 5.

Proof. (By difference method.) Let $v = 6k + 5$ and $X = X_1 \cup X_2$ where

$|X_1| = 5$ and $|X_2| = 6k$. Now, let $X_2 = \mathbb{Z}_{6k}$. Hence, the set of

differences in $\mathbb{Z}_{6k} = \{1, 2, \dots, 3k(\text{half})\}$. As mentioned in the

above construction, we can find difference triples either in

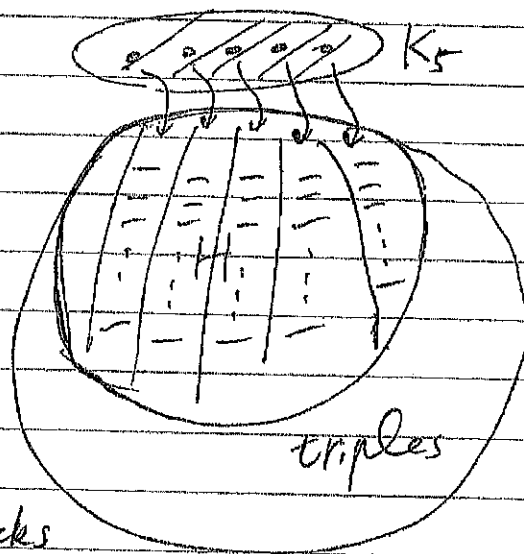
$\{1, 2, \dots, 3k-3\}$ or $\{1, 2, \dots, 3k-4, 3k-2\}$. Hence, after taking away

those triples, we have a 5-regular graph H left defined on

$\mathbb{Z}_{6k}$. Since $3k$ is one of the differences, $\chi'(H) = 5$. The proof then

follows by the same idea as

in recursive construction.



Note. Such a PBD also

exists for $v \equiv 1 \text{ or } 3 \pmod{6}$

since we can take all blocks

of size 3.

Group Divisible Design (3-GDD)

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Problem For which m and n , $K_3 \mid K_{m(n)}$?

Fact 1. If $n=1$, then $m \equiv 1 \text{ or } 3 \pmod{6}$.

Definition (3-sufficient)

A graph G is said to be 3-sufficient if (1) $|G| \geq 3$, (2) G is an even graph and (3) $3 \mid \|G\|$.

Problem (Open) For which 3-sufficient graph G , $K_3 \mid G$?

Nash-Williams Conjecture (Remains open)

If G is 3-sufficient and $\delta(G) \geq \frac{3}{4}|G|$, then $K_3 \mid G$.

Fact 2. If $K_{m(n)}$ is 3-sufficient, then

(1) Either n is even or n is odd and m is odd; and

$$(2) \quad 3 \mid \binom{m}{2} \cdot n^2.$$

Theorem. If $K_{m(n)}$ is 3-sufficient and $m \geq 3$, then $K_3 \mid K_{m(n)}$.

We need several basic facts in order to prove the theorem.

Fact 3. $K_3 \mid K_{3(n)}$. (By using a L.S. of order n .)

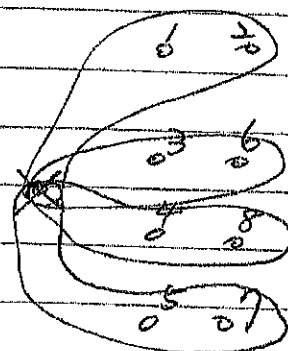
Fact 4. $K_3 \mid K_4(n)$ if and only if n is even.

Proof: ($\Rightarrow$)

Since $m=4$, n must be even in order that each vertex is of even degree.

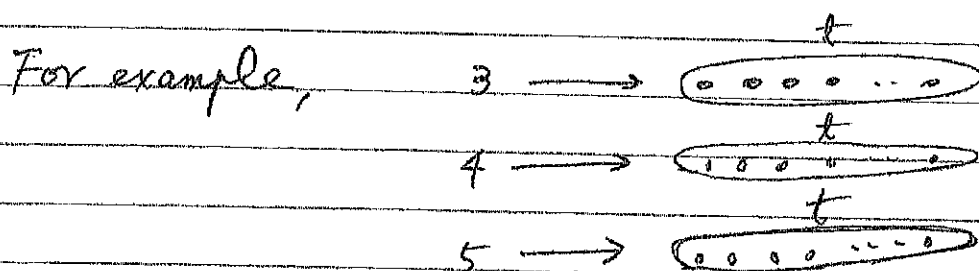
($\Leftarrow$) If $n=2$, then $K_3 \mid K_{4(2)}$. This is a consequence of deleting one vertex of an STS(9). (See it?)

0 1 2	0 3 6	0 4 8	0 5 7
3 4 5	1 4 7	1 5 6	1 3 8
6 7 8	2 5 8	2 3 7	2 4 6



Now, let $n=2t$. The proof follows by

blowing up each vertex into t vertices and use an STS(t) to construct all the K_3 's we need.



As a consequence, we have $8 \cdot t^2$ K_3 's in total. This is also the number K_3 's we desire: $\frac{6 \cdot (2t)^2}{3} = 8t^2$.

Fact 5. If $n \equiv 1$ or $3 \pmod{6}$, then $K_3 \mid K_{n(n)}$ for each positive integer n .

Proof. It is a direct consequence of blowing each vertex of K_n into n vertices. ■

Fact 6. If $m \equiv 0 \text{ or } 4 \pmod{6}$ and n is even, then $K_3 \mid K_{m(n)}$.
(X, B)

Proof. First, we take an $STS(2m+1)$, and delete one vertex from X , then we have $K_3 \mid K_{m(2)}$. Since n is even, we use the same technique as that in Fact 4. This concludes the proof. ■

Fact 7. If $m = 5$ and $3 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. Let $n = 3k$. By the fact that $K_3 \mid K_{5(3)}$, we conclude the proof by blowing each vertex into k vertices. ■

Fact 8. If $m \equiv 5 \pmod{6}$ and $3 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. This is a direct result of the existence of a PBD $(m, \{3, k\}, 1)$ -design and Fact 7. ■

Fact 9. If $m \equiv 2 \pmod{6}$ and $6 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. Let $m = 6k + 2$. Consider $2m+1 \equiv 5 \pmod{6}$. Since a $(2m+1, \{3, 5\}, 1)$ -design exists, we may let it ^{be} as in the following figure.

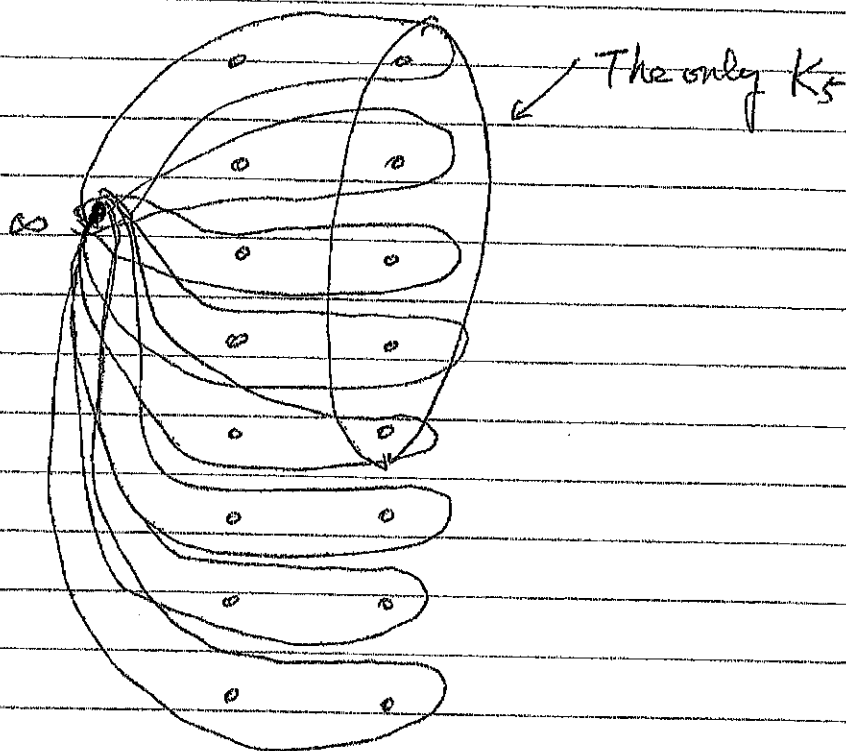


Figure for $(2m+1, \{3, 5\}, 1)$ PBD

Now, by deleting ∞ , we obtain a decomposition of $K_{m(2)}$ into K_3 's and one K_5 . Let $n = 6k$. Then, the proof follows by blowing up each vertex into $3k$ vertices. ■

Theorem (3-GDD)

$K_3 \mid K_{m(n)}$ if and only if $K_{m(n)}$ is 3-sufficient.

Proof. Combining Facts 5, 6, 7, 8, 9; we have the proof. ■

Exercise 5-1 (6 points) prove the above theorem in details.