

Lecture 1 Introduction

组合设计 (Combinatorial Designs)

1-1

(*) Given a finite set $X \neq \emptyset$.

(*) A design (combinatorial) is an ordered pair (X, B) where B is a collection of (multi)-subsets of X .

Note. Since an empty set plays no role in a design, we assume that all subsets are non-empty.

e.g. $X_1 = \{0, 1, 2, 3\}$, $B_1 = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}$.

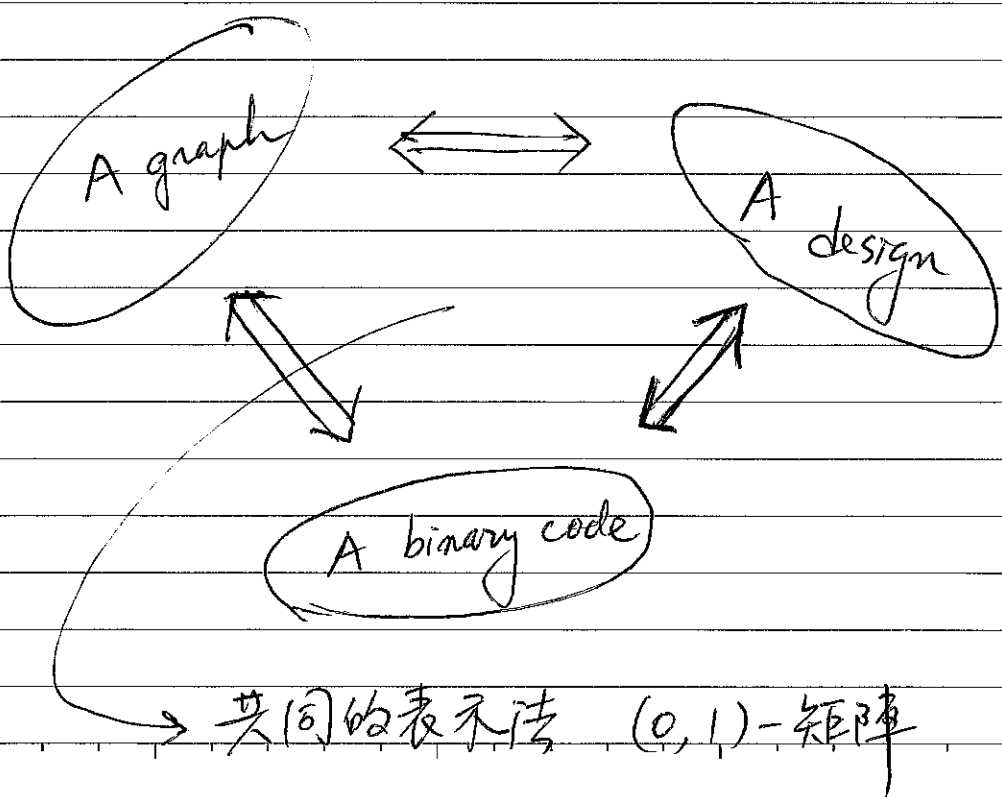
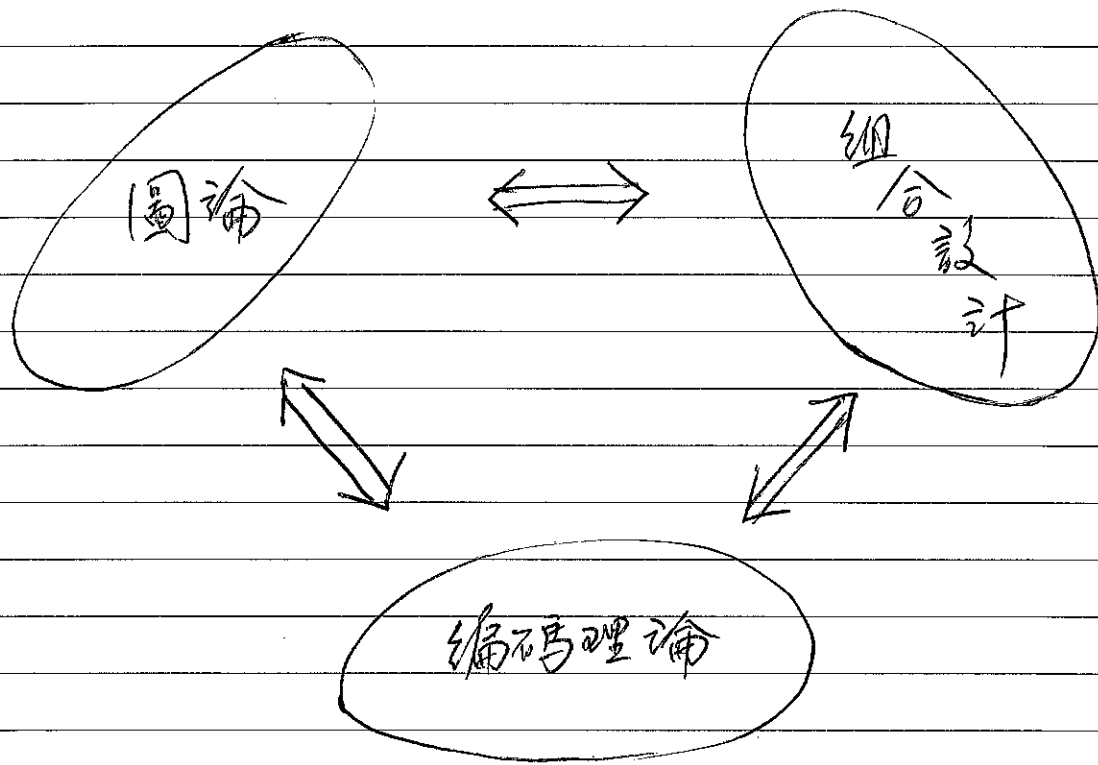
$X_2 = \{0, 1, 2, 3, 4, 5, 6\}$, $B_2 = \{\{0, 1, 3\} + i \mid i \in \mathbb{Z}_7\}$.
(mod 7)

(*) For convenience, we use $\mathbb{Z}_4$ for X_1 and $\mathbb{Z}_7$ for X_2 for above examples.

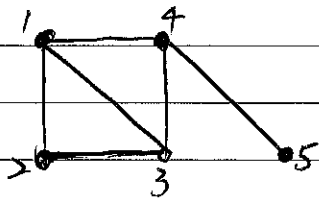
(*) 组合设计研究的目標在於找到滿足特定條件後如何找到 (X, B) .
例如, 上面的例子, (B_1)
 X_1 中任兩個元素都出現在兩個不同的集合中, 而 X_2 中任兩個元素都出現在 B_2 中唯一的一個集合中.

(*) 考慮任兩個元素出現的子集合中的設計, 簡稱為 2-design.
(t) ... (t-design)

背景 設計最早是用在實驗(或解決問題), 統計學中一般稱為
實驗設計 (Experimental Design).



1-1"



Design

$$\Leftrightarrow X = \{1, 2, 3, 4, 5\}$$

$$B = \{\overline{12}, \overline{23}, \overline{13}, \overline{14}, \overline{34}, \overline{45}\}$$

$\downarrow$
 $\{1, 2\}$

$\Updownarrow$ Binary code.

$$\left\{ \begin{array}{l} (1, 1, 0, 0, 0), \\ (0, 1, 1, 0, 0), \\ (1, 0, 1, 0, 0), \\ (1, 0, 0, 1, 0), \\ (0, 0, 1, 1, 0), \\ (0, 0, 0, 1, 1). \end{array} \right.$$

Using the support of $(0, 1)$ -vector

$$\text{supp}(1, 0, 0, 1, 0) = \{1, 4\}$$

共同表示法

	B_1	B_2	B_3	B_4	B_5	B_6
1	1	0	1	1	0	0
2	1	1	0	0	0	0
3	0	1	1	0	1	0
4	0	0	0	1	1	1
5	0	0	0	0	0	1

(•) A Column vector represents an edge in graph,
a "block" in design and a codeword in a binary code.

(*) 探讨实验设计中所隐藏的数学, (Combinatorics), 是组合设计¹⁻²的研究对象。
 ↑
 Check Wiki

(**) 组合设计的应用非常广泛!

→ 魔方的研究
 (Magic square)

历史背景 (外国文献), [本国文献更早一些, 但是没有期刊可以发表]

最早提供和组合设计有相关联的问题是:

Kirkman's 15 girls problem (1840左右)

15个女孩每天排队路上学, 三个一列共5列; 每天一个女孩会认识(如下图)

同一列的另两个女孩。是否可以在七天中安排好不同路队, 让

每个女孩都认识其他的十四位呢?

(Exercise 1-1)
 (3 points)

→ Find a solution for 15 girls problem.

(*) 如果是九个女孩排三列, 以下是一个答案:

0 1 2	0 3 6	0 4 8	0 5 7
3 4 5	1 4 7	1 5 6	1 3 8
6 7 8	2 5 8	2 3 7	2 4 6

对应于 (X_3, B_3) , $X_3 = \mathbb{Z}_9$, B_3 为 12 个 3-subsets, 满足...

(*) A design is incomplete if for each $B \in \mathcal{B}$, $|B| < |X|$.

(Complete otherwise.)

(*) A design $(X, \mathcal{B})$ is balanced if $\forall x \in X$, the number of sets in $\mathcal{B}$ which contains x is a constant, i.e.

$$r(x) = |\{B \mid x \in B, B \in \mathcal{B}\}| = r(y) \text{ for any two elements (varieties) (treatments)}$$

x and y in X . $r(x) = r(y) = r$: replication "number"

e.g. Both $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ are incomplete and balanced.

(*) If all sets in $\mathcal{B}$ are of the same size, say k , then $(X, \mathcal{B})$ is called a "block design" (区组设计).

Again, both $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ are block designs.

(*) We use $\lambda(x, y)$ to denote the number of sets in $\mathcal{B}$ which contain $\{x, y\}$. (frequency of pairs x and y .)

If for any two distinct elements x and y , $\lambda(x, y)$ remains (t distinct).

fixed, then we use λ to denote the frequency number

of 2-design $(X, \mathcal{B})$.
(t -design)

(*) In $(X, \mathcal{B}_1)$, $\lambda = 2$; In $(X_2, \mathcal{B}_2)$ and $(X_3, \mathcal{B}_3)$, $\lambda = 1$

(*) A 2 -design $(X, \mathcal{B})$ with $|X| = v$, $|B| = b$, $|B| = k$ for each $B \in \mathcal{B}$, replication r and frequency λ is denoted as a

2 -(v, b, r, k, λ)-design or 2 -(v, k, λ) in short (?)

Exercise 1-2 Show that in a 2 -(v, b, r, k, λ) design,
(3 points)

(1) $bk = vr$, (2) $\lambda(v-1) = r(k-1)$ and (3) $b \geq v$. ← Important ideas!

Note If the design defined above is a t -design, then we use $S_2(t, k, v)$ to denote the design (or, t -(v, k, λ) in literature).

Fact 1

In a 2 -(v, k, λ) design, $b = \frac{\lambda v(v-1)}{k(k-1)}$ and $r = \frac{\lambda(v-1)}{k-1}$.

Open problem

From the graph theory point of view, if a 2 -(v, k, λ) design exists, then $K_k/\lambda K_2$, i.e., λK_2 can be decomposed into b edge-disjoint K_k 's. This implies that $\frac{\lambda(v-1)}{k-1}$ and $\lambda \binom{v}{2} / \binom{k}{2}$ are integers. Are these two conditions also sufficient?

(*) 我们可以用 $(0,1)$ -矩阵来表一个设计 (X, B) , 然後從矩阵的性质来研究设计的性质。

$$\text{令 } X = \{x_1, x_2, \dots, x_v\}, B = \{B_1, B_2, \dots, B_b\}.$$

定义 (X, B) 的关联矩阵 (Incidence matrix) 如下:

$$N_{v \times b} = [n_{ij}]_{v \times b}, \quad n_{ij} \in \{0, 1\}, \quad 1 \leq i \leq v, \quad 1 \leq j \leq b,$$

$$n_{ij} = 1 \text{ if and only if } x_i \in B_j.$$

$(X, B) : 2-(v, k, \lambda)$ design.

$$N = \begin{bmatrix} n_{ij} \end{bmatrix}_{v \times b}$$

$$N \cdot N^T = \begin{bmatrix} r & \lambda \\ \lambda & r \end{bmatrix}_{v \times v}$$

$$= (r - \lambda) I_v + \lambda J_v$$

单位矩阵 $\swarrow$ 全1矩阵 $\swarrow$

$$\det(N N^T) = (r - \lambda + \lambda v) \cdot (r - \lambda)^{v-1} > 0$$

$$\begin{array}{ccc} & \uparrow & \uparrow \\ & r + \lambda(v-1) & v-1 > k-1 \\ & = r + r(k-1) = rk & \end{array}$$

$$\text{rank } N \geq v \Rightarrow b \geq v$$

$(X, \mathcal{B})$: A design

- (*) An $(X, \mathcal{B})$ is called a partial linear space if any two blocks of $\mathcal{B}$ contain at most one common element.
- (*) If any two elements of a partial linear space $(X, \mathcal{B})$ occur together in a block of $\mathcal{B}$, then $(X, \mathcal{B})$ is a linear space with index 1.

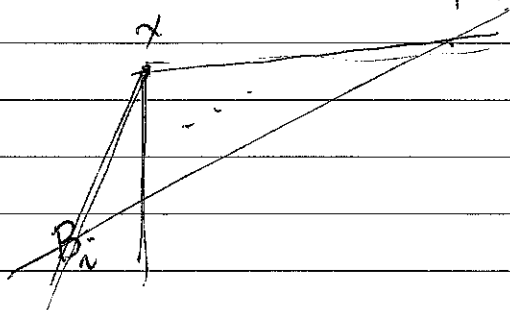
Theorem If $(X, \mathcal{B})$ is a linear space with index 1, then

$$|X| \leq |\mathcal{B}|.$$

Proof. Let $X = \{x_1, x_2, \dots, x_v\}$ and $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$.

Observation 1: If $x \notin B_i \in \mathcal{B}$ for some i , then $r_x \geq |B_i|$.

(x 与 B_i 上的每一元素都要落在另一个 $\mathcal{B}$ 中的 block.)



Assume that $b \leq v$.

$$(a) \quad 1 = \sum_{B \in \mathcal{B}} \frac{1}{b} = \sum_{B \in \mathcal{B}} \left(\sum_{x \notin B} \frac{1}{v - |B|} \right) \cdot \frac{1}{b} = \sum_{B \in \mathcal{B}} \sum_{x \notin B} \frac{1}{b(v - |B|)}$$

$$(b) \quad 1 = \sum_{x \in X} \frac{1}{v} = \sum_{x \in X} \left(\sum_{B \ni x} \frac{1}{b - r_x} \right) \cdot \frac{1}{v} = \sum_{x \in X} \left(\sum_{B \ni x} \frac{1}{v(b - r_x)} \right)$$

$$(c) \quad \forall B \in \mathcal{B}, x \notin B, v \cdot r_x \geq b \cdot |B|. \text{ (By observation and } v \geq b \text{.)}$$

$$\begin{aligned} \frac{1}{b} &= \sum_{x \notin B} \frac{1}{b \cdot (v - |B|)} = \sum_{x \notin B} \frac{1}{b \cdot v - \underbrace{b|B|}_{\lambda}} \\ &\leq \sum_{x \notin B} \frac{1}{b \cdot v - \underbrace{v r_x}_{\lambda}} = \sum_{B \ni x} \frac{1}{v(b - r_x)} = \frac{1}{v} \end{aligned}$$

$\Rightarrow v \leq b$, Hence, $b = v$.

In conclusion, $b \geq v$. (如果假设 $v \geq b$, 则只有 $b = v$ 会成立!)

(*) 這裏的 $B \in \mathcal{B}$ 不一定都要具有一樣多的元素。

顯然在一個 $2-(v, k, \lambda)$ design, 結論是正確的。

(*) 由於 $2-(v, k, \lambda)$ design 的 blocks 會隨著 λ 加大而增加, 因 $b \geq v$ 就更沒問題 (v 不變)。

$$(*) (X_1, B_1) \approx K_3 | 2K_4.$$

$$(X_2, B_2) \approx K_3 | K_7.$$

$$(X_3, B_3) \approx K_3 | K_9.$$

N.C. for the existence of a $2-(v, k, \lambda)$ design :

$$(1) \underline{k \leq v}, \quad (\text{Incomplete designs})$$

$$(2) (k-1) \mid \lambda(v-1), \text{ and}$$

$$(3) \binom{k}{2} \mid \lambda \binom{v}{2}.$$

(*) R. Wilson : The necessary conditions are also sufficient for large v .

(说明) 为了证明上述的必要条件也是充分条件, 我们需要

对所有满足上述条件的 (v, k, λ) 都构造出一个

$2-(v, k, \lambda)$ design. (这是比较困难的部分!)

(X_1, B_1) 是一个 $2-(4, 3, 2)$ design.

(X_2, B_2) 是一个 $2-(7, 3, 1)$ design.

(X_3, B_3) 是一个 $2-(9, 3, 1)$ design.

Exercise 1-3 Find a $2-(13, 3, 1)$ design and a
(3 points) $2-(19, 3, 1)$ design.

Similarly, we can determine the necessary conditions for the existence of a t -(v, k, λ) design.

(*) An $S_t(t, k, v)$ exists. $\Rightarrow$ N.C.

(1) $v \geq k$,

(2) $\binom{k-1}{t-1} \mid \lambda \binom{v-1}{t-1}$, and $(r = \frac{\lambda \binom{v-1}{t-1}}{\binom{k-1}{t-1}})$

(3) $\binom{k}{t} \mid \lambda \binom{v}{t}$.

Again, are these conditions also sufficient?

(**) 研究 $(t \geq 3)$ -design 的存在问题, 相较于 $t=2$ 就要困难许多。

以下是两个 $t=3$ 的例子。

1. $v = 2^n, n \geq 2$

$X = (\mathbb{Z}_2)^n, B = \{ \{ \vec{x}, \vec{y}, \vec{z}, \vec{w} \} \mid \vec{x} + \vec{y} + \vec{z} + \vec{w} = \vec{0}, \vec{x}, \vec{y}, \vec{z}, \vec{w} \in (\mathbb{Z}_2)^n \}$.

2. $v = 10$

$X = E(K_5), B = \{ \{ e_1, e_2, e_3, e_4 \} \mid e_i \in E(K_5), i=1,2,3,4, \text{ and}$

$\langle \{ e_1, e_2, e_3, e_4 \} \rangle \cong \begin{array}{c} \triangle \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} \text{ or } \begin{array}{c} \triangle \\ \hline \triangle \end{array} \text{ or } \begin{array}{c} \square \end{array} \}$.

(*) 我们要学习的内容, 主要是如何建構满足给定条件的设计。