

FGT Lecture 1.1

No.

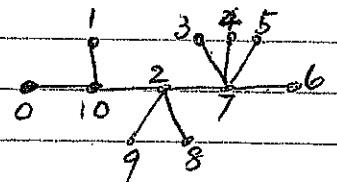
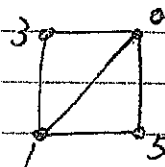
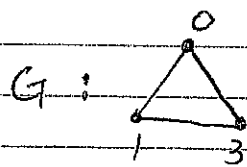
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Graph Labelings

1. Graceful Labeling

(*) A graceful labeling of a graph G is a ¹⁻¹ mapping f , $f: V(G) \xrightarrow{1-1} \{0, 1, 2, \dots, \|G\|\}$ such that the weight of all of edges uv , defined by $|f(u) - f(v)|$, are distinct.

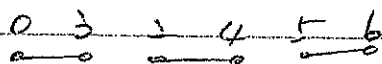
Examples



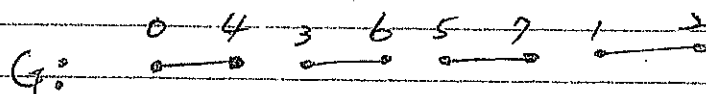
(**) So, in general, we consider the graceful labeling of a connected graph since for tree $|G| = \|G\| + 1$. In order that f to be 1-1 $\|G\| \geq |G| - 1$.

(*) If $\|G\| < |G|$, then, we use $f: V(G) \xrightarrow{1-1} \{0, 1, 2, \dots, |G|\}$ instead.

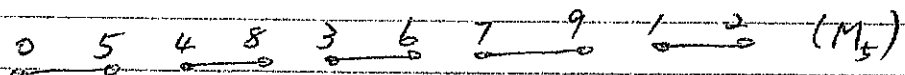
Example



(M_3)



(M_4)



(M_5)

(*) A graph is graceful if it has a graceful (β) labeling.

Conjecture (Ringel-Kotzig) ~ 1980

β : Alex Rosa (1967)

Graceful: S.W. Golomb

Every tree is graceful.

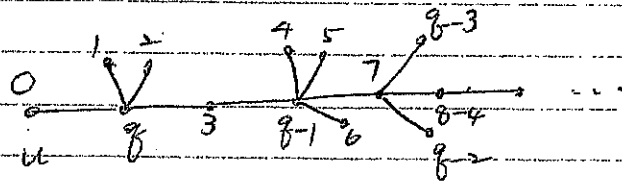
(*) Many results are obtained on special trees. But, no one can prove the conjecture so far.

Theorem

(1) Caterpillars are graceful.

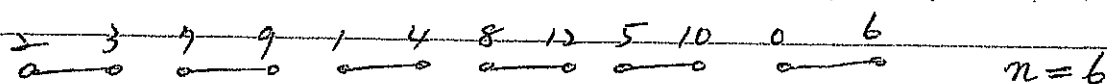
(2) M_n is graceful for all $n \in \mathbb{N}$.

Proof. (1) The proof can be obtained by a direct labeling: start in from the end vertex u as shown in the following figure, let its label be 0, and then its neighbor be q , $\dots$. Then, all weights are distinct.

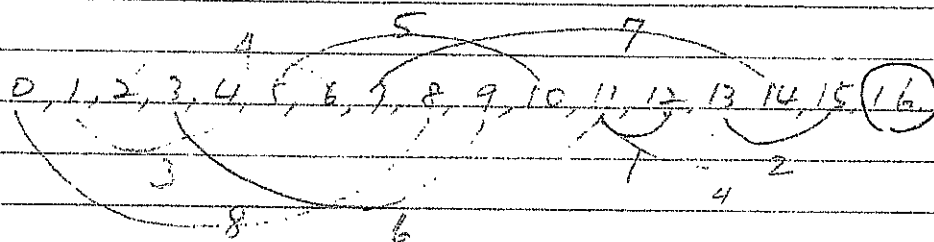
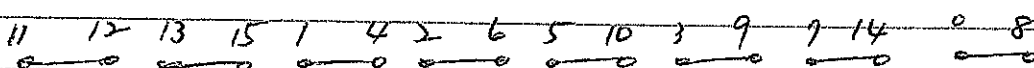


(2) The proof of this result is quite long.

The case when $n \equiv 0$ or $1 \pmod{4}$ is known as Skolem sequence of type A and $n \equiv 2$ or $3 \pmod{4}$ is Skolem sequence here of type B. We shall give some examples and omit the proof.

 $n=6$

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, (11), 12



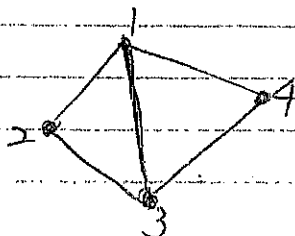
(c) In type A, 16 is not used and in type B, 11 is not used.

(*) Graeful labeling is very useful in constructing Combinatorial designs.

2. Prime labeling

(i) A prime labeling of a graph G of order n is a 1-1, onto mapping $f: V(G) \xrightarrow[onto]{1-1} \{1, 2, \dots, n\}$ such that if $uv \in E(G)$, then $\gcd(f(u), f(v)) = 1$.

Example



Conjecture (Ringel).

Every tree has a prime labeling.

✓(•) Let G be a graph of order n . If $\alpha(G) < \lfloor \frac{n}{2} \rfloor$, then G has no prime labelings.

Proof. All even integers in $\{1, 2, \dots, n\}$ form an independent set in a co- prime graph G_n .

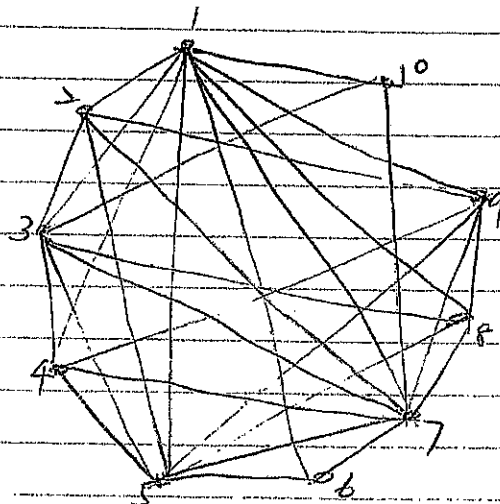
(*) Co-prime graph of order n , G_n .

$$V(G_n) = \{1, 2, \dots, n\} = [n].$$

$$\{i, j\} \in E(G_n) \text{ if } i, j \in [n] \text{ and } \gcd(i, j) = 1.$$

(••) If a graph G_n ^{of order n} has a prime labeling, then $G \leq G_n$.

$n=10$

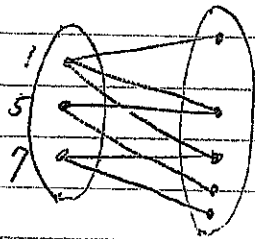


(*) For each n , there exists a prime in $(n, 2n)$. This is known as Bertrand's Postulate or sometimes Bertrand-Chelyshev Theorem.

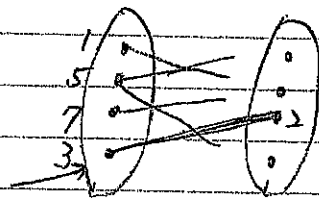
(*) The number of primes in $[n]$ is $\pi(n) \sim \frac{n}{\log n}$ (base e).

(*) If T is a tree of order 8, then T has a prime labeling.

Proof Case 1 $T = (A, B)$ and $|A| \leq 7$.



Case 2 $|A| = |B| = 4$



degree 1

(By Pigeon-hole principle)

(*) If G is a bipartite graph of order 8 and $G \not\cong K_{4,4}$, then G has a prime labeling.

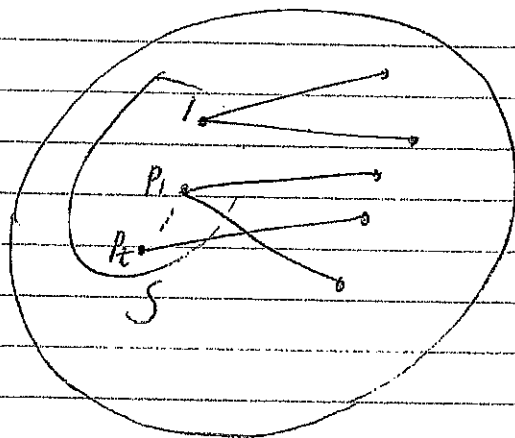
(*) If G has a prime labeling and S is a set of edges in G , then $G - S$ has a prime labeling.

Let $P(a, b)$ denote the set of primes between integers a and b .

Theorem If G is a graph of order n and S is a vertex cover of G such that $|S| \leq P(\lceil \frac{n}{2} \rceil, n) + 1$, then G has a prime labeling.

Proof. Let S be labelled by 1, and primes in $P(\lceil \frac{n}{2} \rceil, n]$. Then, the proof follows. (Since S is a vertex cover, no edges exist

in $\langle G - S \rangle_{G^0}$.)



接 b', b''

3. Magic Labelings (on Edges)

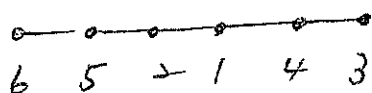
G has a magic labeling if there exists a 1-1 mapping $f: E(G) \rightarrow \{1, 2, 3, \dots, \|G\|\}$ such that for each v , the weight of v , defined by $\sum_{u \in N_G(v)} f(uv)$, is a constant.

(*) If all weights are different, then we have an anti-magic labeling.

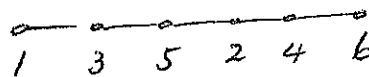
Prime sum labeling (Prime difference labeling)

A mapping $f: V(G) \xrightarrow{1-1} \{1, 2, \dots, |G|\}$ is called a prime sum (difference) labeling of G if for each edge $uv \in E(G)$, $f(u) + f(v)$ is a prime (if $f(u) - f(v) \neq 0$).

e.g.



prime sum labeling



prime difference labeling

Problem 1

Does C_n have a prime sum labeling for each $n \in \mathbb{N}$?

(1, 2, 3, 4), (1, 6, 5, 2, 3, 4), (1, 6, 7, 4, 3, 8, 5, 2), (1, 6, 7, 4, 3, 8, 5, 12, 11, 2)

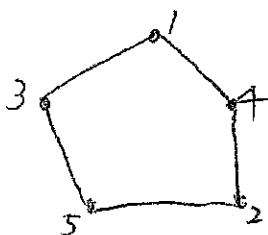
Problem 2

Does C_n have a prime difference labeling for each $n \in \mathbb{N}$?

Fact 1 If n is odd, then C_n does not have a prime sum labeling.

Fact 2 If $n = 4$, C_n has a prime sum labeling, but not a prime difference labeling.

Fact 3 If $n = 5$, then C_n has a prime difference labeling



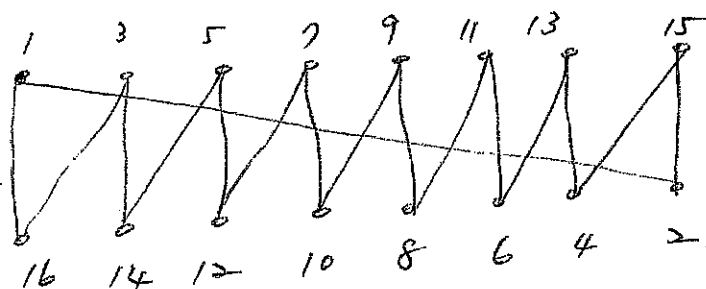
Proposition

If p and $p+2$ are primes, then C_{p-1} has a prime sum labeling provided $p \geq 5$.

Proof. The labeling can be obtained directly:

$$(1, 2, p-2, 4, p-4, 6, p-6, 8, \dots, p-(p-5), p-3, 3, p-1).$$

eg. $p=17$



Conjecture For each even integer $n \geq 4$, C_n has a prime sum labeling.

Proposition

For each $n \geq 5$, the Hamilton path P_n has a prime difference labeling.
cycle(C_n)?

Proof. First, we consider $n \in \{5, 6, 7, 8, 9\}$, the solutions are

$\langle 1, 3, 5, 2, 4 \rangle$, $\langle 1, 3, 5, 2, 4, 6 \rangle$, $\langle 1, 3, 5, 2, 7, 4, 6 \rangle$, $\langle 1, 3, 8, 5, 2, 7, 4, 6 \rangle$

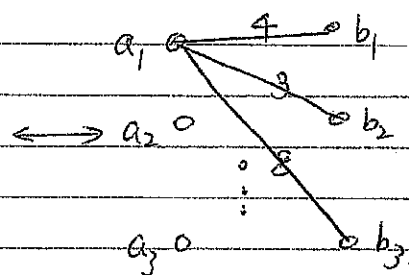
and $\langle 1, 3, 8, 5, 2, 7, 9, 4, 6 \rangle$. Then, the proof follows by put paths together. (?)

$n=1$ and

Example For each $n \geq 3$, $K_{n,n}$ has a magic labeling.

- (*) A magic labeling of $K_{n,n}$ is corresponding to the existence of a magic square (only rows and columns are required to be the same sum).
(weak)

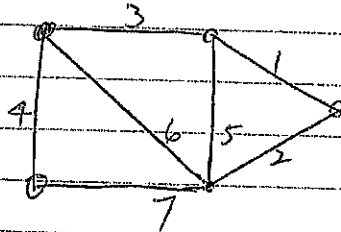
	b_1	b_2	b_3
a_1	4	3	8
a_2	9	5	1
a_3	2	7	6



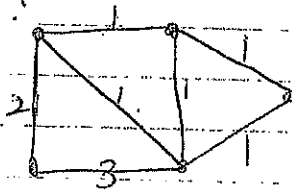
- (*) For $n \neq 2, 6$, we can use a pair of orthogonal Latin squares to construct a magic labeling of $K_{n,n}$.

1. Anti-magic Labelings

G has an anti-magic labeling if there exists a 1-1 mapping $f: E(G) \rightarrow \{1, 2, \dots, \|G\|\}$ such that for each v , the weights $\sum_{u \in N_G(v)} f(uv)$ are all different.

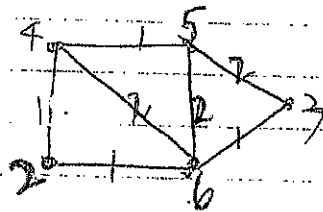


(*) We may also use $f: E(G) \rightarrow \{1, 2, \dots, k\}$ to obtain an anti-magic labeling. $k \leq \|G\|$ in general



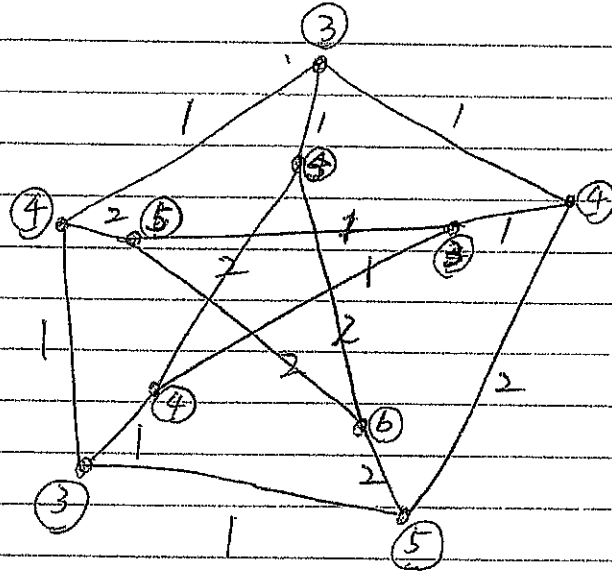
5. Vertex-distinguished labeling

We only require the weight of u is different from the weight of v when uv is an edge of G .



Conjecture $k=3$ is enough!

One more example



Conjecture For all graphs G , a vertex-distinguished labeling, $\{1, 2, 3\}$ -labeling, (Use 1, 2, 3 only!) can be obtained by a

基礎圖論介紹到這裏, 希望你(妳)有興趣, 繼續學習。

Happy New Year!

Merry Christmas!

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