

Lecture 9 Dec. 5, 7

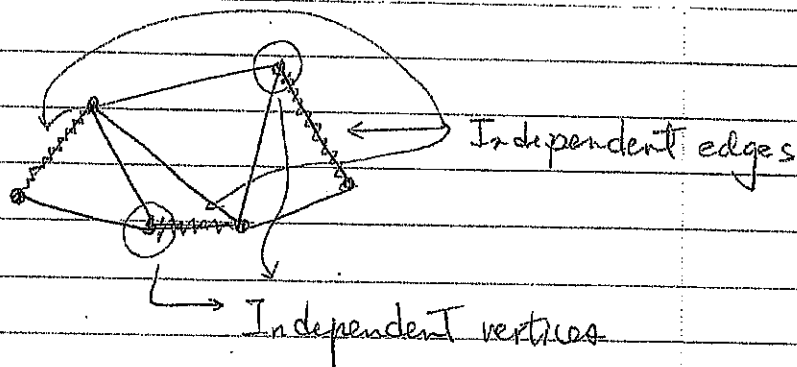
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Independent Sets and Matchings

Definition

A set of vertices is called independent if no two of its vertices (edges) are adjacent. (incident)



Definition (Matching)

A set of independent edges is also known as a matching.

Remark If $\chi(G) = k$, then G has k independent sets which partition $V(G)$. If $\chi'(G) = k$, then G has k matchings which partition $E(G)$.

(*) We are aiming at finding an independent set of a graph which is of maximum size. The size of an independent set (of vertices) is called the independence number of G , denoted by $\alpha(G)$.

Easy facts: $\alpha(K_n) = 1$, $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha(T) \geq \lfloor \frac{|T|}{2} \rfloor$.
(T is a tree.)

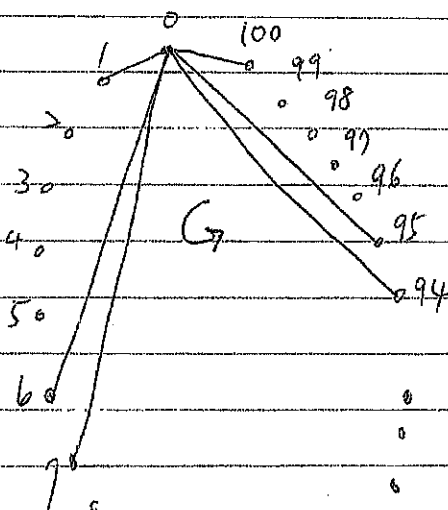
Proposition Let $\chi(G) = k$. Then, $\alpha(G) \geq \frac{|G|}{k}$, or equivalently

$$\chi(G) \cdot \alpha(G) \geq |G|.$$

(*) Finding $\alpha(G)$ for a general graph G is very difficult.

(*) Bonus problem

Let G be a graph defined on $\mathbb{Z}_{101}$. Two vertices i and j are adjacent if and only if $i - j \in \{1, 6, 7, -1, -6, -7\} \pmod{101}$. Find $\alpha(G)$.



Definition (Matching)

A matching of a graph G is a set of independent edges.

We use $\alpha_1(G)$ to denote the maximum size of a matching.

Clearly, a matching has at most $\frac{|G|}{2}$ edges. If M is a matching of size $\frac{|G|}{2}$ when $|G|$ is even, then M is called a perfect matching of G and the graph induced by M is called a 1-factor.

Definition (k -factor)

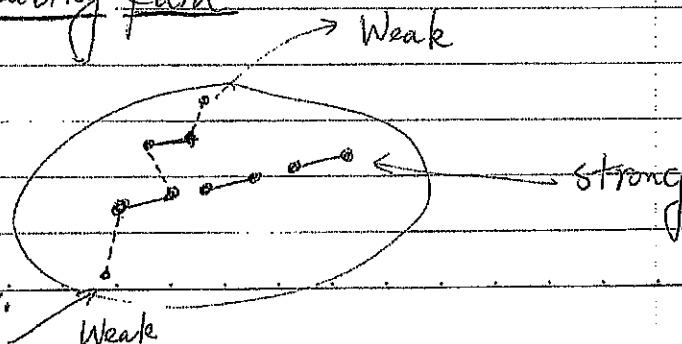
spanning k -regular

A subgraph of a graph G is called a k -factor.

(*) Finding $\alpha_1(G)$ of a general graph G is comparatively easy. (There exists a polynomial time algorithm to find $\alpha_1(G)$.)

Idea: Alternating path

We can enlarge a matching if we have a path: Weak-Weak-



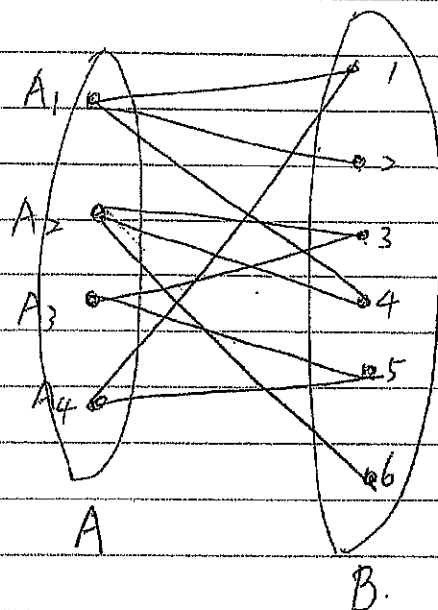
✓ Marriage Problem

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Given a bipartite graph $G = (A, B)$, what is the condition for which G contains a matching saturates A , i.e., a matching incident to every vertex of A .



$$A_1 = \{1, 2, 4\}$$

$$A_2 = \{3, 4, 6\}$$

$$A_3 = \{3, 5\}$$

$$A_4 = \{1, 5\}$$

(*) The reason why this is called "marriage problem" follows from the fact that every "one" in A finds a mate in B , but not the same mate.

(**) This problem can be thought as finding representatives from a collection of sets.

Theorem (P. Hall, 1935)

1. Graph Version Let $G = (A, B)$ be a bipartite graph with

$A = \{a_1, a_2, \dots, a_m\}$. Then G contains a matching saturates A

if and only if for any $1 \leq k \leq m$, $|\bigcup_{j=1}^k N_G(a_i)| \geq k$.

2. Set Version Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a collection of

subsets of $[1, n]$. Then, there exists a vector $(a_1, a_2, \dots, a_m)$

where $a_i \in A_i$, $i = 1, 2, \dots, m$ and $a_i \neq a_j$, for $1 \leq i \neq j \leq m$, if and

only if for any $1 \leq k \leq m$, $|\bigcup_{j=1}^k A_{i_j}| \geq k$. ----- (1)

Remark In set version, $(a_1, a_2, \dots, a_m)$ is called a system of

distinct representatives (SDR in short) and the condition

(1) is known as the Hall's condition.

We shall prove the set version in what follows.

Proof. $(\Leftarrow)$ By induction on the number of sets in $\mathcal{A}$, i.e., m .

Clearly, it is true for $m=1$. Assume that the assertion is

true for $1 \leq m \leq h$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_{h+1}\}$ be a collection

of $n+1$ sets satisfying (1). Now, we consider two cases.

Case 1 For any $1 \leq k \leq n$, the union of k sets in A contains at least $k+1$ elements.

$$A' =$$

Let $a_{n+1} \in A_{n+1}$ and consider $\bigwedge \{A_1 \setminus \{a_{n+1}\}, A_2 \setminus \{a_{n+1}\}, \dots, A_n \setminus \{a_{n+1}\}\}$.

← By observation, condition (1) holds for A' and thus A' has an

SDR $(a_1, a_2, \dots, a_n)$. Combine with a_{n+1} , we have an SDR $(a_1, a_2, \dots, a_{n+1})$

for A .

Case 2 There exists a $1 \leq k \leq n$, such that the union of some k sets contains exactly k elements, say $A' = \{A_1, A_2, \dots, A_k\}$.

Let $S = \bigcup_{i=1}^k A_i$. Consider the collection of sets $A'' = \{A_{k+1} \setminus S, A_{k+2} \setminus S, \dots, A_n \setminus S\}$.

Now, both A' and A'' satisfy (1). By induction and the combination

of their SDR's respectively, we conclude the proof.

(Note that any $\bigwedge$ k subsets of A'' contains at least k elements, for

otherwise we have a contradiction. (?))

(For example, if $(A_{k+1} \setminus S) \cup (A_{k+2} \setminus S)$ contains only one element,

the union of

then $\bigwedge A_1, A_2, \dots, A_k, A_{k+1}, A_{k+2}$ will contain only $k+1$ elements. $\rightarrow$ (1))

Proof of Hall's Theorem (Graph version)

($\Rightarrow$) Easy to see.

($\Leftarrow$) Suppose not. That's, if $G=(A,B)$ satisfies Hall's condition, there exists a maximum matching M of size smaller than $|A|$, i.e., $|M| < |A|$. Let $a \in A$ and a is not incident to edges of M (see Fig. below). Clearly, $N_G(a) \subseteq B'$, for otherwise, we have larger size matching (a is saturated).

Let $S \subseteq A$ be the set of vertices which are reachable from a by M -alternating paths in A and T in B resp.

(*) T 中的点如果再接回不在 $M(A)$ 的边, 则会产生

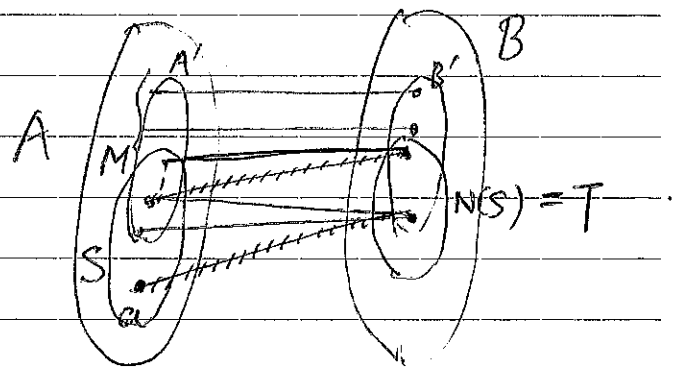
一条 $a \cdots b \xrightarrow{a'} \cdots$ 的路径, 因此, 利用

交换的方式可以找到

$|M|+1$ 的 matching. $\rightarrow \Leftarrow$

因此每一边都接到 M 的边,
(T)

也就是 $S \setminus \{a\}$ 的点, $|T| = |N(S)| = |S| - 1 < |S|$. $\rightarrow \Leftarrow$



Theorem (König, ~1930)

Let G be a k -regular bipartite graph. Then, G has a matching saturates A .

Proof. Note that if $G = (A, B)$ is regular, then $|A| = |B|$.

($|E| = k \cdot |A| = k \cdot |B|$) Let $A = \{a_1, a_2, \dots, a_n\}$ and

$B = \{b_1, b_2, \dots, b_m\}$. Now, we shall use Hall's Theorem to prove the theorem.

Let S be a subset of A , $S = \{a_1, a_2, \dots, a_s\}$. By assumption,

$N_G(S) = \bigcup_{i=1}^s N_G(a_i) = \{b_1, b_2, \dots, b_t\} = T$ and $\sum_{i=1}^s |N_G(a_i)| = k \cdot s$.

Hence, $\sum_{i=1}^s |N_G(a_i)| = \sum_{j=1}^t |N_{\langle S, T \rangle_G}(b_j)| = k \cdot s$. But,

$|N_{\langle S, T \rangle_G}(b_j)| \leq k$. This implies that $t \geq s$, and

thus $|N_G(S)| \geq |S|$. The proof follows by using Hall's Theorem.

—— (再叙述一次)

Discrete Version of Hall's Theorem (Set Version)

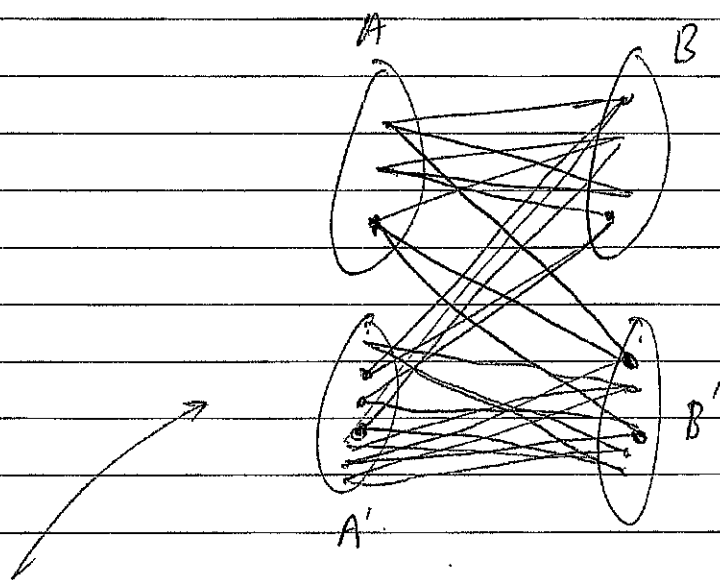
Definition (System of Distinct Representatives, SDR)

Let $\mathcal{S}$ be a collection of subsets of S , $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$.

Theorem If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.

Proof Claim We can add more vertices if necessary to $G = (A, B)$ to obtain a $\Delta(G)$ -regular bipartite graph $\tilde{G}$. (?)

Then, by using the proof in p.13, $\tilde{G}$ has an edge-coloring using $\Delta(G)$ colors. Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) = \Delta(G)$.
This implies that $\chi'(G) = \Delta(G)$.



Can you find a way to prove this claim?

$$(a_1, a_2, \dots, a_n)$$

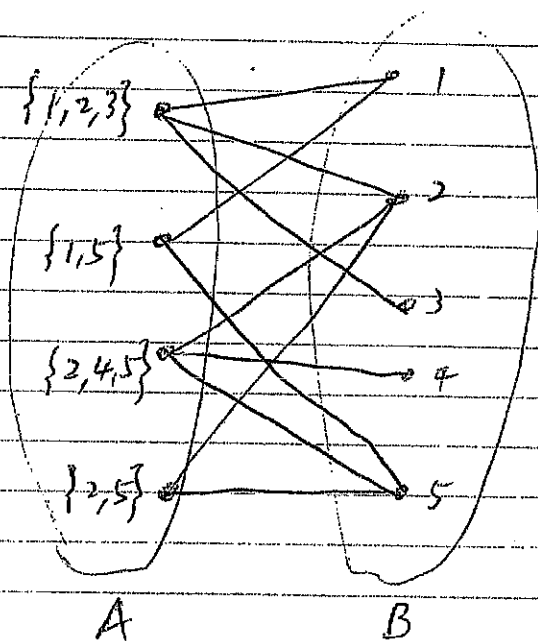
An ordered n -tuple is called a system of distinct representatives (SDR) if

$a_i \in S_i$, $i=1, 2, \dots, n$; and all a_i 's are distinct.

e.g. $S = \{\{1, 2, 3\}, \{1, 5\}, \{2, 4, 5\}, \{2, 5\}\}$ and $S = \{1, 2, 3, 4, 5\}$.

Then, $(1, 5, 4, 2)$ is an SDR of S .

(i) The relationship between sets and graphs.



Let A be the vertex set
each $\underset{\text{vertex}}{\wedge}$ of them is the set
in S and B be the set
of elements in S .

(i) e is an edge of (A, B)
if and only if the
containment holds.

Applications

- Let $M_{n \times n}$ be a $(0, 1)$ -matrix such that each row and each column has exactly k 1's. Then, there exists a permutation matrix hidden in M , i.e., a sub-matrix with exactly one

$$\begin{bmatrix} 1 & 0 & \textcircled{1} \\ \textcircled{1} & 1 & 0 \\ 0 & \textcircled{1} & 1 \end{bmatrix}$$

2. In fact, M can be partitioned into k permutation matrices

3. The above result can be transformed to a k -regular bipartite graph (A, B) with $|A| = |B| = n$.

4. This graph $G = (A, B)$ is of Class 1, i.e., $\chi'(G) = k$.

(*) Let G be a bipartite graph, Then, $\chi'(G) = \Delta(G)$.

Can you prove this?