

FGT Lecture 8

Nov, 28, 30

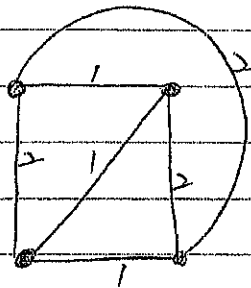
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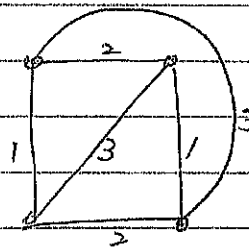
Edge-coloring

Definition A k -edge-coloring π of a graph G is a mapping from $E(G)$ into $\{1, 2, \dots, k\}$. π is a proper k -edge-coloring if incident edges receive distinct image, i.e., $\pi(e) \neq \pi(f)$ provided $|e \cap f| = 1$.

Examples



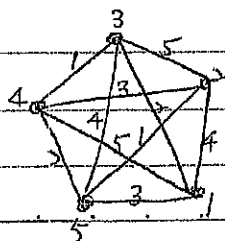
2-edge-coloring



proper 3-edge-coloring

Fact 1 If G has a proper k -edge-coloring, then $k \geq \Delta(G)$.

Fact 2 If K_5 has a proper k -coloring, then $k \geq 5$.



(Each color occurs at most "on" 2 edges!)

Fact 3 A graph G is "overfull" if $\Delta(G) \cdot \lfloor \frac{|G|}{2} \rfloor < \|G\|$.
If G is overfull, then $\chi'(G) > \Delta(G)$.

Definition (Chromatic index)

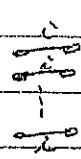
The chromatic index of a graph G is the minimum number k such that G has a proper k -edge-coloring.

The chromatic index of G is denoted by $\chi'(G)$.

In what follows, all edge-colorings are "proper".

Fact 3 $\chi'(K_n) = \begin{cases} n-1, & n \text{ is even;} \text{ and} \\ n, & n \text{ is odd.} \end{cases}$

Remark If M_i is a set of edges in G colored " i ", then M_i

is a matching, i.e., a set of independent edges. 

Proof of Fact 3

Let $V(K_n) = \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\} \pmod{n}$. Let $\pi(\{i, j\}) = t$
Consider n is odd.

provided $i + j \equiv 2t \pmod{n}$. Then π is an n -coloring for

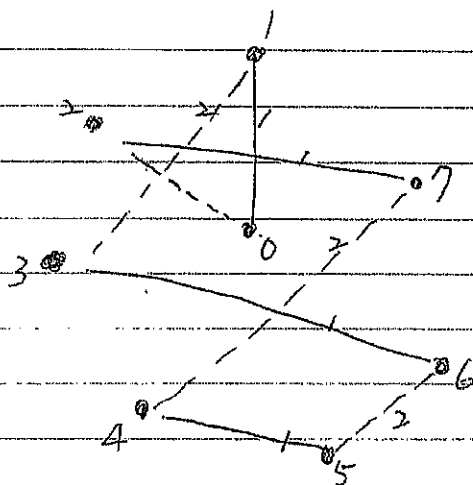
the case n is odd. This implies that $\chi'(K_n) \leq n$ when

n is odd. Since each color can occur at most $\frac{n-1}{2}$ times,

$\chi'(K_n) \geq \left\lceil \frac{\binom{n}{2}}{\frac{n-1}{2}} \right\rceil = n$. Hence, $\chi'(K_n) = n$ if n is odd.

n is even

Since $\Delta(K_n) = n-1$, $\chi'(K_n) \geq n-1$ (?). So, it suffices to find an $(n-1)$ -edge-coloring of K_n when n is even. The coloring pattern can be seen as the following case $n=8$.



In fact, the coloring π can be $\pi(\{i, j\}) = t$ if (1) $i = 0$ and $j = t$ (2) $i + j \equiv 2t \pmod{n-1}$. Since we have an $(n-1)$ -edge-coloring of K_n (n is even), $\chi'(K_n) \leq n-1$. $\square$

Important!

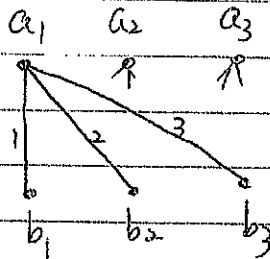
✓ Fact 4 $\chi'(K_{m,n}) = \max\{m, n\}$, Consider $m \leq n$.

It suffices to prove that $\chi'(K_{n,n}) = n$. Since an n -edge-coloring

of $K_{n,n}$ is corresponding to the existence of a Latin square of order n , the proof follows. (?) SEAS

Example

	b_1	b_2	b_3
a_1	1	2	3
a_2	2	3	1
a_3	3	1	2



Latin Square

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3

Latin Rectangle

Definition A Latin square of order n based on $[1, n]$ is an $n \times n$ array such that each element from $[1, n]$ occurs exactly once in each row and each column.

Theorem (Vizing, 1964) (The proof is omitted in this course.)

If G is a simple graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Definition (Class 1 and Class 2)

A graph G is of Class 1 if $\chi'(G) = \Delta(G)$ and Class 2 otherwise.

(*) Determine whether a graph is of Class 1 or 2 is a very difficult but important problem in Chromatic Theory.

~~Viz~~ Use the non-existence of Hamilton cycles in the Petersen graph!

Proposition Petersen graph is of Class 2.

(1st)

Proof. Note that Petersen graph is a 3-regular graph. Let

P denote the graph. Then $\Delta(P) = 3$. Assume that P is of

Class 1, i.e., we have $\chi'(P) = 3$. Now, consider the subgraph G of

P induced by the edges colored 1 and 2. By observation, G

is a 2-regular spanning subgraph of G . The only possibility

for such a graph G is a Hamilton cycle. (?) This is not possible

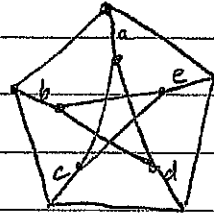
since P contains no Hamilton cycles. $\rightarrow \leftarrow$

(2nd)

Proof. Let the edges a, b, c, d and e be link-edges in Petersen graph, see figure below.

Suppose that $\chi'(P) = 3$.

p:



Then, each color will occur exactly 5 times.

Then, there exists a color "1" in

$\{1, 2, 3\}$, which occurs in link-edges at least two times.

Case 1 "1" occurs 5 times.

In this case the two C_5 's (in and out) must be 2-edge-colorable, not possible.

Case 2 "1" occurs 4 times.

Now, there is no edge which can be colored by 1, not possible.

Case 3 "1" occurs 3 times.

Then, only one edge in the two C_5 's can be colored "1", and we have only 4 1's, not possible.

Case 4 "1" occurs 2 times.

Again, in total "1" can occur 4 times, not possible either.



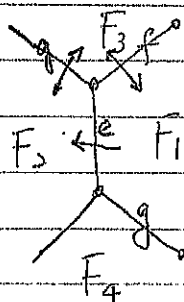
2-connected

✓ Theorem Let G be a planar, ^{2-connected} cubic graph, _(3-regular) Then $\chi'(G) = 3$.

Proof. Since G is planar, $\chi_f(G) \leq 4$. So, we can use $\{(0,0), (0,1), (1,0), (1,1)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ to color the faces such that incident faces

(Face coloring)

receive distinct colors. Now, if e is the boundary of faces F_1 and F_2 , we color e by using the addition, defined on $\mathbb{Z}_2 \times \mathbb{Z}_2$.



Clearly, we use only three colors ⁱⁿ $\{(0,1), (1,0), (1,1)\}$ since adjacent faces will receive distinct colors and thus their sum is not $(0,0)$.

Moreover, if e and f are incident edges, then the corresponding faces adjacent to F_1 will be different, see figure above. This implies

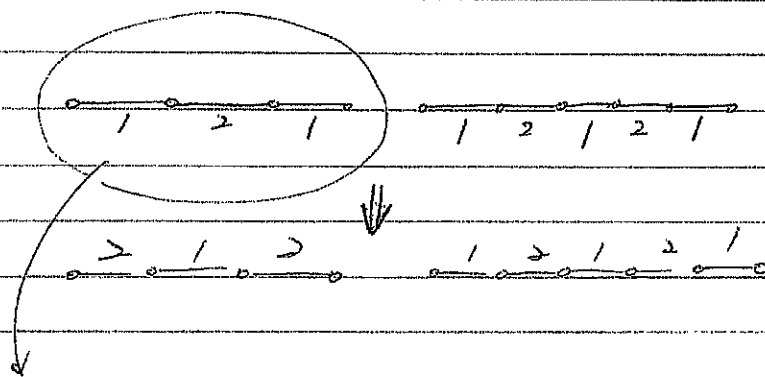
the colors of e and f are distinct. (e, f, g are of distinct colors!)

(*) If we don't know $\chi_f(G) \leq 4$, then the proof of this theorem is very difficult. (!)

(**) If G has a k -edge-coloring, then G has an equitable k -edge-coloring such that the difference of colors occurred in the coloring is at most 1. $\left(\left| |\varphi^{-1}(i)| - |\varphi^{-1}(j)| \right| \leq 1 \right)$

Proof. By color-switching method.

Remark. If color 1 occurs five times and color 2 occurs three times, then we put these eight edges together to induce a subgraph, see the figure below.



This configuration exists if we have more color "1" than color "2".

(*) 接近差距大于或等于2的距離。