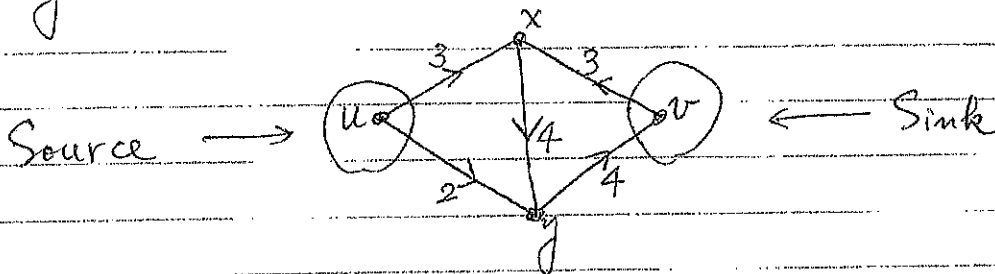


NetworksDefinition

A network N is a digraph D with two distinguished vertices u and v , called the source and sink of N , resp., and a nonnegative integer-valued function c on $A(D)$, i.e. $c: A(D) \rightarrow \mathbb{R}^+ \cup \{0\}$. The digraph is called the underlying digraph of N . The function c is the capacity function of N and its value $c(\vec{a}) = c(x, y)$ on the arc $\vec{a} = (x, y)$ is referred to as the capacity of $\vec{a}$.

e.g.

Definition (Flow)

A flow in a network N (with underlying digraph D , source u , sink v , and capacity function c) is an integer-valued

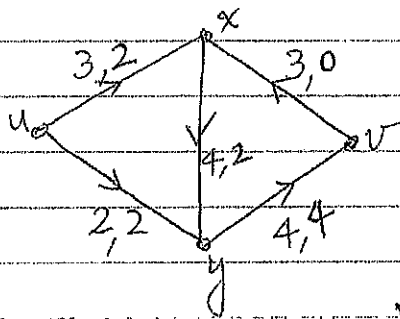
function f on $A(D)$ such that for every $\vec{a} \in A(D)$,

$$0 \leq f(\vec{a}) \leq c(\vec{a}), \text{ and}$$

$$\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x) \quad \forall x \in V(D) \setminus \{u, v\}. \quad \dots (*)$$

$(*)$ is known as the "conservation equation" of the network.

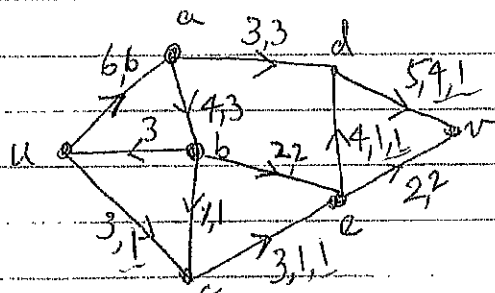
Example of network with flow
(1)



val f: The value of a flow f in a network, which is defined as the net flow out of the source of N ; equivalently, the net flow into the sink of N .

$$\text{val } f = 4$$

Another example (2)



What is "maximum flow" from u to v ?

Notation (X, Y)

$$(X, Y) = \{(x, y) \in A(D) \mid x \in X, y \in Y\}.$$

Definition (Cut)

Let N be a network with underlying digraph D , source u , sink v . A cut in N is $(X, \bar{X})$ where $\bar{X} = V(D) \setminus X$, $u \in X$ and $v \in \bar{X}$.

Definition (Capacity of a cut)

Let $K = (X, \bar{X})$ be a cut of network N with capacity c . Then, the capacity of the cut K , denoted $\text{cap } K = c(X, \bar{X})$

$$= \sum_{(x, y) \in (X, \bar{X})} c(x, y).$$

In example 12).

Let $X = \{u, a, b, c\}$, then $c(X, \bar{X}) = 8$.

$\bar{X} = \{d, e, v\}$. (How about we change the direction of (b, e) to (e, b) ?)

Theorem Let f be a flow in a network N and let

$K = (X, \bar{X})$ be a cut in N . Then

$$\text{val } f = f(X, \bar{X}) - f(\bar{X}, X) \leq \text{cap } K.$$

Minimum cut

A cut K in a network N is called a minimum cut if $\text{cap } K \leq \text{cap } K'$ for every cut K' in N .

(Fact) Let f be a flow and K be a cut in N . If $\text{val } f = \text{cap } K$, then f is a maximum flow and K is a minimum cut.

(A flow f in N is called a maximum flow if $\text{val } f \geq \text{val } f'$ for every flow f' in N .)

Proof. It can be proved (?) that if f^* is a maximum flow and K^* is a minimum cut, then $\text{val } f^* \leq \text{cap } K^*$. Now,

f is a flow, hence $\text{val } f \leq \text{val } f^*$; K^* is a minimum cut,

hence $\text{cap } K^* \leq \text{cap } K$. Since $\text{val } f = \text{cap } K$ and

$\text{val } f \leq \text{val } f^* \leq \text{cap } K^* \leq \text{cap } K$, we conclude

that $\text{val } f = \text{val } f^*$ and $\text{cap } K = \text{cap } K^*$. ■

(*) If there exists a K and an f s.t. $\text{cap } K = \text{val } f$, then K is a minimum cut and f is a maximum flow.

(*) In the following theorem, let s be the source and t be the sink.

Theorem. (Ford and Fulkson, 1956-1962)

In any network N defined on D , the value of a maximum flow equals the capacity of a minimum cut.

Proof.

Clearly, if there exist no cuts such that its capacity of the cut is $\text{val } f$, then f does not exist. (?) So, it suffices to claim that if the value of a maximum flow f is v , then there exists a cut K , such that $\text{cap } K = \text{val } f = v$.

Define a subset $S \subseteq V(D)$ recursively as follows. Let $s \in S$. If $x \in S$, and $c(x, y) > f(x, y)$ or $f(y, x) > 0$, then let $y \in S$. We shall prove that (S, S') is a cut with capacity v . First, we claim $t \notin S$. Suppose not, i.e., $t \in S$. Hence, we can find a sequence of vertices in N such that $s = x_0, x_1, \dots, x_l = t$. Moreover, if we let $\varepsilon_i = \max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\}$, $i = 0, 1, \dots, l-1$, then $\varepsilon_i > 0$. Let $\varepsilon = \min\{\varepsilon_i\}$. Now, let $f^*(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \varepsilon$ if $c(x_i, x_{i+1}) - f(x_i, x_{i+1}) = \varepsilon_i > 0$ and $f^*(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \varepsilon$ if $f(x_{i+1}, x_i) = \varepsilon_i > 0$. As a consequence, f^* is a flow from s into t with value $\text{val } f^* = v + \varepsilon$, a contradiction. (See Figure 1) By the definition of a flow,

$$\text{val } f = v = \sum_{x \in S, y \in S'} f(x, y) - \sum_{x \in S', y \in S} f(x, y). \quad \text{---(1)}$$

6-4''

Again, by the definition of S , if $x \in S$ and $y \in S'$, then $c(x, y) = f(x, y)$ and $f(y, x) = 0$. This implies that (1) = $\sum_{x \in S, y \in S'} c(x, y) = \text{cap}(S, S') = v$, the proof follows. ■

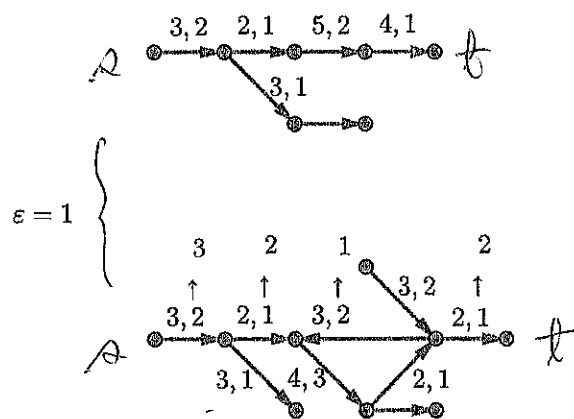


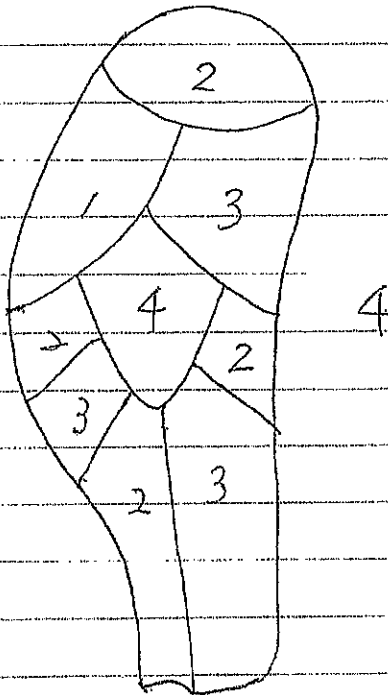
Figure 5.1/ augmenting flow f^*

Graph Coloring

NO.

6-5

DATE



Map Coloring Problem

1852, Oct 23, Augustus De Morgan ^{sent a letter} → William Rowan Hamilton
(University College, London) (Trinity College, Dublin)

... My pupil says he guessed 4 colors are enough to

color the compartments differently so that figures with any
(of England)

portion of common boundary are differently colored. ...

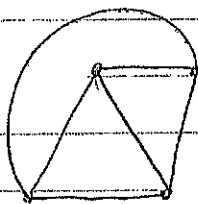
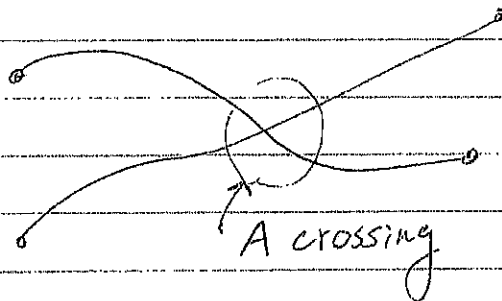
Student: Frederick Guthrie

Learned this idea from his brother: Francis Guthrie
(He provided an incomplete proof.)

Definition (Region Colorable)

A plane graph G is said to be n -region colorable if the regions of G can be colored with n or fewer colors so that adjacent regions are colored differently.

Note A plane graph G is a graph which can ^{be} drawn on a plane such that no two edges have a crossing.



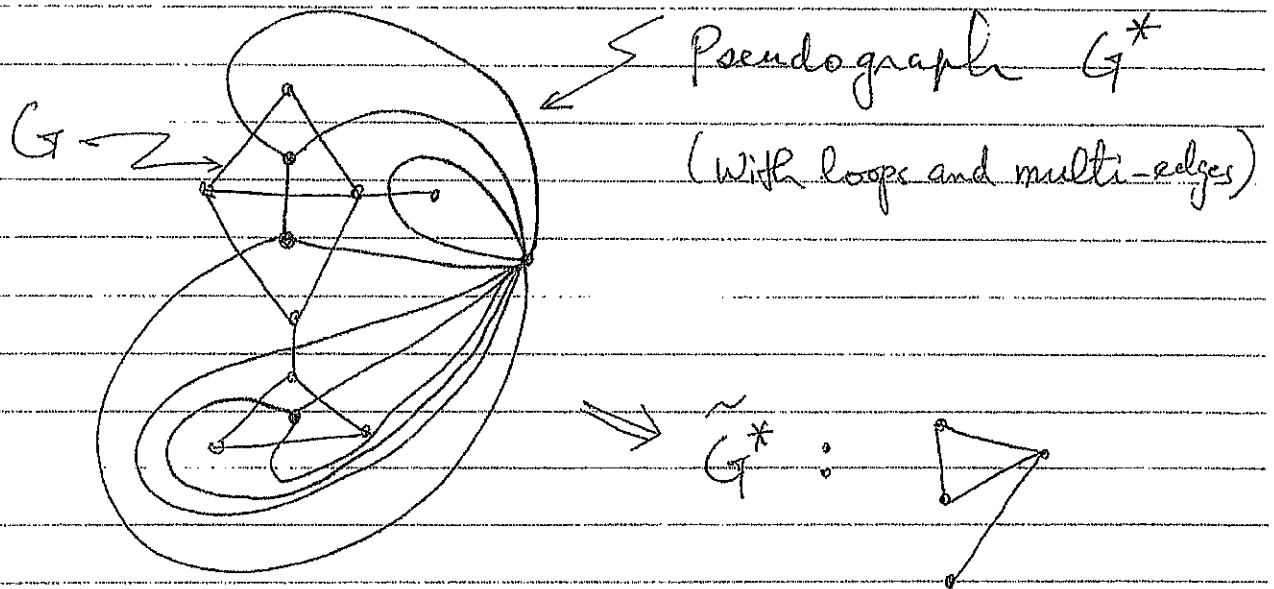
K_4 is a plane graph, but not K_5 .

Remark We shall talk about crossings of graphs later.

The Four Color Conjecture

Every map (plane graph) is 4-region colorable.

Now, it is known as the 4-color Theorem, 4CT in short.



Step 1 For each region of a plane graph, define a vertex.

Step 2 Join two vertices with an edge (or loop) if they have a common boundary.

Step 3 This pseudograph is known as the dual of the plane graph (G^*) .

Fact $(G^*)^* \cong G$.

Fact $\tilde{G}^*$ is called the underlying graph of G^* if loops are deleted and replace the multiple edges by a single edge.

So, we can use the vertex coloring of $\tilde{G}^*$ to color the regions of a plane graph G .

Definition (Vertex coloring)

A vertex k -coloring φ is a mapping $\varphi: V(G) \rightarrow [1, k]$ such
(k -coloring in short)

that $\varphi(u) \neq \varphi(v)$ if uv is an edge of G . The minimum

number k is called the chromatic number of G denoted by
such that G has a k -coloring

$\chi(G)$. (If G has a k -coloring, G is said to be k -colorable.)

(Fact 1) If $G \geq K_k$, then $\chi(G) \geq k$.

(Fact 2) If G is a bipartite graph, then $\chi(G) = 2$ provided
 G contains at least one edge.

(Fact 3) If G has a k -coloring $\varphi: V(G) \rightarrow \{1, 2, \dots, k\}$, then
for each $c \in [1, k]$, $\varphi^{-1}(c) \subseteq V(G)$ is an independent
set.

(Fact 4) $\chi(K_n) = n$ and $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even; and} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

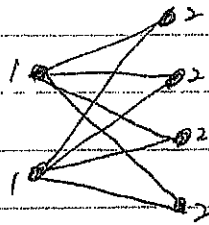
(Fact 5) If $h \geq k$ and G is k -colorable, then G is also h -colorable.

Remark 1 If G has a k -coloring φ such that

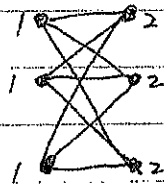
$$||\varphi^{-1}(c)| - |\varphi^{-1}(d)|| \leq 1 \text{ for any } c, d \in [1, k], \text{ then } \varphi$$

is an equitable k -coloring.

Remark 2 If G has a k -coloring, then G "may not" have an equitable k -coloring or equitable $(k+1)$ -coloring.



$$\chi(K_{2,4}) = 2$$



$$\chi(K_{3,3}) = 2$$

No equitable 3-coloring of

$K_{3,3}$ exists. (?)

Definition (critically n -chromatic)

For each $n \geq 2$, G is critically n -chromatic if $\chi(G) = n$

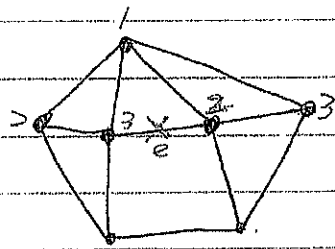
and $\chi(G - v) = n - 1$ for all $v \in V(G)$.

Example K_n is critically n -chromatic.

C_{2n+1} is critically 3-chromatic.

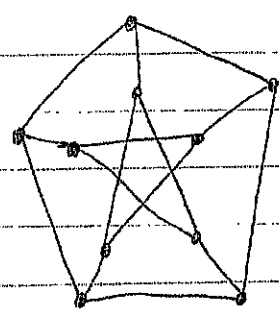
Definition (Minimally n -chromatic)

A graph G is minimally n -chromatic if $\chi(G) = n$ and $\chi(G - e) = n - 1$ for all edges $e \in E(G)$.



Critically 4-chromatic

Not minimally
4-chromatic
 $\chi(G - e) = 4$



Not critically 3-chromatic

Not
minimally
3-chromatic

Theorem (Brooks) 1941

If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.

Proof. (See 6-13)

(*) It is easier to consider the graphs which ~~are~~ are not regular.
connected

Theorem (Halin, ¹⁹⁶⁷ Szekeres and Wilf)

For every graph G , $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraph G' of G . (Notice that if G is not a regular graph, then $\chi(G) \leq 1 + \delta(G) \leq \Delta(G)$.)

Proof. First, we claim that if G is critically n -chromatic, then

$\delta(G) \geq n-1$. Suppose not. Let $\delta(G) \leq n-2$ and $\deg(v) = \delta(G)$ where

$v \in V(G)$. By assumption, $\chi(G-v) = n-1$. Now, G has an $(n-1)$ -coloring

since v can be colored with the color missing in its neighbors at most

$(n-2)$ vertices. This is a contradiction to $\chi(G) = n$.

Now, we are ready for this theorem. Let H be an induced

subgraph of G which is n -critical. (We can find this graph by

deleting vertices if necessary. So, $\delta(H) \leq \max_{G' \leq G} \delta(G')$. By

the fact that $\delta(H) \geq n-1$, $\max_{G' \leq G} \delta(G') \geq n-1 = \chi(G)-1$.

Hence $\chi(G) \leq 1 + \max_{G' \leq G} \delta(G')$. □

Theorem (Gallai, 1968)

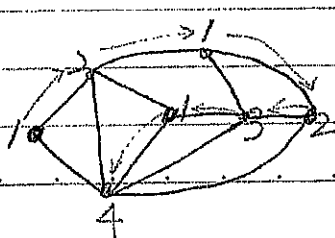
For any graph G , $\chi(G) \leq 1 + m(G)$, where $m(G)$ denotes the length of a longest path in G . (If G does have a Hamilton path, then $\chi(G)$ is smaller.)

Proof. If H is an n -critical induced subgraph of G , then $\delta(H) \geq n-1$ and therefore H contains a path of length $n-1$.

This implies that $m(G) \geq n-1 = \chi(G)-1$ and $\chi(G) \leq m(G)+1$. □

Greedy Coloring Algorithm

Let $\{1, 2, \dots, k\}$ be the colors we plan to use. Color a vertex with smaller integer if it is available.



(*) A graph G is d -degenerate if for each subgraph H of G , $\delta(H) \leq d$.

Theorem If G is d -degenerate, then $\chi(G) \leq d+1$.

Proof. By induction on $|G|$. Clearly, it's true for $|G|=2$.

Assume that the assertion is true for $|G|=n \geq 2$. Consider a d -degenerate graph G of order $n+1$. Since G is d -degenerate, let $v \in V(G)$ and $\deg_G(v) \leq d$. Consider $G-v$. $G-v$ is also d -degenerate and $|G-v|=n$. By induction $G-v$ is $(d+1)$ -colorable. By the fact that $\deg_G(v) \leq d$, we conclude that G is also $(d+1)$ -colorable by assigning v the color which is missing in its neighbor colored in $G-v$. ■

(*) The proof of Brooks' Theorem can be done by considering two cases either G is regular or G is not regular. The latter case is similar to the proof of the above theorem and the other case needs more effort. (Try yourself!) NAN PAO