

# FGT Lecture 4

4-1

## Distance in Graphs

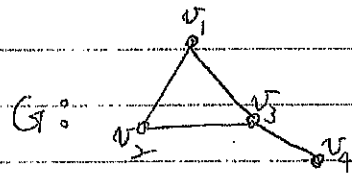
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### Definition (Adjacency Matrix)

Let  $G$  be a graph with  $V(G) = \{v_1, v_2, \dots, v_n\}$ . Then the adjacency matrix  $A_G$  of the graph  $G$  is an  $n \times n$  matrix such that

$$A_G(i,j) = \begin{cases} 1 & \text{if } v_i \sim_G v_j; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Example.



$$A_G = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

### Remark

1.  $A_G$  is a symmetric matrix with "0"-diagonal.

2. The eigenvalues of  $A_G$  can be "found" and they are reals.

(\*) The study of the relationship of eigenvalues of  $G$  and (for distinct matrices of  $G$ )

"structure of  $G$ " is known as "Algebraic Graph Theory".

3. We are interested in "distance", so, we focus on  $A_G^k$ .

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

$v_2 \rightarrow v_3$   
3-walk

## Generalized "Adjacent Matrices"

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$G$  (Simple graph)

$$V(G) = \{v_1, v_2, \dots, v_n\}$$

1. Adjacent Matrix :  $A$ .

2.  $\bar{A} = J - A - I$ .

3. Laplacian Matrix  $L = D - A$ ,  $D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & d_n \end{bmatrix}$ ,

$$d_i = \deg_G(v_i).$$

4. Signless Laplacian Matrix

$$\mathcal{L} = D + A.$$

5. Seidel Matrix  $S = \bar{A} - A = J - 2A - I$ .

(\*) For each matrix mentioned above, we find its

spectrum :  $\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_t \\ m_1 & m_2 & \dots & m_t \end{pmatrix}$ .

of "matrices"



Can we see the graph " $G$ " from its spectrum?

Theorem (Number of  $k$ -walks) (A walk with  $k$  edges is a  $k$ -walk.)

Let  $G$  be the graph with  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $A_G$  be its adjacency matrix. Then, the number of different  $k$ -walks from  $v_i$  to  $v_j$  is equal to  $A_G^k(i, j)$ .

Proof. By induction on  $k$  and it is clear for  $k=1$ .

Assume the assertion is true for  $1 \leq h < k$ . Consider the number

of  $(h+1)$ -walks in  $A_G^{h+1} = (A_G^h)A_G$ . By multiplication of two

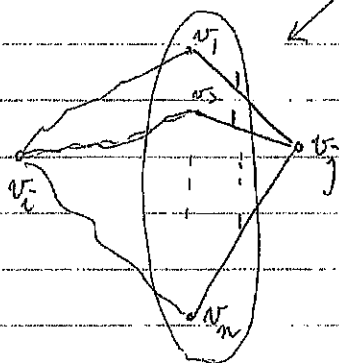
matrices  $A_G^{h+1}(i, j) = (A_G^h(i, 1), A_G^h(i, 2), \dots, A_G^h(i, n)) \cdot \begin{matrix} \swarrow \text{Inner product} \\ (A_G(1, j), A_G(2, j), \dots, A_G(n, j)) \end{matrix}$ .

$$\begin{pmatrix} A_G(1, j), A_G(2, j), \dots, A_G(n, j) \end{pmatrix} = \sum_{t=1}^n A_G^h(i, t) \cdot A_G(t, j) \quad (1)$$

(transpose of the  $j$ -th column)

Since the summation of (1) shows the number of  $(h+1)$ -walks from

$v_i$  to  $v_j$ , we have the proof. ■



Through  $v_1, v_2, \dots, v_n$  respectively

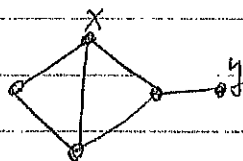
## Facts

1. If  $\deg_G(v_x) = 1$ , then  $A_G^{2m+1}(x, x) = 0$  for  $m = 0, 1, 2, \dots$ .
2. The number of triangles in  $G$  is equal to  $\frac{C_3(G)}{1} = \frac{1}{6} \sum_{i=1}^n A_G^3(i, i) = \frac{1}{6} \text{tr}(A^3)$ .  
(trace)
3. For any two vertices  $v_i$  and  $v_j$  in  $V(G)$ , if there exists a  $k$  such that  $A_G^k(i, j) > 0$ , then  $G$  is a connected graph.
4. The number of 4-cycles in  $G$  is equal to  $C_4(G) = \frac{1}{8} (\text{tr}(A^4) - 2\|G\| - 2 \sum_{i+j} A^2(i, j))$ .
5. More ...  $C_5, C_6, C_7, C_8, \dots$ .

## Review

1. The length of a path (or cycle, walk, trail, circuit) is the number of edges on the path (or ...).
2. The distance of two vertices  $x$  and  $y$  in a graph, denoted  $\text{dist}(x, y)$ , is the length of a shortest path between  $x$  and  $y$ .  
(min. # of edges)

$y$ .



$$\text{dist}(x, y) = 2.$$



## Distance in Graphs



For reference about Metric Space

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### Definition (Metric space)

A metric space (defined on  $M$ ) is an ordered pair  $(M, d)$

where  $M$  is a set and  $d$  is a metric on  $M$ , i.e., a function

$d: M \times M \rightarrow \mathbb{R}$  such that for any  $x, y, z \in M$ , we have

1.  $d(x, y) = 0 \Leftrightarrow x = y$ ; (identity of indiscernibles)

2.  $d(x, y) = d(y, x)$ ; and (symmetry)

3.  $d(x, z) \leq d(x, y) + d(y, z)$ . (triangle inequality)

Note  $d(x, y) \geq 0$  (non-negativity or separation axiom)

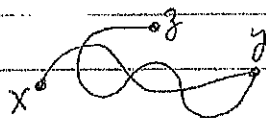
Proof.  $2d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0$ . ■

Fact The "distance" defined on graph  $G$ ,  $d$ , is a metric.

⑥  $d: V(G) \times V(G) \rightarrow \mathbb{R}$

①  $d(x, y) = 0$  iff  $x = y$ , ②  $d(x, y) = d(y, x)$ , and

③  $d(x, y) + d(y, z) \geq d(x, z)$ . (Why?)



If there exists a walk from  $x$  to  $z$ , then there exists a path from  $x$  to  $z$ , and the length of path is not larger than that of the walk.

Problem Find the diameter of a graph  $G$ . 4-5  
(connected)

(Using the adjacency matrix of  $G$ ,  $A$ .)

Fact 1 Let  $V(G) = \{v_i \mid i = 1, 2, \dots, n\}$ .

If  $d(v_i, v_j) = h$ , then  $A^h(i, j) \neq 0$ . ( $h \geq 1$ )

Fact 2

If  $d(v_i, v_j) = h$ , then  $A^{h'}(i, j) = 0$  for each  $1 \leq h' < h$ .

Fact 3

Let  $\text{diam}(G) = t$  be the diameter of  $G$ . Then, for each  $t' < t$ , there exist  $i$  and  $j$  such that  $A^{t'}(i, j) = 0$ .

Fact 4

For all  $i \neq j$ , there exists an  $1 \leq s \leq t$  such that  $A^s(i, j) \neq 0$ . (Clearly, let  $s = d(v_i, v_j)$ .)  
 $\text{diam}(G)$

\* (Fact 5)

Let  $A = \sum_{k=1}^t A^k$ . Then,  $A(i, j) > 0$ . Moreover, if  $s$  is the smallest positive integer such that  $(\sum_{k=1}^s A^k)(i, j) > 0$ , then  $\underline{s = \text{diam}(G)}$ .

How to determine the diameter of a graph  $G$ ?

策略一

For each  $v \in V(G)$ , find  $\text{ecc}(v)$ . Then,  $\text{diam}(G)$  is equal to  $\max\{\text{ecc}(v) \mid v \in V(G)\}$ .

策略 = (同學提供)

Pick an arbitrary vertex  $u \in V(G)$  and find a vertex  $u'$  such that  $d(u, u') = \text{ecc}(u)$ . Then,  $\text{diam}(G)$  is equal to  $\text{ecc}(u')$ .

Note that if this idea is good, then we have a quicker way to find  $\text{diam}(G)$ .

Proof. Give it a try!

策略三, Use adjacency matrix of  $G$ .

(\*) If  $\text{diam}(G) \geq 3$ , then  $\text{diam}(\bar{G}) \leq 3$ .

(\*\*) Almost all graphs are of diameter 2.  
 机率  $(G^p)$

(\*) If  $\text{diam}(G) \geq 3$ , then  $\text{diam}(\bar{G}) \leq 3$ .

Proof. Since  $\text{diam}(G) \geq 3$ , there exist two vertices

$u$  and  $v$  such that  $u$  and  $v$  do not have a common (not adjacent)

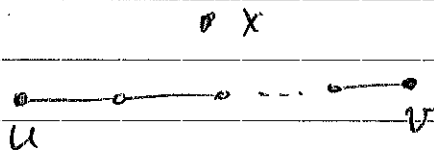
neighbor. Consider any two vertices  $x$  and  $y$  in  $\bar{G}$ .

At least one of  $xu$  and  $xv$  is not an edge of  $G$ , so does  $yu$  and  $yv$ .

If

(1)  $xu, yu$  are not edges of  $G$ , then  $xu, yu \in E(\bar{G})$ ,

$\text{dist}(x, y) \leq 2$ .



$y$

If

(2)  $xu, yv$  are not edges of  $G$ , then we have a

path  $\langle x, u, v, y \rangle$ ,  $\text{dist}(x, y) \leq 3$ . ▣

(\*\*) (Hint)

Consider  $v_i, v_j$  in  $G^p$ . Let  $X_{ij}$  (indicator variable)

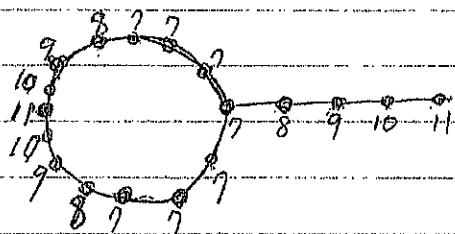
be 1 if and only if  $v_i$  and  $v_j$  do not have a common

neighbor. (There are  $n-2$  vertices in check!)

$$P(X_{ij}=1) = (1-p^2)^{n-2}, \quad \begin{matrix} E(X) = E(\sum_{i,j} X_{ij}) = \binom{n}{2} \cdot (1-p^2)^{n-2} \\ \downarrow 0 \quad \checkmark \end{matrix}$$

Fact For each pair of <sup>positive</sup> integers  $a$  and  $b$  such that  $a \leq b \leq 2a$ , there exists a graph  $G$  with  $\text{rad}(G) = a$  and  $\text{diam}(G) = b$ .

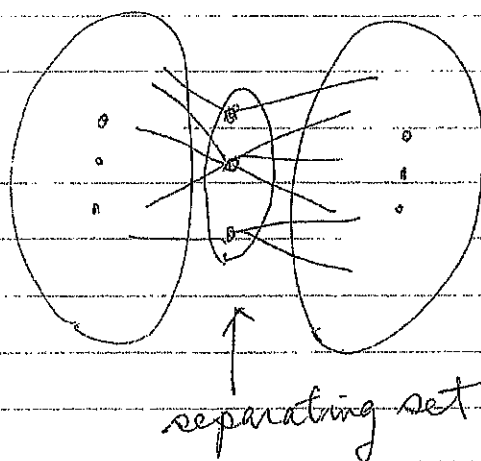
For example,  $a=7$  and  $b=11$ . (A cycle of length  $2a+1$  with a "tail"  $P_{b-a}$ .)



(\*)  $a=7, b=15$  is not possible, why?

Exercise Find a 3-connected graph  $G$  such that  $\text{rad}(G)=7$  and  $\text{diam}(G)=11$ .

Review A graph  $G$  is  $k$ -connected if any separating set is of size at least  $k$ .



附件 (\*) 如何算不一樣歐拉迴路的個數

Definition (Branching or out-tree) (有向圖)!

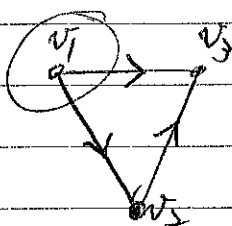
A branching or out-tree is an orientation of a tree having a root of indegree 0 and all other vertices are of indegree 1. (An in-tree is an out-tree with edges reversed.)

(證明省略)

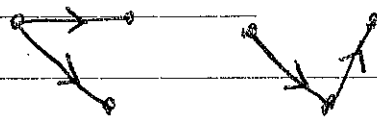
Theorem (Directed Matrix-Tree theorem, Tutte 1948)

Given a digraph  $G$ , let  $\bar{Q} = \bar{D} - A'$  and  $Q^+ = D^+ - A'$ , where  $\bar{D}$  and  $D^+$  are the diagonal matrices of indegrees and outdegrees in  $G$ , and  $A'(i,j)$  is the number of edges from  $v_i$  to  $v_j$ . The number of spanning (directed) out-trees of  $G$  rooted at  $v_i$  is the value of each cofactor in the  $i$ th row of  $\bar{Q}$  (column) ( $Q^+$ ).

(e.g.)



Two out-trees rooted at  $v_1$



$$\bar{Q} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$v_1 \rightarrow v_i$

Rooted at  $v$ ,

(\*) Each co-factor of the 1<sup>st</sup> row  $\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$  is 2.

(\*) In an eulerian graph, each vertex is the root of an out-tree and the number of distinct out-trees (in-tree) rooted at  $v_1, v_2, \dots, v_n$  respectively is a constant.

(\*) In an eulerian graph, an in-tree determines an eulerian circuit.

不是唯一

Algorithm (Rooted at  $v$ )

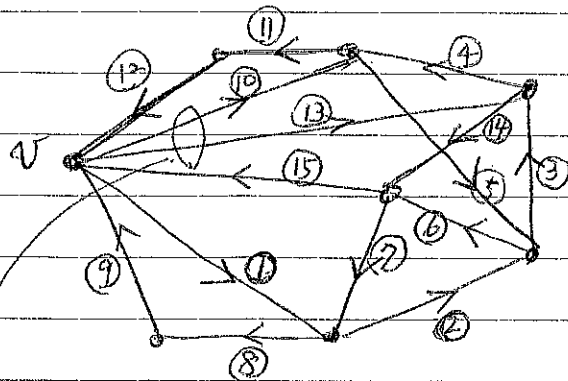
Step 1

離開  $v$  的走法, 除非不得已再走 in-tree

的有向邊。

Step 2. 由於進去的有向邊個數相等, 最後回到  $v$ .

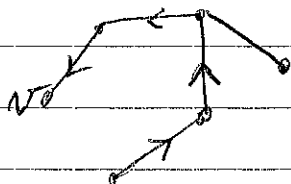
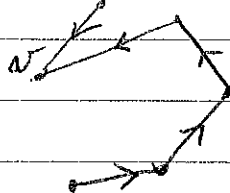
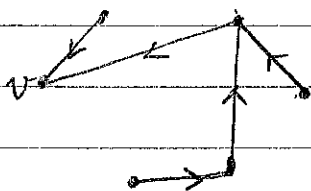
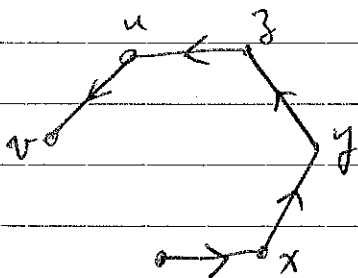
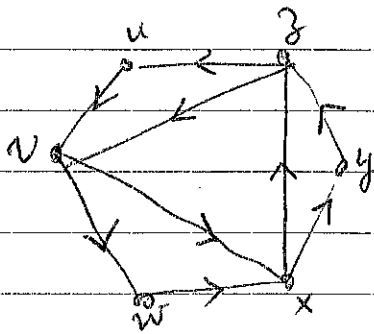
紅色 in-tree



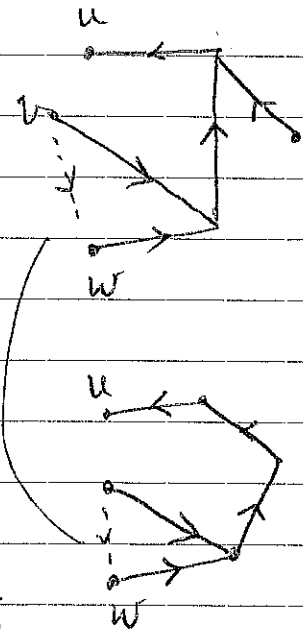
out-degree 大於 3 時, 就有不同的選擇!

(\*) Each in-tree generates  $\prod_{u \in V(G)} (\deg^+(u) - 1)!$  different eulerian circuits.

(\*) 由於循環次序的關係, 使得無論 Root 的位置在那裏, 都會產生相同個數的 in-trees (or out-trees)



Root  $v$ , 4



為一種選擇

Root  $u$ , 4

Theorem (van Aardenne-Ehrenfest and de Bruijn, 1951)

In an eulerian digraph with  $d_i = \deg^+(v_i) = \deg^-(v_i)$ ,  $i = 1, 2, \dots, n$ , the number of eulerian circuits is

$c \cdot \prod_{i=1}^n (d_i - 1)!$ , where  $c$  counts the in-trees or out-trees from any vertex.

Proof. As explained in previous pages. ■