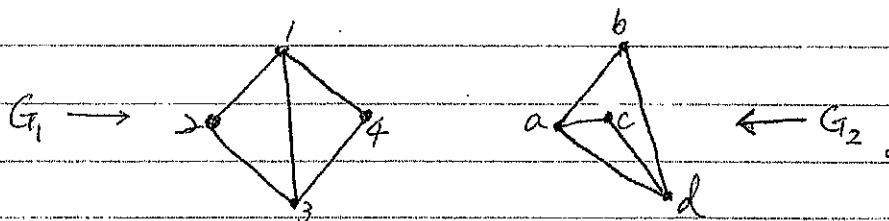


Graph Isomorphism for Simple Graphs (Sep. 19)

Given two graphs, for example the following:



Are they "the same" or not?

Fact If $|G_1| \neq |G_2|$ or $\|G_1\| \neq \|G_2\|$, then $G_1 \not\cong G_2$.

Observation

- (1) G_1 and G_2 are of the same order, i.e., $|G_1| = |G_2|$.
- (2) G_1 and G_2 are of the same size, i.e., $\|G_1\| = \|G_2\|$.
- (3) G_1 and G_2 are of the same "incidence" structure:

$$1 \rightarrow a, 3 \rightarrow d, 2 \rightarrow c, 4 \rightarrow b.$$

Review: Two sets A and B are of the same cardinality

if there exists a bijection between A and B , i.e., $\exists$

$$f: A \xrightarrow[\text{onto}]{1-1} B. \quad (A \text{ and } B \text{ can be infinite sets.})$$

(*) 非圖論敘述

You are you, you are the only one in the world just like you!

Definition (Isomorphic Graphs)

Two graphs G_1 and G_2 are isomorphic if there exists a bijection $f: V(G_1) \rightarrow V(G_2)$ such that

$$\begin{array}{ccc} u \sim_{G_1} v & \iff & f(u) \sim_{G_2} f(v) \\ \parallel & & \parallel \\ uv \in E(G_1) & & f(u)f(v) \in E(G_2) \end{array}$$

$$uv =_{\text{def}} \{u, v\}$$

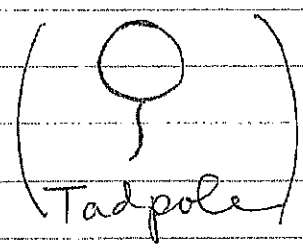
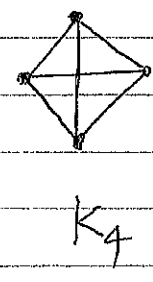
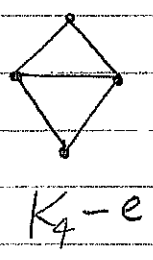
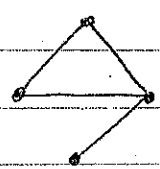
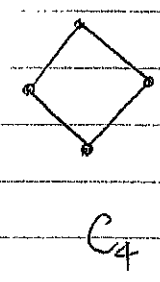
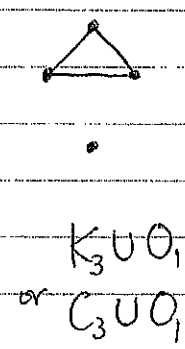
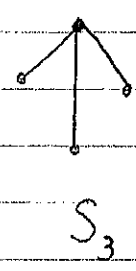
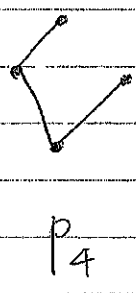
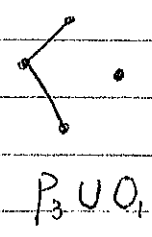
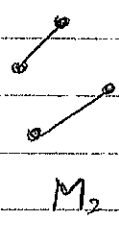
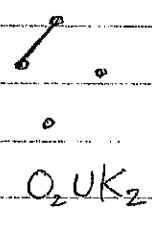
We use $G_1 \cong G_2$ to denote the fact that G_1 is isomorphic to G_2 .

Fact $G_1 \cong G_2 \Rightarrow$ $\left\{ \begin{array}{l} \textcircled{0} \overline{G_1} \cong \overline{G_2} \quad (*) \\ \textcircled{1} |G_1| = |G_2| \\ \textcircled{2} \|G_1\| = \|G_2\| \\ \textcircled{3} \text{ Many others } \dots \\ \quad \text{(properties)} \end{array} \right.$

Fact Two graphs are non-isomorphic if they are not isomorphic.

Fact Isomorphism is an equivalence relation defined on $G \times G$ where G is the set of all simple graphs. (graphs)

Example: Non-isomorphic graphs of order 4
(simple)

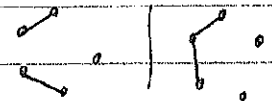
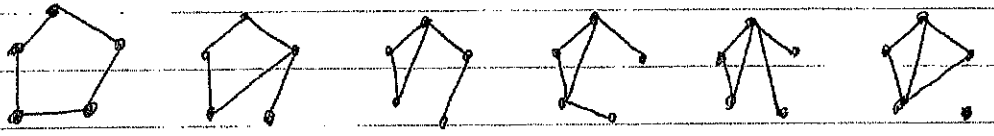


Problem: (How many non-isomorphic graphs are of order 5?)

Solution attached, see next page.

There are 34 non-isomorphic graphs of order 5.
(simple)

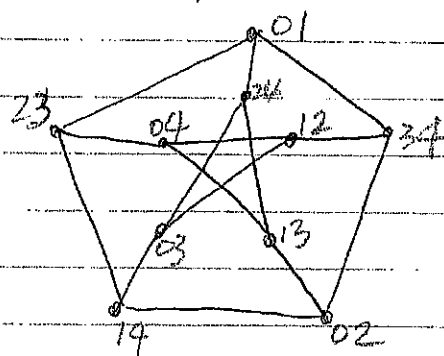
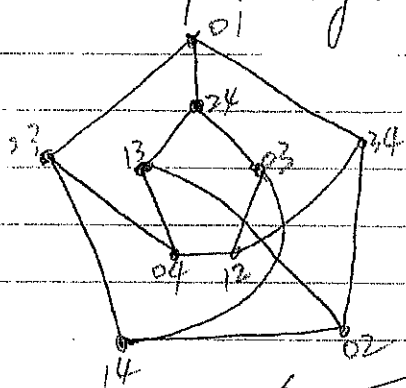
Idea First, we use the number of edges to partition the set of graphs of order 5.

1. $e=0$,		1
2. $e=1$		1
3. $e=2$		2
4. $e=3$		4
5. $e=4$		6
6. $e=5$		6
7. $e=6 \approx e=4$		6
8. $e=7 \approx e=3$		4
9. $e=8 \approx e=2$		2
10. $e=9 \approx e=1$		1
11. $e=10 \approx e=0$		1

(Why?)

(*) If $G_1 \cong G_2$, then $\bar{G}_1 \cong \bar{G}_2$. (Complement!)

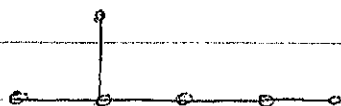
Are the following two graphs isomorphic?



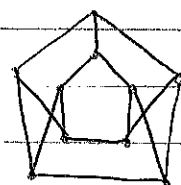
Yes!

Petersen Graph

Are the following two graphs isomorphic?



No!



No!

Fact Let $V = \binom{\mathbb{Z}_5}{2} = \{ \text{2-element subsets of } \mathbb{Z}_5 \}$

$$= \{ 01, 02, 03, 04, 12, 13, 14, 23, 24, 34 \}$$

and $E = \{ AB \mid A, B \in V \text{ and } A \cap B = \emptyset \}$. The (V, E) is isomorphic

to Petersen graph. (Kneser Graphs?)

Famous Graphs

(!!) So, how to determine whether two graphs are isomorphic or not. (Isomorphism Disease)

Possible approach

Definition (Degree sequence)

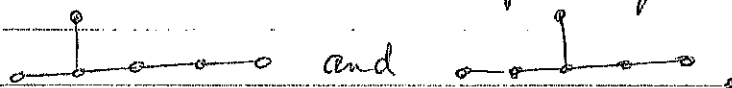
Let G be a graph and $V(G) = \{v_i \mid i = 1, 2, \dots, n\}$. Let $d_i = \deg_G(v_i)$ and $d_1 \geq d_2 \geq \dots \geq d_n$. Then, $\langle d_1, d_2, \dots, d_n \rangle$ is the degree sequence of G .

Review 1. $\forall v \in V(G)$, $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. They are the neighborhood of v and closed neighborhood of v respectively.

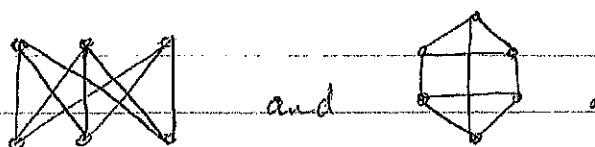
2. $|N(v)|$ is known as the degree of v . If the graph has a loop passing v , then this loop contributes two degrees. denoted by $\deg_G(v)$

3. A vertex is even (or odd) if its degree is even (or odd).

Example 1. $\langle 3, 2, 2, 1, 1, 1 \rangle$ is the degree sequence of both



2. $\langle 3, 3, 3, 3, 3, 3 \rangle$ is the degree sequence of both



Can we say something about isomorphic graphs and degree sequences?

Conclusion (Trivial)

Two graphs may have the same degree sequence, but they are not isomorphic.

(Fact) If two graphs are isomorphic, then they have the same degree sequence.

= Sep 21

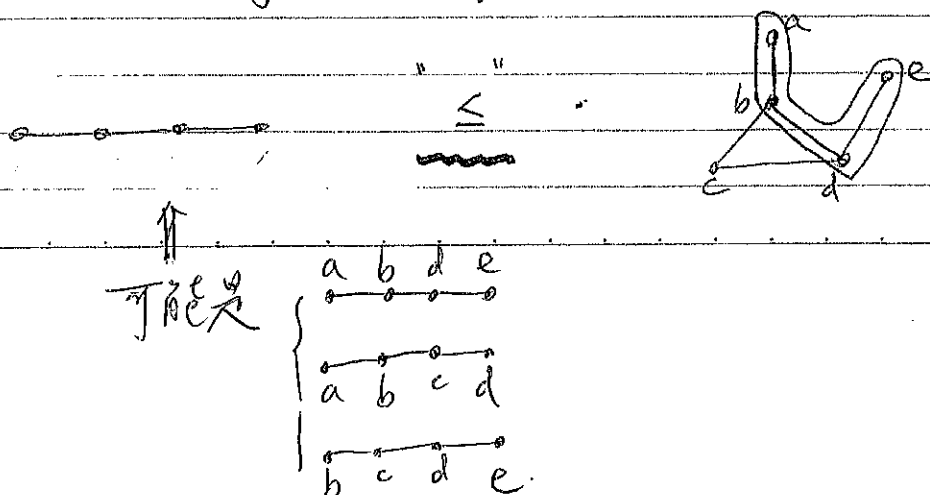
Subgraphs

Definition (Strong sense)

Let G and H be two graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Then H is a subgraph of G .

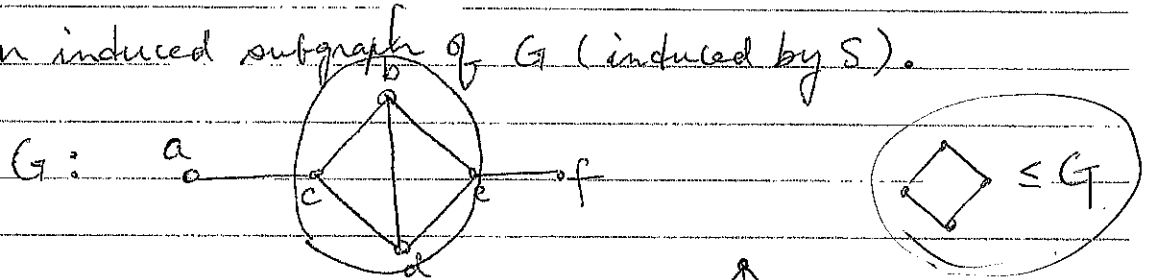
Definition (General version)

H is a subgraph of G if H is isomorphic to a subgraph (strong sense) of G , denoted by $H \leq G$.

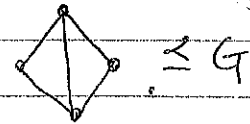


Definition (Induced subgraphs)

Let G be a graph and $S \subseteq V(G)$ be a non-empty (or generated) subset. The graph induced by S , $\langle S \rangle_G$, is a subgraph of G such that $u \sim_{\langle S \rangle_G} v$ if and only if $u \sim_G v$. $\langle S \rangle_G$ is called an induced subgraph of G (induced by S).



Let $S = \{b, c, d, e\}$. $\langle S \rangle_G$:



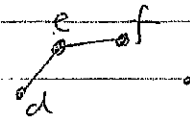
$\langle S \rangle_G \subseteq G$.

↑ induced subgraph

Definition (Edge-induced subgraphs)

Let G be a graph and $\tilde{E} \subseteq E(G)$. Then the edge-induced subgraph of $\tilde{E}$, $\langle \tilde{E} \rangle_G$, is the subgraph of G whose edge set is $\tilde{E}$ and vertex-set is the set of vertices in G which are incident to an edge in $\tilde{E}$.

For example, let $\tilde{E} = \{de, ef\}$, then $\langle \tilde{E} \rangle_G$:



(Fact) If two graphs G_1 and G_2 are isomorphic, then H is a subgraph of G_1 if and only if H is a subgraph of G_2 .

Definition (H-free graphs).

A graph G is said to be H-free if H is not a sub-graph of G.

Example. Petersen graph is C_3 -free and also C_4 -free. (Can you prove them?)

Review. A cycle with $n \geq 3$ vertices is denoted by C_n .

Definition (Extremal graphs)

Consider all graphs on n vertices. Let

$ex(n; H) = \max\{|G| \mid G \in \mathcal{G}_n \text{ and } G \text{ is } H\text{-free}\}$. The

graphs G of order n and size $ex(n; H)$ are known as extremal graphs which forbids H.

Example. $n=6$, $ex(6; C_3) = 9$. (why?)

Note Two steps to show

Step 1. There exists a graph of ^{order 6 and} size 9 which is C_3 -free.



$$(ex(6; C_3) \geq 9)$$

Step 2. All graphs of order 6 and size ≥ 10 contain a C_3 .
(Harder step!) $(ex(6; C_3) \leq 9)$

(*) Prove this theorem

$$ex(n; K_3) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Theorem (Mantel, 1907)

Every graph of order n and size greater than $\left\lfloor \frac{n^2}{4} \right\rfloor$ contains a $K_{3,2}$ - δ .

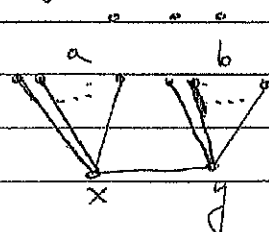
Since $K_3 \not\subseteq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \emptyset$. This implies that $\deg_G(x) + \deg_G(y) \leq |G| = n$. (Figure 1) Now, consider

$$\sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) = \sum_{x \in V(G)} (\deg_G(x))^2 \quad (\text{Two-way counting}).$$

$\nearrow$
 $n \cdot K_3 \nsubseteq n \quad \searrow$

$$\leq n \cdot \|G\| = n \cdot e(G)$$

By Cauchy's inequality, $(2e(G))^2 = \left(\sum_{x \in V(G)} \deg_G(x) \right)^2 \leq n \cdot \sum_{x \in V(G)} (\deg_G(x))^2$



$$\leq n^2 \cdot e(G).$$

Hence, $e(G) \leq \frac{n^2}{4}$. $\square$

Figure 1. $\deg_G(x) = a+1$, $\deg_G(y) = b+1$

(2nd proof.)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2)

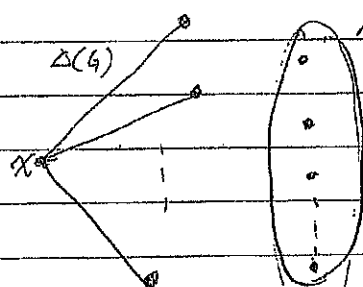
Since $K_3 \not\subseteq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies

$$\|G\| \leq \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G)).$$

$\|G\|$ will take a maximum when $\Delta(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Hence, we have the

proof. $\square$

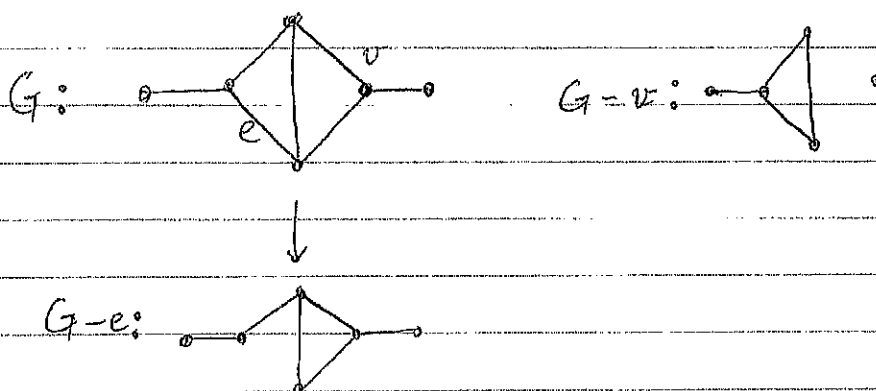
Problem $G \not\subseteq C_4$
 $|G| = 13$
 Find $\max\{\|G\|\}$.



Remark To determine $ex(n; H)$ for a given H is a very difficult problem in general. But, for some special H , say $K_k, P_n, \dots$, $ex(n; H)$ can be determined.

Definition (Deletion of vertices (and/or) edges)

Let $v \in V(G)$. The graph $G-v$ is obtained by deleting the vertex v and all the edges incident to v . The graph $G-e$ where $e \in E(G)$ is a subgraph of G with the deletion of e from $E(G)$.



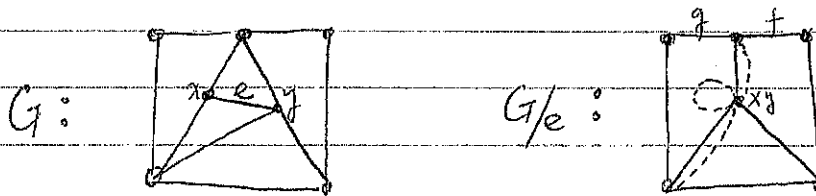
Let $S \subseteq V(G)$ and $T \subseteq E(G)$. $G-S$ and $G-T$ can be defined accordingly. (Take away vertices (or edges) one by one.)

Definition (Graph minors)

A graph M is called a minor of G if M can be obtained from G by contracting edges, deleting vertices and edges.

Review (Edge-contraction)

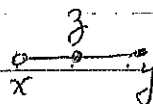
Given an edge $xy \stackrel{=}{=} e$ of a graph G , the graph G/e is obtained from G by contracting e ; that is to identify the vertices x and y and ^{deleting} resulting loops and duplicate edges.



Example K_4 is a minor of the above G .
(Contracting e , f and g .)

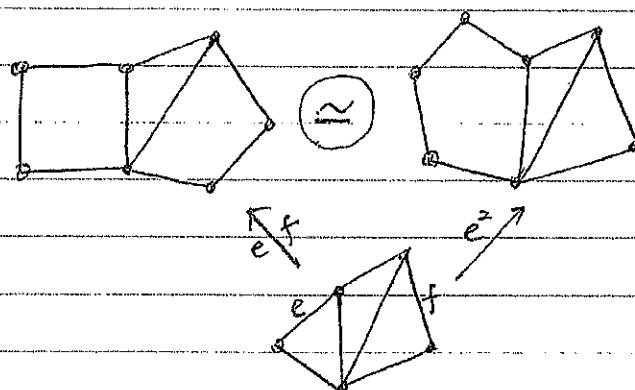
Definition (Subdivision)

A subdivision of an edge xy is obtained by adding a new vertex z such that we have edges xz and zy .



Definition (Homeomorphic Graphs)

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

Remark : Two cycles are homeomorphic.

高元徽/高, Problem 2*

Determine the exact value of $ex(n; C_4)$.

國立陽明交通大學 112 學年度
第一學期行事曆^[※]

112年3月29日 111學年度第6次行政會議通過

教育部臺教高(一)字第1120044516號核備

年	週次	日期						星期	辦理事項
		日	一	二	三	四	五		
一 一 二 年				1	2	3	4	5	八 1 二 112學年度第1學期學期開始
		6	7	8	9	10	11	12	八 10 四 國際學位生春季班入學申請開始
		13	14	15	16	17	18	19	
		20	21	22	23	24	25	26	
		27	28	29	30	31			月 九 1 五 暑期課程結束
	1	3	4	5	6	7	8	9	2 六 大一新生宿舍進住(2日-3日)
	2	10	11	12	13	14	15	16	4 一 初選第三階段(新生優先)(4日-7日)、新生地震防災演練(配合新生入學輔導)
	3	17	18	19	20	21	22	23	4 一 研究生轉所申請繳交截止日(112學年度第1學期生效)
		24	25	26	27	28	29	30	5 二 新生開學典禮；新生入學輔導週(4日-8日)
									8 五 111學年度第2學期畢業離校截止
									11 一 第一學期上課開始
									11 一 選課加退選、校際選修申請(11日-22日)
一 一 三 年	4	1	2	3	4	5	6	7	13 三 學雜費繳交截止
	5	8	9	10	11	12	13	14	15 五 教師更改學生111學年度第2學期成績截止
	6	15	16	17	18	19	20	21	18 一 交通安全宣導活動(18日-21日)
	7	22	23	24	25	26	27	28	22 五 學生抵免學分申請截止
	8	29	30	31					23 六 調整上班
									28 四 教師節(停課不停班) ← 停課
									29 五 中秋節
									月 30 六 國際學位生春季班入學申請截止
	4	1	2	3	4	5	6	7	十 9 一 國慶日調整放假
	5	8	9	10	11	12	13	14	十 10 二 國慶日
	6	15	16	17	18	19	20	21	20 五 休學、退學及畢業生退2/3學雜費截止
	7	22	23	24	25	26	27	28	20 五 推廣教育開班計畫書收件截止
	8	29	30	31					30 一 學期預警啟動
一 一 三 年									月 30 一 期中考試(10月30日-11月3日)
	8			1	2	3	4		十 6 一 學期預警課程登錄
	9	5	6	7	8	9	10	11	一 20 一 學期預警輔導
	10	12	13	14	15	16	17	18	一 29 三 全校運動會預賽
	11	19	20	21	22	23	24	25	
	12	26	27	28	29	30			月 十 1 五 休學、退學及畢業生退1/3學雜費截止
	12				1	2			十 6 三 全校運動會(停課)
	13	3	4	5	6	7	8	9	八 五 停修課程申請截止日
	14	10	11	12	13	14	15	16	二 20 三 校務會議
	15	17	18	19	20	21	22	23	二 20 三 國際學位生秋季班申請入學開始
	16	24	25	26	27	28	29	30	月 25 一 學期考試(25日-29日)
	17	31							
一 一 三 年	17	1	2	3	4	5	6		一 1 一 開國紀念日
	18	7	8	9	10	11	12	13	一 2 二 初選第一階段(2日-5日)
		14	15	16	17	18	19	20	八 一 初選第二階段(8日-11日)
		21	22	23	24	25	26	27	一 15 一 寒假開始
		28	29	30	31				一 19 五 112學年度第2梯次逕修讀博士學位申請截止日
									月 31 三 112學年度第1學期學期結束

備註

註1. 國定假日如有規定，則以公告辦理。

註2. 輔系、雙主修申請期限說明：

(1) 學士班學生：自二年級起至修業年限最後一年第一學期(不含延長修業年限)加退選截止前提出申請。

(2) 碩、博士班學生：自第二學期起至第六學期結束前提出申請。

學生自行完成審查流程後，送註冊組登錄後即獲得修讀資格。

註3. 轉所申請期限說明：

研究所學生修業滿一學期以上，經原系所同意後，得申請轉所。

申請程序於開學前提出並經核定，於當學期生效；於學期中提出申請並經核定者，次學期生效。

註4. 逕修讀博士學位申請如教學單位另有申請時程，請依教學單位規定辦理。