

L.4. Orthogonal Latin Squares

Definition (Orthogonal L.S.)

based on $\mathbb{Z}_n$

Use $\mathbb{Z}_n$ throughout of this lecture.

Two Latin squares of order n , $L = [l_{ij}]$ and $M = [m_{ij}]$, are orthogonal if $\{(l_{ij}, m_{ij}) \mid 1 \leq i, j \leq n\} = \mathbb{Z}_n^2$, denoted by $L \perp M$.

e.g.

0	1	2
1	2	0
2	0	1

L

$\perp$

0	1	2
2	0	1
1	2	0

M

Let $\alpha(L)$ denote the Latin square which is obtained from L by permuting the entries of L with α (permutation of $\mathbb{Z}_n$).

Then, we have

Proposition 1 If $L \perp M$, then $\alpha(L) \perp \beta(M)$ for any two permutations α and β of $\mathbb{Z}_n$.

e.g. Let $\alpha = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

Then, we have

1	2	0
2	0	1
0	1	2

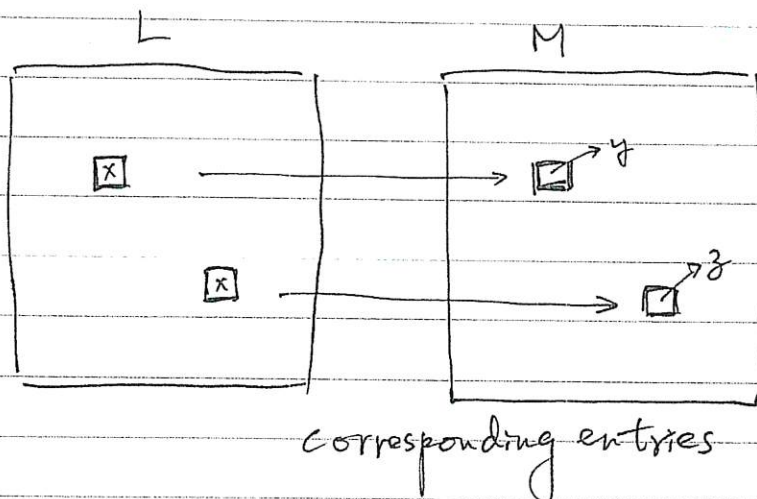
$\alpha(L)$

$\perp$

0	2	1
1	0	2
2	1	0

$\beta(M)$

Two Finger's Rule



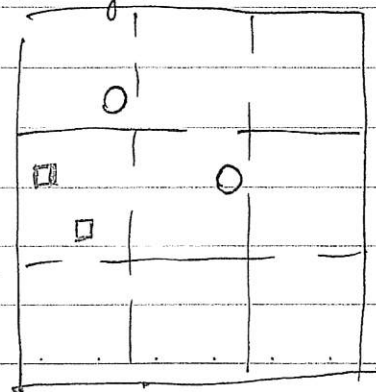
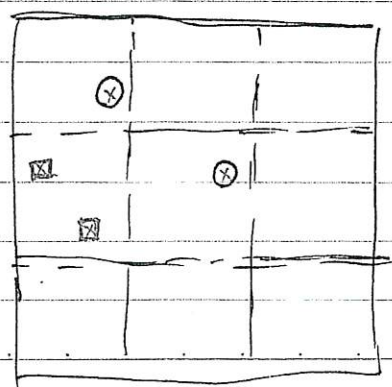
$L \perp M \Leftrightarrow y \neq z$ in M whenever their corresponding entries are the same entry.

$$\Leftrightarrow l_{i,j} = l_{i',j'} \Rightarrow m_{i,j} \neq m_{i',j'}$$

Proposition 2 If $L_1 \perp L_2$ (of order m) and $M_1 \perp M_2$ (of order n),
 then $L_1 \otimes M_1 \perp L_2 \otimes M_2$ (of order mn). $\Rightarrow (L_1 \otimes L_2, M_1 \otimes M_2, N_1 \otimes N_2$
 and more.)

✓ Exercise 1.8 (3 points) Prove Proposition 2.

Consider the following two cases and use two finger's rule.



Proposition 3 If n is a prime power, then there exists $n-1$ Latin squares of order n which are mutually orthogonal.

Note. $L_1, L_2, \dots, L_k$ are mutually orthogonal if for any two $1 \leq i \neq j \leq k$, $L_i \perp L_j$.

Proof. Since n is a prime power, we have a finite field $GF(n)$,

$\langle F, +, \cdot \rangle$. Let $F^* = F \setminus \{0\}$. For convenience, let $F = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$.

Now, we define $L_{i,j}^{(h)} = \alpha_i + \alpha_h \cdot \alpha_j$ where $\alpha_h \in F^*$. Now, since for $0 \leq i, j \leq n-1$, $i \neq i'$ implies that $L_{i,j}^{(h)} \neq L_{i',j}^{(h)}$ and $j \neq j'$ implies that $L_{i,j}^{(h)} \neq L_{i,j'}^{(h)}$, $L^{(h)}$ is a Latin square. As to the orthogonality of two Latin

squares, we can also use two finger's rule.

Assume that $L_{i,j}^{(h)} = L_{i',j'}^{(k)}$. Consider $1 \leq k \neq h \leq n-1$.

Suppose that $L_{i,j}^{(h)} = L_{i',j'}^{(k)}$. Then, we have

$$\begin{cases} \alpha_i + \alpha_h \cdot \alpha_j = \alpha_{i'} + \alpha_h \cdot \alpha_{j'} \\ \alpha_i + \alpha_k \cdot \alpha_j = \alpha_{i'} + \alpha_k \cdot \alpha_{j'} \end{cases}$$

$$\Rightarrow (\alpha_h - \alpha_k) \alpha_j = (\alpha_h - \alpha_k) \alpha_{j'} \Rightarrow \alpha_j = \alpha_{j'} \Rightarrow \alpha_i = \alpha_{i'} \rightarrow \leftarrow$$

Hence $L^{(h)} \perp L^{(k)}$.

Some facts on Finite fields

1. A finite field exists if and only if n is a prime power.

2. $\langle \mathbb{Z}_n, +, \cdot \rangle$ is a finite field if and only if n is a prime.

3. Let $n = p^m$ where p is a prime and $m \geq 1$.

Then a finite field of order n can be constructed by using an irreducible polynomial $g(x)$ of degree m (over $\mathbb{Z}_p$), i.e., $GF(n) \cong \mathbb{Z}_p[x] / \langle g(x) \rangle$.

4. All finite fields of the same order are isomorphic.

5. If $\langle F, +, \cdot \rangle$ is a finite field, then $\langle F^*, \cdot \rangle$ is a cyclic group, i.e., $\langle F^*, \cdot \rangle \cong \langle \alpha \rangle$, generated by an element of F^* , α .

6. e.g. $x^3 + x + 1$ is irreducible over $\mathbb{Z}_2$.

$\mathbb{Z}_2[x] / \langle x^3 + x + 1 \rangle$ is a finite field of order 8.
 $\parallel$
 $\{0, 1, x, x+1, x^2, x^2+x, x^2+x+1, x^2+x^2+1\} / (\text{mod } x^3+x+1)$
 $\parallel$
 $\emptyset$

Definition (A complete family of $MOLS(n)$)

For order n , $n-1$ mutually orthogonal Latin squares form a complete family of $MOLS(n)$.

Fact 1. If n is a prime power, then we have a complete family of $MOLS(n)$.

Note. So far, only for prime power n , we can find a complete family of $MOLS(n)$.

Note. It is known that there does not exist a complete family of $MOLS(n)$ for $n=6$ and 10 .

Observation (Three $MOLS(4)$!)

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

 $\perp$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

 $\perp$

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

Can we find the 3rd one by using the above two $MOLS(4)$?

Solve the 16 cards problem!

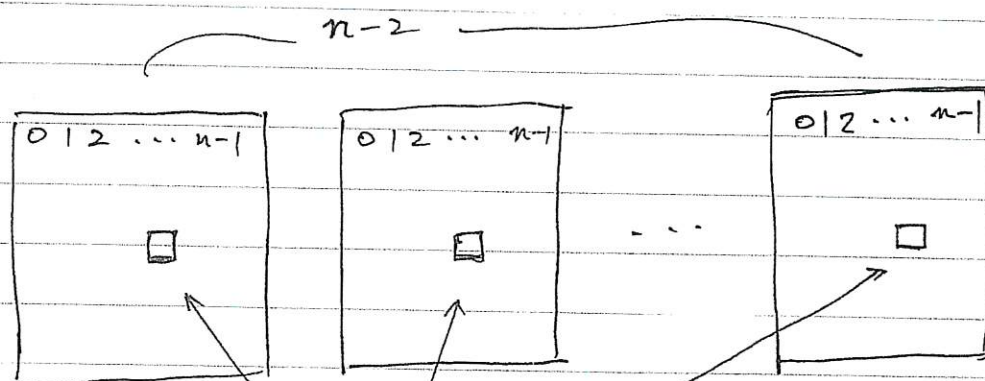
Fact 2 For each n , there are at most $n-1$ mutually orthogonal Latin squares.

Proof. By Proposition 1, we can assume all mutually orthogonal Latin squares do have the same row $(0, 1, 2, \dots, n-1)$. Then, consider the $(2, 1)$ cell; no two of the squares have the same entry. ^(?) Hence, we have ^{at most} $n-1$ distinct Latin squares which are mutually orthogonal. ■

Proposition 4. If there exist $n-2$ MOLS(n), then we can find $n-1$ MOLS(n).

✓ Exercise 1-9 prove Proposition 4. (3 points)

Idea:



must have distinct entries
and we have "one" left!

Why "Euler" made the following conjecture?

Euler's Conjecture on MOLS.

For each $n \equiv 2 \pmod{4}$, there do not exist two mutually orthogonal Latin square of order n .
 $\left(\begin{array}{l} n > 1 \text{ and} \\ \text{If } n \not\equiv 2 \pmod{4}, \text{ then either} \\ n \text{ is a prime or } n \text{ has a prime} \\ \text{factor larger than } 2. \end{array} \right.$

Fact 3. It is ^{not} true for $n=2$ and 6 (only!). ($n=1$ is trivial.)

Fact 4 If $n \not\equiv 2 \pmod{4}$, then we can find at least two MOLS(n).

Proof. Case 1. $n \equiv 0 \pmod{4}$

In this case, $n = 2^k \cdot m$ where $k \geq 2$ and m is an odd integer. If $m=1$,

then n is a prime power, the proof follows. On the other hand,

if $m > 1$, then $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where p_i 's are odd primes. Now,

by using Proposition 2, we can construct two MOLS(n) by using direct product of two mutually orthogonal Latin squares of order $2^k, p_1^{e_1}, p_2^{e_2}, \dots, p_k^{e_k}$ respectively.

Case 2. $n \equiv 1 \text{ or } 3 \pmod{4}$

[^] this case has been included in Case 1.
[^] The proof of

Bonus (5 points) prove that there do not exist two mutually orthogonal Latin squares of order 6. (See next page for reference.)

Euler's Conjecture was disproved by Parker, Bose and Shrikhande in the year 1959. The following two $\text{MOLS}(10)$ was proposed by E.T. Parker.

4	0	9	8	3	2	7	5	6	1
2	3	7	5	4	0	9	8	1	6
8	1	6	9	0	4	5	3	2	7
9	8	1	4	5	6	3	2	7	0
0	9	8	6	1	3	2	7	4	5
7	2	3	1	6	5	4	0	9	8
5	4	0	3	2	7	6	1	8	9
6	5	4	2	7	1	8	9	0	3
1	6	5	7	8	9	0	4	3	2
3	7	2	0	9	8	1	6	5	4

5	4	0	1	2	7	8	9	3	6
3	1	6	4	8	5	9	2	0	7
0	9	8	7	3	6	1	4	5	2
2	5	4	3	6	1	7	8	9	0
9	8	7	6	1	0	4	5	2	3
1	6	3	5	9	2	0	7	4	8
8	7	2	9	0	4	5	3	6	1
4	0	9	2	7	8	3	6	1	5
7	2	5	0	4	3	6	1	8	9
6	3	1	8	5	9	2	0	7	4

For $n \equiv 2 \pmod{4}$ and $n \geq 10$, we need to apply ideas from pairwise balanced design to prove that two $\text{MOLS}(n)$ do exist. (So, we will provide a proof later.)

(1984)

D.R. Stinson, A short proof of the non-existence of a pair of orthogonal Latin squares of order six,
J. Combin. Th. A36, 373-376.

About

0	1	2	3	4	5
1	2	3	5	0	4
2	5	0	4	1	3
~~3~~	4	~~5~~	2	5	0
4	0	5	1	3	2
5	3	4	0	2	1

0	1	2	3	4	5
5	0	4	2	3	1
3	4	1	5	2	0
~~4~~	3	~~5~~	1	0	2
2	5	3	0	1	4
1	2	0	4	5	3

(3,4) and (1,5) are the only two repeated
ordered pairs

Definition Two L.S. of order n defined on the same set S
are r -orthogonal if, when they are superimposed, exactly
 r different ordered pairs of S^2 occur among the n^2 ordered
pairs of entries.

So, the above example is a pair of 34-orthogonal
L.S. of order 6.

In application, we can use another term to describe orthogonal Latin squares.

Definition (Orthogonal Array)

An orthogonal array of order n with depth k is a $k \times n^2$ array $A = [a_{ij}]$ such that for any two rows, the ordered pairs obtained from these two rows are exactly all ordered pairs of $\mathbb{Z}_n^2$. ($a_{ij} \in \mathbb{Z}_n$)

e.g. $OA(4,3)$

0	0	0	1	1	1	2	2	2
0	1	2	0	1	2	0	1	2
0	1	2	2	0	1	1	2	0
0	2	1	2	1	0	1	0	2

}}

0	0	0
1	1	1
2	2	2

0	1	2
0	1	2
0	1	2

0	1	2
2	0	1
1	2	0

0	2	1
2	1	0
1	0	2

Fact 5 The existence of an $OA(k, n)$ is equivalent to the existence of $k-2$ MOLS(n).

Fact 6: An $OA(k, n)$ has at most n^2 columns and $n+1$ rows.

Proof. The first fact comes from the number of ordered pairs is at most n^2 and the second fact is a consequence of there are at most $n-1$ MOLS(n).

In application, regularly a part of orthogonal array is sufficient. Therefore, we can use the so-called partial orthogonal array of order m with depth k . In such an array, (defined on $\mathbb{Z}_n$) the ordered pairs are required to be distinct, not necessarily all pairs in $\mathbb{Z}_n^2$. Here, $m \leq n^2$ (as the case in an $OA(k, n)$), but k may be larger than $n+1$.

e.g. $n=3$, $m=3$, $k=5$

0	0	0
0	1	2
0	1	2
0	2	1
1	2	0

Three
 $\approx$ Orthogonal partial
 Latin squares

(*) If $m = n^2$, then $k \leq n+1$.

(**) If we use each column as a codeword, then $d(C) \geq k-1$.