

Dec. 13 ~

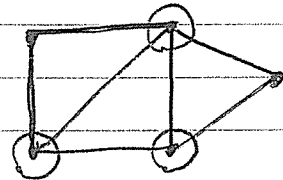
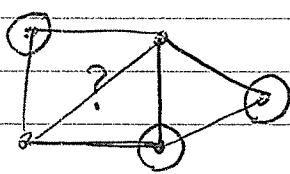
FGT Lecture 9

Vertex Cover

Definition (Vertex cover)

A set of vertices S is called a vertex cover of a graph G if each edge of G is incident to at least one vertex in S .

We use $\sigma(G)$ to denote $\min\{|S| \mid S \text{ is a vertex cover of } G\}$, $\sigma(G)$ is known as the cover number.



Can $\sigma(G)$ be "2"?

$$(*) \quad \sigma(G) \geq \frac{\|G\|}{\Delta(G)}.$$

(*) If $G = (A, B)$ is bipartite graph, then $\sigma(G) \leq \min\{|A|, |B|\}$.

(*) Finding $\sigma(G)$ of a general graph G is NP-complete.

$$(**) \quad \sigma(G) \geq \alpha_1(G)$$

(matching number)

Proof: If G has a matching of size k , then we need at least k vertices to cover these k edges. (No vertex can cover two edges of a matching!)

(*) We can use a greedy algorithm to find an upper bound of $\alpha(G)$.

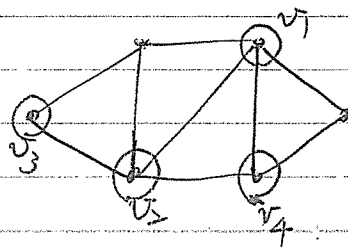
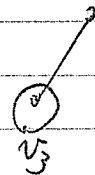
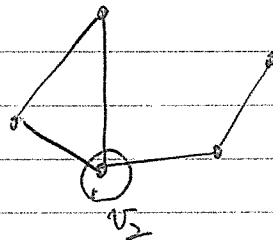
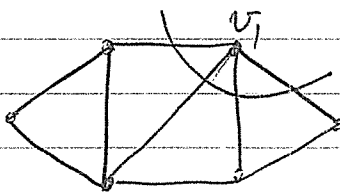
Step: Find a vertex for the cover and then delete all the edges incident to this vertex, and continue.

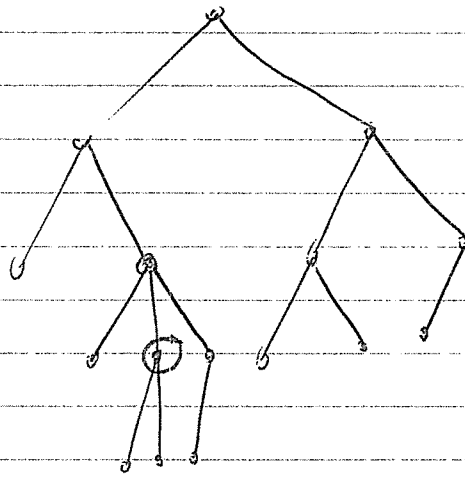
(**) Revise version

Step: For each induced subgraph H of G , deleting the vertex with $\Delta(H)$.

Start with a vertex v_1 of degree $\Delta(G)$, delete v_1 and put $v_1 \in S$. Then, consider v_2 , s.t., $\deg_{G-v_1}(v_2)$ is maximum.

Again, add v_2 to S and continue this process until no edges left.
($\{v_1, v_2\} \subseteq S$)




$$\begin{cases} \frac{\|p\|}{3} = 5 \\ \alpha_1(p) = 5 \end{cases} \Rightarrow \alpha(p) \geq 5.$$


Starting from ^{the parent of} leaves!

Theorem (König¹⁹³¹-Egerváry^{later})

Let G be a bipartite graph with no isolated vertices.

Then, $\alpha(G) = \alpha_1(G)$.

How can we find a matching G with maximum number of edges?

(如何在圖中找到邊數最多的配對 (matching)?)

(*) 對於一個一般圖，決定最大的獨立集 (Independence number) 是一個

$$\alpha(G)$$

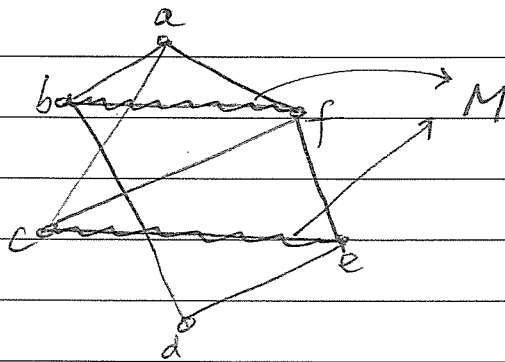
非常困難的問題；然而決定最多的配對邊就比較容易。

Definition (Alternating path)

Let M be a matching in G . An alternating path with respect to

M is a path starting at an unmatched vertex and then contains, alternately, edges from $E(G) \setminus M$ and from M .

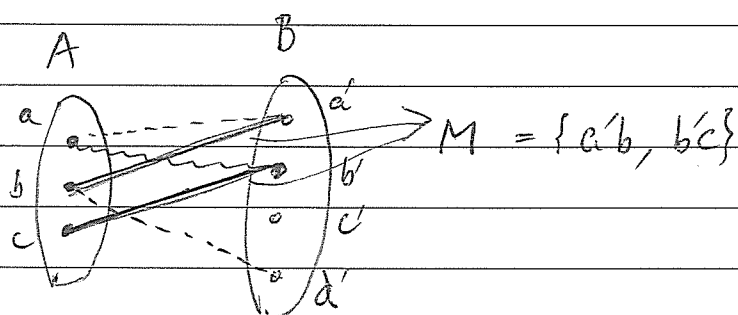
(從不屬於配對邊的頂出發，經過一個不是配對的邊，然後配對的邊，然後不是配對邊，再配對邊，...。)



$\langle a, b, f, c, e, d \rangle$

(*) 如果 path 的頭尾都不是配對邊，則該配對^數可以再增加！
例如：ab, fc, ed 為三邊的配對。

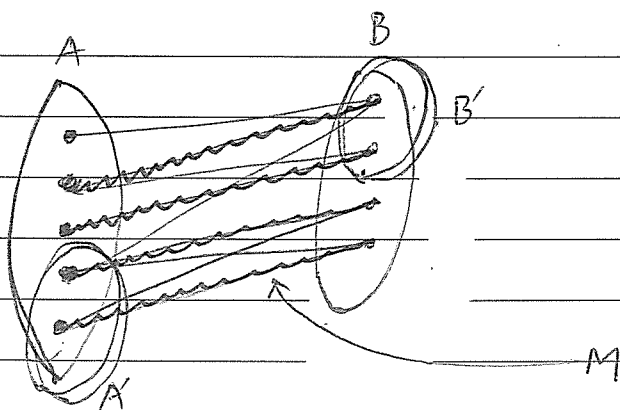
(*) 如果我们考虑二部图 (Bipartite graph) $G=(A,B)$, 希望找到通过 A 所有点的配对; Alternating path 就可以由 A 出发 (如果 B 有 A 中的点不在配对上)。



① $\langle a, a', b, b' \rangle$ is an alternating path
 $\Rightarrow M' = \{aa', bb', c'c\}$

② and $\langle a, b', c \rangle$ is also an alternating path

(*) 如果找到 M 已经是最多边了, 则所有的 alternating paths 都是 ② 的形态。
 最長



(*) $A' \cup B'$ 将会是这个图的最小的 Cover.

↓ cover
 ↑ matching #

Theorem (König, 1931)

If G is a bipartite graph, then $\alpha_1(G) = \sigma(G)$.

Proof. It suffices to show that a matching M with $\alpha_1(G)$ edges does have a vertex cover with $\alpha_1(G)$ vertices. Let $G = (A, B)$. If M saturates A , then the proof follows. (A is the cover we need.) Otherwise there are vertices in A such that they are not incident to any edge in M . So, we start from those vertices. First, we shall choose one end vertex from each edge in M . If an alternating path (exactly) ends in B , then put the end in B' . Clearly, this vertex is incident to an edge of M . For the other edges (not incident to B'), let the end vertices in A be A' , see the above example.

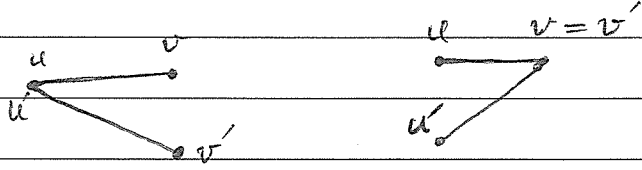
Claim $A' \cup B'$ is a vertex cover of G .

Let $uv \in E(G)$; we show that either $u \in A'$ or $v \in B'$.
($u \in A$ and $v \in B$)

First, if $uv \in M$, then by definition, either $v \in B'$ or $u \in A'$.

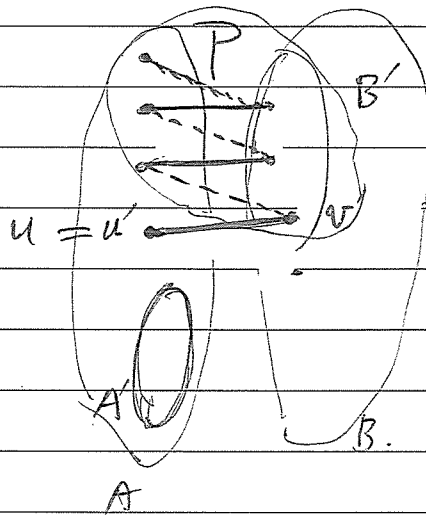
(从没有碰到 M 的 A 出发, 下一个边就是 M 的边, 走得到的在 B' 中, 走不到的在 A' 中。)

Assume that $uv \notin M$. Since M is a maximal matching, there exists an edge $u'v' \in M$ such that either $u = u'$ or $v = v'$ but not both. First, if $u \neq u'$, and $v = v'$, then by definition of B' , $v \in B'$. (done)



On the other hand, let $u = u'$ and $v \neq v'$. Moreover, $u = u'$ is not in A' . (Otherwise, we are done.) Then, since $u'v' \in M$, $v' \in B'$. (M中的边, 有一端在 A' , 另一端就不在 B' .)

Hence, there exists an alternating path P ends in v' .



Now, if v is on the path of P , then $v \in B'$, done. Otherwise (matched)

$Pv'u'v$ is an alternating path. Since M is maximal, v must be matched and $v \in B'$. This concludes the proof.

For reference

4

Proof. We shall apply max-flow min-cut theorem to prove the theorem. First, we define a network as in Figure 1. Let

$A = \{a_1, a_2, \dots, a_m\}$, $B = \{b_1, b_2, \dots, b_n\}$ and $G = (A, B)$. Then, the network is defined by letting u and v be source and sink respectively, $\text{cap}(u, a_i) = 1$ for $i = 1, 2, \dots, m$, $\text{cap}(b_j, v) = 1$ for $j = 1, 2, \dots, n$, and $\text{cap}(a_i, b_j) = |G| + 1$ if $a_i b_j$ is in $E(G)$. (Note that all arcs are from A to B .)

Since $\sigma(G) \geq \alpha_1(G)$ as mentioned above, it suffices to show that $\alpha_1(G) \geq \sigma(G)$.

Now, let f be a maximum flow. It is easy to see that $\text{val } f = \alpha_1(G)$. This is due to the fact that all the arcs from u and into v are of capacity 1. (No two arcs can be out of two vertices in A and ended in a vertex of B .)

So, it is left to consider the minimum cut, let it be

$K = (X, \bar{X})$ where $u \in X$, $v \in \bar{X}$, $A \cap \bar{X} = A'$ and $B \cap X = B'$, see

Figure 2. Hence, K contains arcs from u to A' , $A \setminus A'$ to $B \setminus B'$ and

B' to v . Notice that $\text{cap } K \leq |G|$, for example, let $\bar{X} = \{u\}$.

This implies that the following (*) is true and $A' \cup B'$ is a vertex cover and $\text{cap } K = |A'| + |B'| = \text{val } f = \alpha_1(G)$. By the fact,

$$|A'| + |B'| \geq \alpha(G),$$

we have $\alpha_1(G) \geq \alpha(G)$ ■

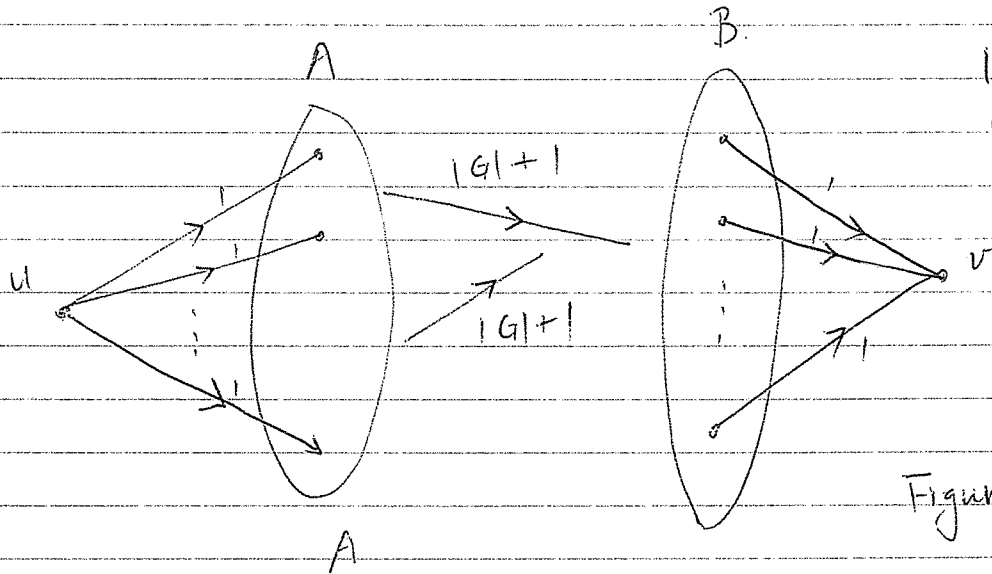


Figure 1. Network

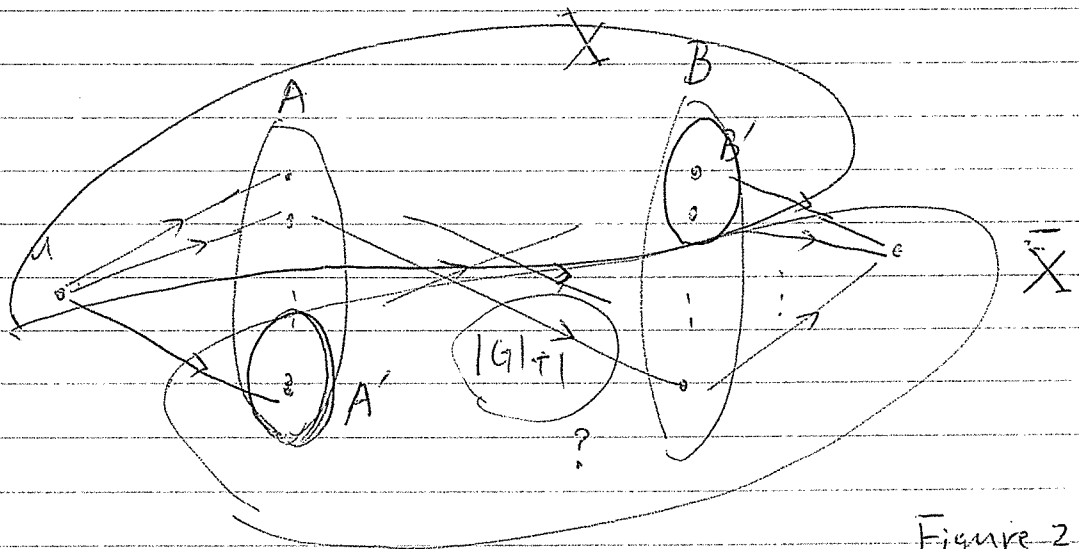


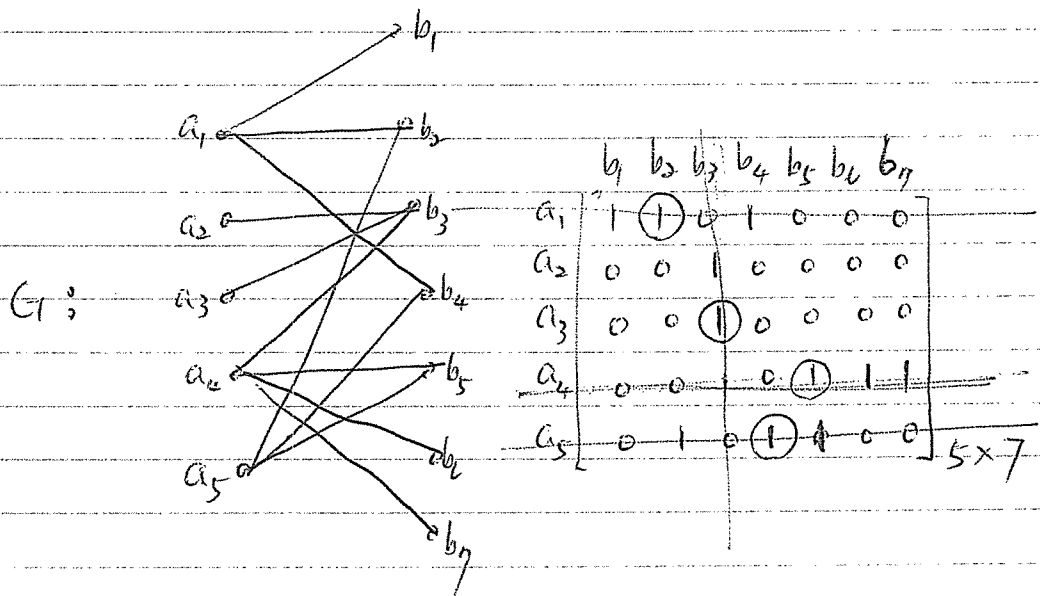
Figure 2. Cut

(*) In $(X, \bar{X})$, there exist no edges from $A \setminus A'$ to $B \setminus B'$. For otherwise, it is not a minimum cut. This implies that all edges are incident vertices in $A' \cup B'$. $A' \cup B'$ is a vertex cover.

(*) Since finding $\alpha_1(G)$ for a graph G is not a difficult problem in the sense of algorithm, finding $\alpha(G)$ of a bipartite graph is also not difficult.

(**) A bipartite graph $G = (A, B)$ with $|A| = m$ and $|B| = n$ can be represented by an $m \times n$ $(0, 1)$ -matrix where its row indices are vertices of A and column indices are vertices of B .

Example.



Now, $\alpha(G)$ is the minimum number of lines cover all 1's
(= 4)

and $\alpha_1(G)$ is the maximum number of 1's which are not in the
(= 4) (independent 1's)
same row and the same column.

Theorem

Let G be a graph of order p which has no isolated vertices.

Then, $\alpha(G) + \sigma(G) = p$.

Proof. Let S be a vertex cover ^{with $\alpha(G)$ vertices} of G . Then, $V(G) \setminus S$ is an independent

set. Hence, $|V(G) \setminus S| \leq \alpha(G)$ and thus $p - |S| \leq \alpha(G)$. This

implies that $p \leq \alpha(G) + |S| = \alpha(G) + \alpha(G)$. On the other hand,

let T be an independent set of G such that $|T| = \alpha(G)$. Then

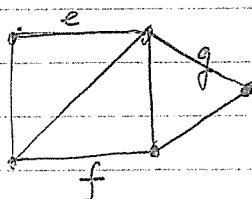
$V(G) \setminus T$ is vertex cover of G . By the fact $\alpha(G) \leq |V(G) \setminus T| = p - \alpha(G)$,
(if 一边没置到, T 就不是独立集) (min.)

we have $p \geq \alpha(G) + \alpha(G)$.

Definition (Edge-cover)

An edge cover of a graph is a set of edges M such that all vertices of G are incident to M , i.e., for each $v \in V(G)$, v is incident to an edge in M .

e.g.



$\{e, f, g\}$ is an edge cover.

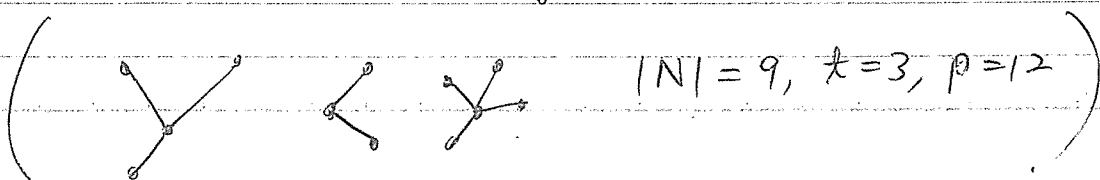
The edge cover number of G , $\sigma(G) = \min \{ |M| \mid M \text{ is an edge cover} \}$

$$(.) \quad \alpha_1(G) \geq \left\lceil \frac{|G|}{2} \right\rceil. \quad (\text{Each edge can cover two vertices.})$$

Theorem $\alpha_1(G) + \alpha_1(G) = p. \quad (G \text{ is a connected graph.})$

Proof. Let M be a matching in G with $\alpha_1(G)$ edges. Then, for each vertex not in M , v , is incident to a vertex in M if v is in an edge of G . Assume that there t vertices not in M , i.e., $p = 2|M| + t$. Now, by taking every edge in the matching M and the set of t edges not in M but incident to M , we have an edge cover with $|M| + t$ edges. This implies that $\alpha_1(G) \leq |M| + t$. As a consequence, we have $p = 2|M| + t = \alpha_1(G) + |M| + t \geq \alpha_1(G) + \alpha_1(G)$.

On the other hand, let N be an edge cover of G with minimum number of edges, i.e., $|N| = \alpha_1(G)$. Notice that $\langle N \rangle_G$ is a disjoint union of stars. (You can not find P_4 in $\langle N \rangle_G$.) Assume that there are t stars. Then, $p = |N| + t$. By the fact that in $\langle N \rangle_G$ we can find a matching of size t , $p \leq |N| + \alpha_1(G)$. ($\alpha_1(G) \geq t$.)



↑
maximum
number

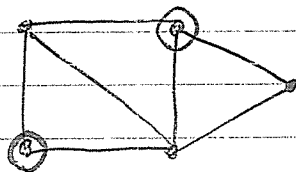
Combining the above two theorems, we have

$$(1) \quad \alpha(G) + \sigma(G) = \alpha_1(G) + \sigma_1(G) = p = |G|.$$

(2) If G is a bipartite graph, then $\alpha(G) = \alpha_1(G)$ and thus $\sigma_1(G) = \sigma(G)$.

Definition (Dominating set)

A set of vertices D in G is a dominating set if $|D| = |G|$ or for each vertex v not in D , v is incident to a vertex of D .



Definition (Domination number)

The domination number of G , $\gamma(G)$, is defined as

$$\gamma(G) = \min. \{ |D| \mid D \text{ is a domination set of } G \}.$$

(*) Finding the domination number of a graph G is a very difficult problem.

Theorem For every graph $\tau(G) \leq \alpha(G)$.

Proof. Let S be an independent with $\alpha(G)$ vertices. If G has no edges, then $\alpha(G) = |G|$ and thus $\tau(G) \leq \alpha(G)$. On the other hand, if $E(G) \neq \emptyset$, then for each vertex v not in S , v must be incident to a vertex in S . For otherwise, we have a larger independent set than S . $\rightarrow \leftarrow$ This implies that $\tau(G) \leq \alpha(G)$.

(*) We can find an independent set by Greedy algorithm. (Dominating set is more complicated to find.)

Step. Find one vertex for D , delete all its neighbors and then continue.

(**) In average, in every $\Delta(G)+1$ vertices we have a vertex for independent set. That is, every vertex is of probability larger or equal to $\frac{1}{\Delta(G)+1}$. Hence, $\alpha(G) \geq \frac{|G|}{\Delta(G)+1}$.

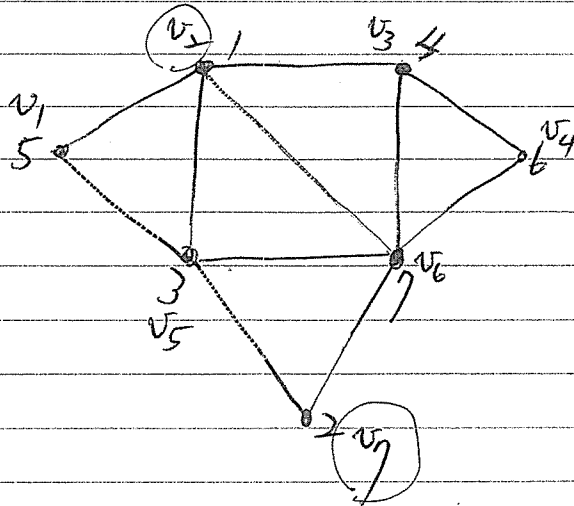
(*) Based on expectation idea, $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)}$.

(Every vertex of $\deg_G(v)$ has a probability $\frac{1}{1 + \deg_G(v)}$ to be selected into the independent set.)

The idea of proof

Let G be a graph of order p . Now, we let $f: V(G) \xrightarrow{\text{r.v.}} \{1, 2, \dots, n\}$ be a function defined randomly, see Figure below for an example.

Now, let S be the set ($\subseteq V(G)$) such that $x \in S$ if $f(x)$ is smaller than $f(y)$ for each $y \in N_G(x)$. So, starting from $v_1, \dots, v_7$, $S = \{v_2, v_7\}$. Clearly, S is an independent set since in S , none of them is a neighbor of the other vertices. Now, since each vertex is chosen with probability $\frac{1}{1 + \deg_G(v)}$, the proof follows by expectation rule.



(*) Since $\alpha(G)$ is maximized we have a lower bound for $\alpha(G)$.

(*) Can you use a probabilistic method to find a dominating set of G ?

Fact 3 $\gamma(G) \geq \rho(G)$.

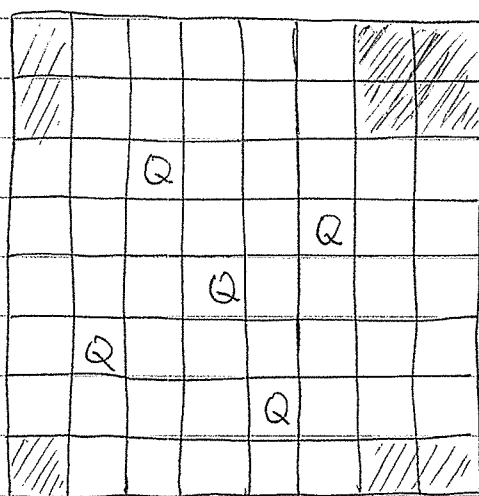
Proposition If $\delta(G) \geq 2$, then $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$.
(G contains no isolated vertices.)

Proof. Let D be a dominating set of G such that $|D| = \gamma(G)$.

Now, consider $W = V(G) \setminus D$. For each vertex $v \in D$, v is adjacent to a vertex in W since D is a minimal dominating set. Hence,

W is a dominating set of G and therefore $|W| \geq |D|$. This

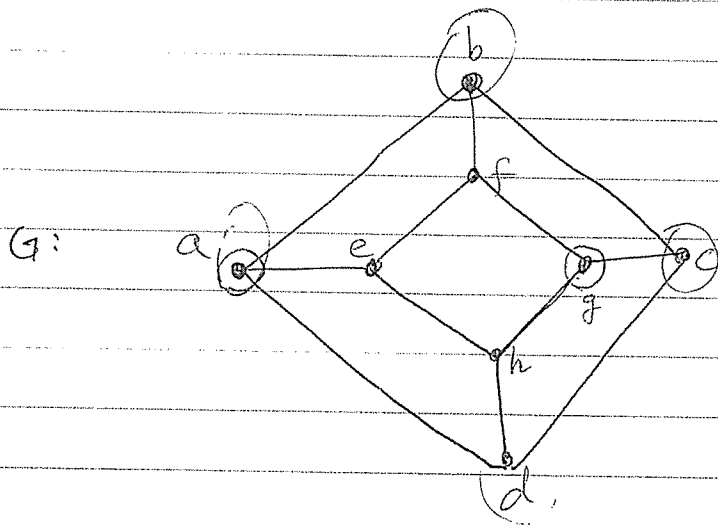
concludes the proof. ■



Five-queens problem

(*) Five queens can dominate the chessboard.

Why four queens are not enough?



$\{a, g\}$ is a dominating set of G .

Dominating sets of G can be obtained by using generating functions:

$$(a + b + e + d)(b + a + f + c)(c + b + g + d)(d + a + h + c)(e + a + f + h) \cdot$$

$$(f + b + e + g)(g + c + f + h)(h + d + e + g)$$

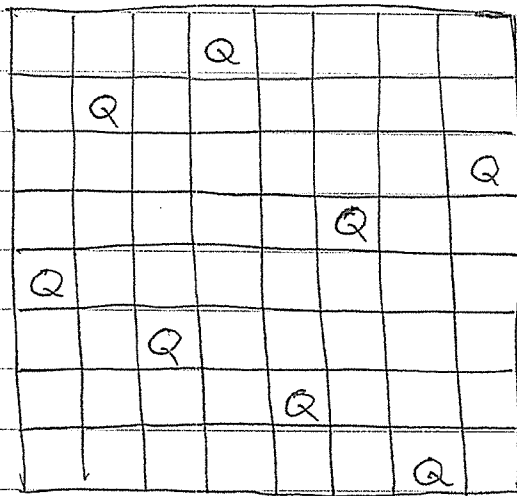
$$= \underline{a^4 \cdot g^4 + b^4 h^4 + b^3 \cdot f \cdot h^4 + \dots} \quad (\text{Many terms})$$

Each of them provides a dominating set! (?)

Relationship between $\alpha(G)$ and $\gamma(G)$ Proposition

If S is a maximal independent set of G , then S is a dominating set of G . Moreover, $\alpha(G) \geq \gamma(G)$.

Proof. Since S is a maximal independent set, for each $v \in V(G) \setminus S$, v is adjacent to a vertex in S . This implies that S is a dominating set and thus $|S| \geq \gamma(G)$. By choosing S such that $|S| = \alpha(G)$, the proof follows. $\blacksquare$



Independent 8 queens on a chessboard.