

FGT Lecture 8

Week 8

Dec. 3

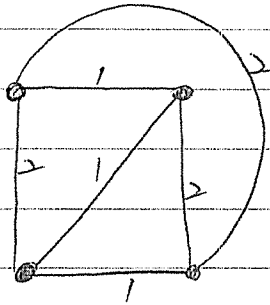
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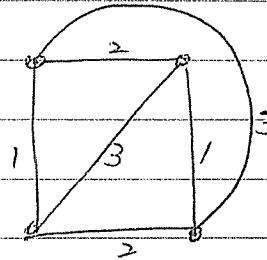
Edge-coloring

Definition A k -edge-coloring π of a graph G is a mapping from $E(G)$ into $\{1, 2, \dots, k\}$. π is a proper k -edge-coloring if incident edges receive distinct image, i.e., $\pi(e) \neq \pi(f)$ provided $|e \cap f| = 1$.

Examples



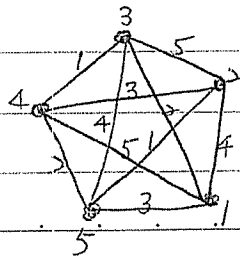
2-edge-coloring



proper 3-edge-coloring

Fact 1 If G has a proper k -edge-coloring, then $k \geq \Delta(G)$.

Fact 2 If K_5 has a proper k -coloring, then $k \geq 5$.



(Each color occurs at most "on" 2 edges!)

Definition (Chromatic index)

The chromatic index of a graph G is the minimum number k such that G has a proper k -edge-coloring.

The chromatic index of G is denoted by $\chi'(G)$.

In what follows, all edge-colorings are "proper".

Fact 3 $\chi'(K_n) = \begin{cases} n-1, & n \text{ is even;} \text{ and} \\ n, & n \text{ is odd.} \end{cases}$

Remark If M_i is a set of edges in G colored "i", then M_i is a matching, i.e., a set of independent edges.

Proof of Fact 3

Let $V(K_n) = \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\} \pmod{n}$. Let $\pi(\{i, j\}) = t$.
Consider n is odd.

provided $i + j \equiv 2t \pmod{n}$. Then π is an n -coloring for

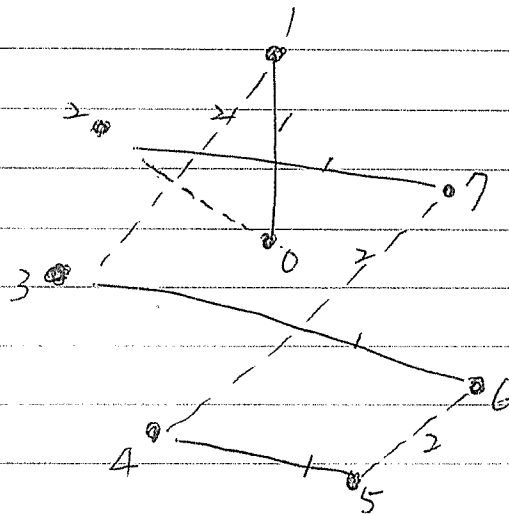
the case n is odd. This implies that $\chi'(K_n) \leq n$ when

n is odd. Since each color can occur at most $\frac{n-1}{2}$ times,

$$\chi'(K_n) \geq \left\lceil \frac{\binom{n}{2}}{\frac{n-1}{2}} \right\rceil = n. \text{ Hence, } \chi'(K_n) = n \text{ if } n \text{ is odd.}$$

n is even

Since $\Delta(K_n) = n-1$, $\chi'(K_n) \geq n-1$ (?). So, it suffices to find an $(n-1)$ -edge-coloring of K_n when n is even. The coloring pattern can be seen as the following case $n=8$.



In fact, the coloring π can be $\pi(\{i,j\}) = t$ if (1) $i=0$ and $j=t$ (2) $i+j \equiv 2t \pmod{n-1}$. Since we have an $(n-1)$ -edge-coloring of K_n (n is even), $\chi'(K_n) \leq n-1$. $\square$

Important!

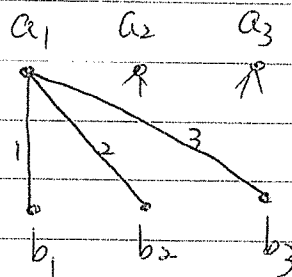
✓ Fact 4 $\chi'(K_{m,n}) = \max\{m, n\}$.

It suffices to prove that $\chi'(K_{n,n}) = n$. Since an n -edge-coloring

of $K_{n,n}$ is corresponding to the existence of a Latin square of order n , the proof follows. (?)

Example

	b_1	b_2	b_3
a_1	1	2	3
a_2	2	3	1
a_3	3	1	2



Latin Square

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3

Latin Rectangle

Definition A Latin square of order n based on $[1, n]$ is an

$n \times n$ array such that each element from $[1, n]$ occurs exactly once in each row and each column.

Theorem (Vizing, 1964) (The proof is omitted in this course.)

If G is a simple graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Definition (Class 1 and Class 2)

A graph G is of Class 1 if $\chi'(G) = \Delta(G)$ and Class 2 otherwise.

(*) Determine whether a graph is of Class 1 or 2 is a very difficult but important problem in Chromatic Theory.

(**) Use the non-existence of Hamilton cycles in the Petersen graph!

Proposition Petersen graph is of Class 2.

Proof. Note that Petersen graph is a 3-regular graph. Let

P denote the graph. Then $\Delta(P) = 3$. Assume that P is of

Class 1, i.e., we have $\chi'(P) = 3$. Now, consider the subgraph G of

P induced by the edges colored 1 and 2. By observation, G

is a 2-regular spanning subgraph of G . The only possibility

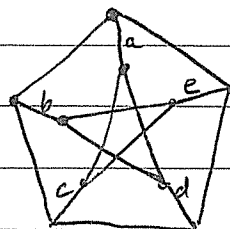
for such a graph G is a Hamilton cycle. (?) This is not possible

since P contains no Hamilton cycles. $\rightarrow \leftarrow$

Proof. Let the edges a, b, c, d and e be link-edges in Petersen graph, see figure below.

Suppose that $\chi'(P) = 3$.

P :



Then, each color will occur exactly 5 times.

Then, there exists a color "1" in

$\{1, 2, 3\}$, which occurs in link-edges at least two times.

Case 1 "1" occurs 5 times.

In this case the two C_5 's (in and out) must be 2-edge-colorable, not possible.

Case 2 "1" occurs 4 times.

Now, there is no edge which can be colored by 1, not possible.

Case 3 "1" occurs 3 times.

Then, only one edge in the two C_5 's can be colored "1", and we have only 4 1's, not possible.

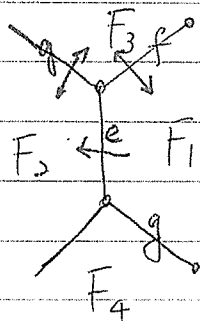
Case 4 "1" occurs 2 times.

Again, in total "1" can occur 4 times, not possible either.

2-connected

Theorem Let G be a planar, ^{2-connected} cubic graph, _(3-regular) Then $\chi'(G) = 3$.

Proof. Since G is planar, $\chi_f(G) \leq 4$. So, we can use $\{(0,0), (0,1), (1,0), (1,1)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ _(Face coloring) to color the faces such that incident faces receive distinct colors. Now, if e is the boundary of faces F_1 and F_2 , we color e by using the addition, _{of colors on F_1 and F_2} defined on $\mathbb{Z}_2 \times \mathbb{Z}_2$.



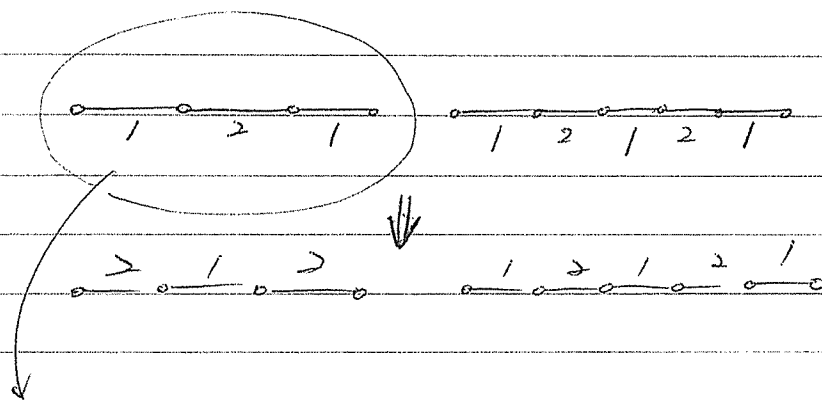
Clearly, we use only three colors _{in} $\{(0,1), (1,0), (1,1)\}$ since adjacent faces will receive distinct colors and thus their sum is not $(0,0)$. Moreover, if e and f are incident edges, then the corresponding faces adjacent to F_1 will be different, see figure above. This implies the colors of e and f are distinct. (e, f, g are of distinct colors!)

(*) If we don't know $\chi_f(G) \leq 4$, then the proof of this theorem is very difficult. (!)

(**) If G has a k -edge-coloring, then G has an equitable k -edge-coloring such that the difference of colors occurred in the coloring is at most 1.

Proof. By color-switching method.

Remark. If color 1 occurs five times and color 2 occurs three times, then we put these eight edges together to induce a subgraph, see the figure below.



This configuration exists if we have more color "1" than color "2".

Independent Sets and Matchings

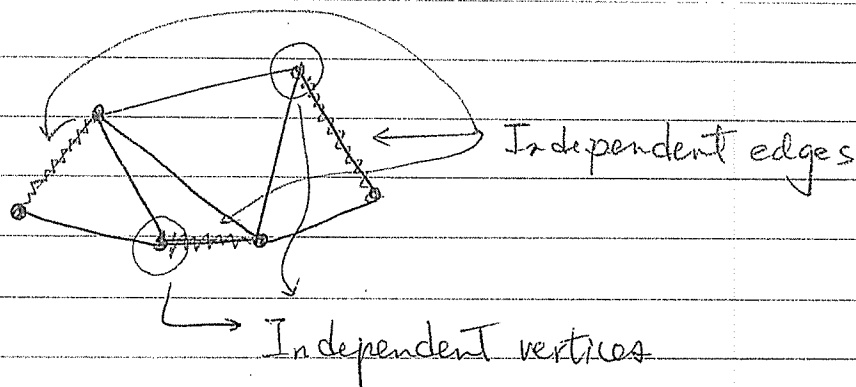
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Definition

A set of vertices is called independent if no two of its vertices (edges) are adjacent. (incident)



Definition (Matching)

A set of independent edges is also known as a matching.

Remark If $\chi(G) = k$, then G has k independent sets

which partition $V(G)$. If $\chi'(G) = k$, then G has k matchings

which partition $E(G)$.

(*) We are aiming at finding an independent set of a graph which is of maximum size. The ^{maximum} size of an independent set (of vertices) is called the independence number of G , denoted by $\alpha(G)$.

Easy facts: $\alpha(K_n) = 1$, $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha(T) = \lfloor \frac{|V|}{2} \rfloor$.
(T is a tree.)

Proposition Let $\chi(G) = k$. Then, $\alpha(G) \geq \frac{|G|}{k}$, or equivalently

$$\chi(G) \cdot \alpha(G) \geq |G|.$$

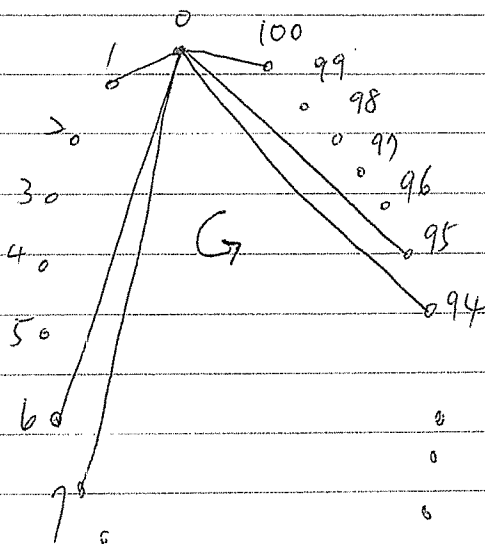
(*) Finding $\alpha(G)$ for a general graph G is very difficult.

(*) Bonus problem

Let G be a graph defined on $\mathbb{Z}_{101}$. Two vertices

i and j are adjacent if and only if $i - j \in \{1, 6, 7, -1, -6, -7\}$

(mod 101). Find $\alpha(G)$.



Definition (Matching)

A matching of a graph G is a set of independent edges.

We use $\alpha_1(G)$ to denote the maximum size of a matching.

Clearly, a matching has at most $\frac{|G|}{2}$ edges. If M is a matching of size $\frac{|G|}{2}$ when $|G|$ is even, then M is called a perfect matching of G and the graph induced by M is called a 1-factor.

Definition (k -factor)

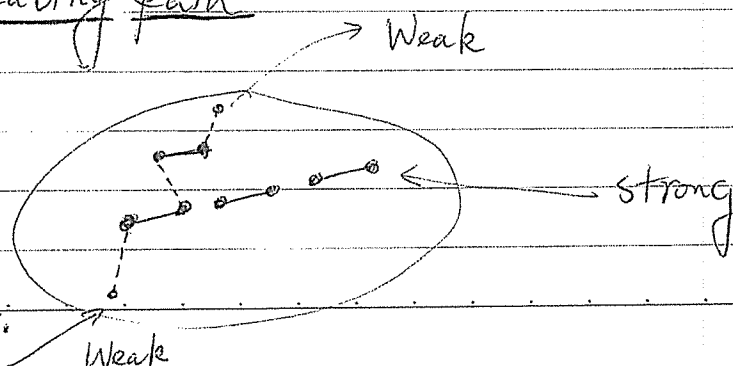
spanning k -regular

A $\wedge$ subgraph of a graph G is called a k -factor.

(*) Finding $\alpha_1(G)$ of a general graph G is comparatively easy. (There exists a polynomial time algorithm to find $\alpha_1(G)$.)

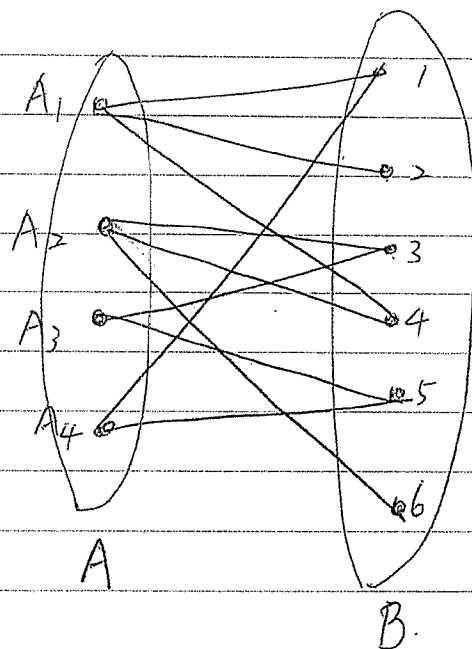
Idea: Alternating path

We can enlarge a matching if we have a path: Weak — Weak — Weak



Marriage Problem

Given a bipartite graph $G = (A, B)$, what is the condition for which G contains a matching saturates A , i.e., a matching incident to every vertex of A .



$$A_1 = \{1, 2, 4\}$$

$$A_2 = \{3, 4, 6\}$$

$$A_3 = \{3, 5\}$$

$$A_4 = \{1, 5\}$$

(*) The reason why this is called "marriage problem" follows from the fact that every "one" in A finds a mate in B , but not the same mate.

(**) This problem can be thought as finding representatives from a collection of sets.

Theorem (P. Hall, 1935)

1. Graph Version Let $G = (A, B)$ be a bipartite graph with

$A = \{a_1, a_2, \dots, a_m\}$. Then G contains a matching saturates A

if and only if for any $1 \leq k \leq m$, $|\bigcup_{j=1}^k N_G(a_j)| \geq k$.

2. Set Version Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a collection of

subsets of $[1, n]$. Then, there exists a vector $(a_1, a_2, \dots, a_m)$

where $a_i \in A_i$, $i = 1, 2, \dots, m$ and $a_i \neq a_j$, for $1 \leq i \neq j \leq m$, if and

only if for any $1 \leq k \leq m$, $|\bigcup_{j=1}^k A_j| \geq k$. ----- (1)

Remark In set version, $(a_1, a_2, \dots, a_m)$ is called a system of distinct representatives (SDR in short) and the condition

(1) is known as the Hall's condition.

We shall prove the set version in what follows.

Proof. $(\Leftarrow)$ By induction on the number of sets in $\mathcal{A}$, i.e, m .

Clearly, it is true for $m=1$. Assume that the assertion is

true for $1 \leq m \leq h$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_{h+1}\}$ be a collection

of $k+1$ sets satisfying (1). Now, we consider two cases.

Case 1 For any $1 \leq k \leq h$, the union of k sets in $\mathcal{A}$ contains at least $k+1$ elements.

$$\mathcal{A}' =$$

Let $a_{k+1} \in A_{k+1}$ and consider $\bigwedge \{A_1 \setminus \{a_{k+1}\}, A_2 \setminus \{a_{k+1}\}, \dots, A_k \setminus \{a_{k+1}\}\}$.

← By observation, condition (1) holds for $\mathcal{A}'$ and thus $\mathcal{A}'$ has an

SDR $(a_1, a_2, \dots, a_k)$. Combine with a_{k+1} , we have an SDR $(a_1, a_2, \dots, a_{k+1})$

for $\mathcal{A}$.

Case 2 There exists a $1 \leq k' \leq h$, such that the union of some k' sets contains exactly k' elements, say $\mathcal{A}' = \{A_1, A_2, \dots, A_{k'}\}$.

Let $S = \bigcup_{i=1}^{k'} A_i$. Consider the collection of sets $\mathcal{A}'' = \{A_{k'+1} \setminus S, A_{k'+2} \setminus S, \dots, A_h \setminus S\}$.

Now, both $\mathcal{A}'$ and $\mathcal{A}''$ satisfy (1). By induction and the combination

of their SDR's respectively, we conclude the proof.

(Note that any $\bigwedge$ k subsets of $\mathcal{A}''$ contains at least k elements, for

otherwise we have a contradiction. (?))

(For example, if $(A_{k'+1} \setminus S) \cup (A_{k'+2} \setminus S)$ contains only one element,

the union of
then $\bigwedge A_1, A_2, \dots, A_{k'}, A_{k'+1}, A_{k'+2}$ will contain only $k'+1$ elements. $\rightarrow \leftarrow (1)$

1935

Theorem (p. Hall) A bipartite graph $G = (A, B)$ contains a

matching saturates A if and only if for every $S \subseteq A$, $N_G(S)$

$= \bigcup_{x \in S} N_G(x)$ contains at least $|S|$ elements of B , i.e., $|N_G(S)| \geq |S|$.

Proof. (1st) $(\Rightarrow)$ By the existence of a matching saturates A .

$(\Leftarrow)$ Now, it suffices to prove that there are $|A|$ vertex-

disjoint A - B edges (and thus a matching saturates A). Suppose not.

Then, there exists a subset A_1 of A and a subset B_1 of B such that
 $(A_1 \cup B_1)$ is a vertex cover of G (König-Egervary Thm.)
 there is no edge between $A \setminus A_1$ and $B \setminus B_1$, see Figure 1, and

$|A_1| + |B_1| < |A|$. (The number of independent edges is less than $|A|$.)

Hence, there are no edges between $A \setminus A_1$ and $B \setminus B_1$, equivalently

$N_G(A \setminus A_1) \subseteq B_1$. Then, $|N_G(A \setminus A_1)| \leq |B_1| < |A| - |A_1| = |A \setminus A_1|$. $\Rightarrow$ Hall's condition 不对!

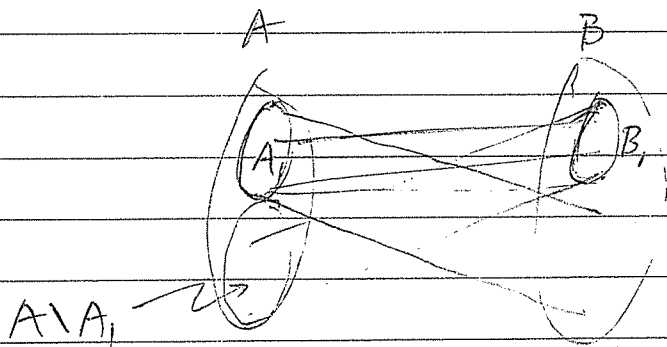


Figure 1.

如果 A 的某
 个顶点找不到对
 应的点 (B 中), 则
 存在 A_1 .

Theorem (König, ~1930)

Let G be a k -regular bipartite graph. Then, G has a matching saturates A .

(A, B)

Proof. Note that if $G = (A, B)$ is regular, then $|A| = |B|$.

($|G| = k \cdot |A| = k \cdot |B|$) Let $A = \{a_1, a_2, \dots, a_n\}$ and

$B = \{b_1, b_2, \dots, b_n\}$. Now, we shall use Hall's Theorem to prove the theorem.

Let S be a subset of A , ^{say} $S = \{a_1, a_2, \dots, a_s\}$. By assumption

$$N_G(S) = \bigcup_{i=1}^s N_G(a_i) = \{b_1, b_2, \dots, b_t\} = T \text{ and } \sum_{i=1}^s |N_G(a_i)| = k \cdot s.$$

$$\text{Hence, } \sum_{i=1}^s |N_G(a_i)| = \sum_{j=1}^t |N_{\langle S, T \rangle_G}(b_j)| = k \cdot s. \text{ But,}$$

$$|N_{\langle S, T \rangle_G}(b_j)| \leq k. \text{ This implies that } t \geq s, \text{ and}$$

thus $|N_G(S)| \geq |S|$. The proof follows by using Hall's Theorem.

—— (再叙述一次)

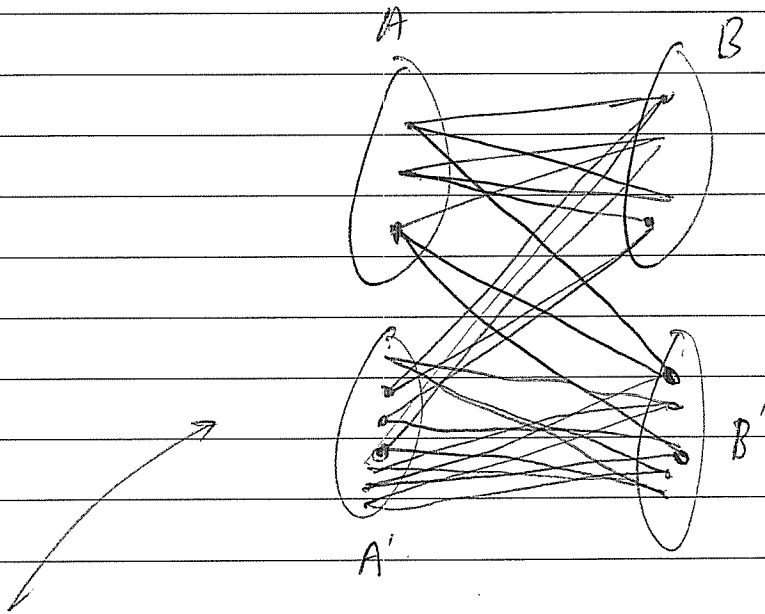
Discrete Version of Hall's Theorem (Set Version)Definition (System of Distinct Representatives, SDR)

Let S be a collection of subsets of S , $S = \{S_1, S_2, \dots, S_n\}$.

Theorem If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.

Proof. Claim We can add more vertices if necessary to $G = (A, B)$ to obtain a $\Delta(G)$ -regular bipartite graph $\tilde{G}$. (?)

Then, by using the proof in p.13, $\tilde{G}$ has an edge-coloring using $\Delta(G)$ colors. Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) = \Delta(G)$. This implies that $\chi'(G) = \Delta(G)$.



Can you find a way to prove this claim?

$$(a_1, a_2, \dots, a_n)$$

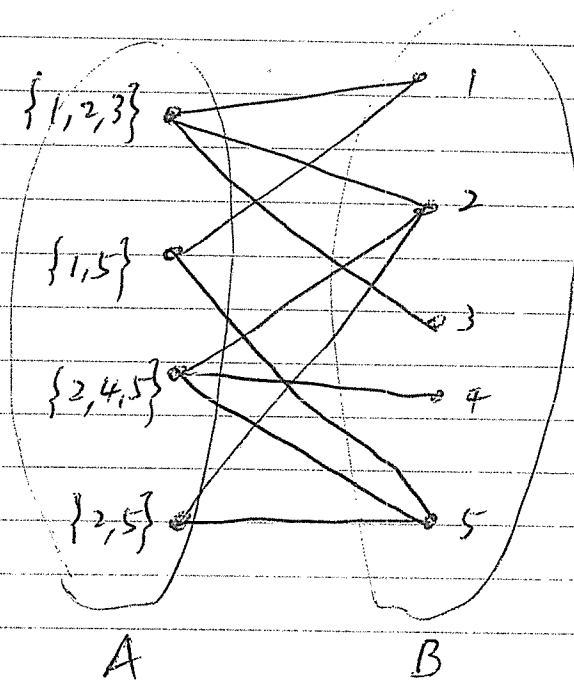
An ordered n -tuple is called a system of distinct representatives (SDR)

if $a_i \in S_i$, $i=1, 2, \dots, n$; and all a_i 's are distinct.

e.g. $S = \{\{1, 2, 3\}, \{1, 5\}, \{2, 4, 5\}, \{2, 5\}\}$ and $S = \{1, 2, 3, 4, 5\}$.

Then, $(1, 5, 4, 2)$ is an SDR of S .

(*) The relationship between sets and graphs.



Let A be the vertex set
vertex
each of them is the set

in S and B be the set
of elements in S .

(*) e is an edge of (A, B)

if and only if the
containment holds.

Applications

- Let $M_{n \times n}$ be a $(0, 1)$ -matrix such that each row and each column has exactly k 1's. Then, there exists a permutation matrix hidden in M , i.e., a sub-matrix with exactly one

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

2. In fact, M can be partitioned into k permutation matrices
 3. The above result can be transformed to a k -regular bipartite graph (A, B) with $|A| = |B| = n$.
 4. This graph $G = (A, B)$ is of Class 1, i.e., $\chi'(G) = k$.
- (*) Let G be a bipartite graph, Then, $\chi'(G) = \Delta(G)$.
- Can you prove this?