

Theorem (Euler)

Let  $G$  be a <sup>connected</sup> planar graph with  $p$  vertices,  $q$  edges and  $f$  faces (regions). Then  $p - q + f = 2$ .

Proof. By induction on  $q$ . Since  $G$  is connected,  $G$  has at

least  $p-1$  edges. (?) If  $G$  has  $p-1$  edges and  $G$  is connected, (See next pages.)

then  $G$  is a tree which contains no cycles. This implies that

$f = 1$  and thus  $p - (p-1) + 1 = 2$ . The assertion is true

for "minimal" graphs. Let the hypothesis is true for  $\|G\| \geq p-1$ .  $k =$

Now, consider  $G$  with  $k+1$  edges. Clearly,  $G$  contains a cycle.

Let  $e$  be a cycle edge. Since  $G$  is a connected planar

graph, <sup>(with  $q$  faces)</sup>  $G - e$  is also a connected planar graph. Moreover,

$\|G - e\| = k$  and  $G - e$  has  $k$  edges and  $q-1$  faces. By

induction  $p - k + q - 1 = 2$  and thus

$p - (k+1) + q = 2$ . This concludes the proof.  $\square$

## Tree

- A graph with no cycles is called a forest.
- A connected forest is a tree.
- A graph with no cycles is acyclic.
- (••) A connected acyclic graph is a tree.
- (\*) A tree contains at least two degree one vertices.  
(Leaves)

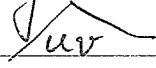
Proof. Let  $T$  be a tree and  $u, v \in V(T)$  such that  $d(u, v) = \text{diam}(T)$ . Then, both  $u$  and  $v$  are degree vertices. For otherwise, either we have a larger diameter or a cycle.

(\*\*) If  $T$  is a tree, then  $\|T\| = |T| - 1$ .

Proof. By induction on  $|T|$ . The initial case is easy <sup>Trees</sup> to check. Assume that the assertion is true for <sub>of order  $k$</sub> .

Let  $T$  be a tree of order  $k+1$ . Now, by deleting a vertex of degree 1 and apply the induction hypothesis, we have the proof.

The following properties are "good" for Trees.

- (1) For any two vertices  $u$  and  $v$  in a tree  $T$ , there exists a unique path from  $u$  to  $v$ .
  - (2) A tree is 1-edge-connected, i.e., every edge of a nontrivial tree is a bridge. ( $\forall e \in E(T)$ ,  $T - e$  is disconnected.)
  - (3) A connected graph with minimum number of edges is a tree.
  - (4) Adding any edge to a tree where  $u$  and  $v$  are not adjacent, we obtain a "unique" cycle. 
  - (5) A graph with exactly one cycle is called a unicyclic graph.
  - (6) We can set any vertex of a tree as its root.
  - (7) A tree is a bipartite graph, so is a forest.
  - (8) If  $G$  is a connected graph, then  $G$  contains a spanning tree as its subgraph. (!)
- (Delete cycle edges until there exist no cycles.)

Theorem If  $G$  is a planar graph, then  $\|G\| = 3|G| - 6$ .

Proof By observation, if  $G$  has maximum size, then each region of  $G$  is a triangle. Since each edge of  $G$  is in the boundary of exact two regions,  $3 \cdot f = 2 \cdot q$  where  $f$  is the number of regions and  $q$  is the size of  $G$ , i.e.,  $q = \|G\|$ . Now, by Euler's formula  $p - q + f = 2$  equivalently

$$|G| - \|G\| + \frac{2}{3}\|G\| = 2$$

$$\Rightarrow 3|G| - 6 = \|G\|. \quad (G \text{ is a maximal planar graph!}) \quad \blacksquare$$

Corollary If  $G$  is a planar graph, then  $\|G\| \leq 3|G| - 6$ .

Corollary In any <sup>planar</sup> graph, there exists at least one vertex of degree smaller than 6.

Corollary The degree sum of a planar graph is at most  $6|G| - 12$ .

This corollary is very useful.

### Theorem (5-color Theorem)

If  $G$  is a planar graph, then  $\chi(G) \leq 5$ .

Proof. It suffices to prove the theorem for maximal planar graph. By induction on  $|G|$  and it is true for graphs of smaller orders. Assume the assertion is true for all graphs (planar) of order  $k$  and let  $G$  be a maximal planar graph of order  $k+1$ .

Since  $G$  is maximally planar,  $\delta(G) \geq 3$ . By the collary mentioned above, we have three cases to consider:  $\delta(G) = 3$  or  $4$  or  $5$ .

Case 1.  $\delta(G) = 3$

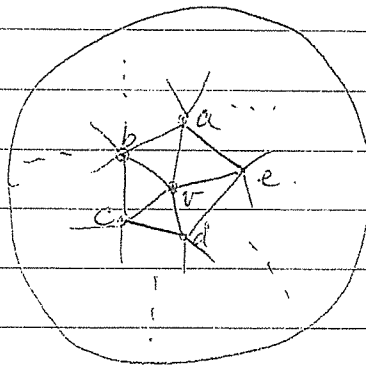
Case 2.  $\delta(G) = 4$

} Easy to see. (?)

Case 3.  $\delta(G) = 5$

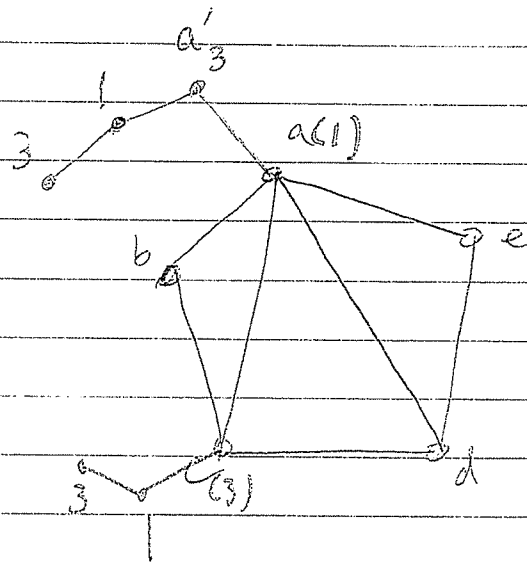
Let  $\deg(v) = 5$ .

Consider  $\tilde{G} = (G - v) + ac + ad$ .



By induction hypothesis,  $\chi(\tilde{G}) \leq 5$ . Now, if the set of vertices  $\{a, b, c, d, e\}$  uses at most 4 colors, then the vertex  $v$  in  $V(G)$  can be colored by a distinct color, and we have the proof.

On the other hand, if  $\{a, b, c, d, e\}$  uses five colors, say  $\varphi(a) = 1, \varphi(b) = 2, \varphi(c) = 3, \varphi(d) = 4$  and  $\varphi(e) = 5$ , then we need recolor  $\tilde{G}$  as follows.



First, we consider the chains starting from  $a$  and  $c$  respectively in which the vertices are colored 1 and 3. If one of them terminate somewhere different from the other say  $\langle a, \dots \rangle$  vertex ( $a$  or  $c$ ), then recolor  $a$  by 3 and  $a'$  by 1,  $\dots$ , etc.

Therefore, the colors used are 4 and we have a color for  $v$ .

On the other hand, if those two chains are the same, then instead of changing the colors, we consider the chains starting at  $b$  and  $d$  respectively (colored  $2, 4, 2, \dots$ ). Since  $\tilde{G}$  is a planar graph, these two chains can not be the same, and thus we can recolor one of them, say from  $2, 4, 2, \dots$  to  $4, 2, 4, \dots$ . Again, we have an extra color for  $v$ . This concludes the proof. 

(\*) The fact that in a planar graph  $G$ ,  $\delta(G) \leq 5$  is very important.

### Definition ( $d$ -degenerate)

A graph  $G$  is called  $d$ -degenerate if for each subgraph  $H$  of  $G$ ,  $\delta(H) \leq d$ .

Therefore, a tree is 1-degenerate and a planar graph is 5-degenerate. Clearly, an  $r$ -regular graph is  $r$ -degenerate.

### Theorem (about vertex-coloring)

If  $G$  is  $d$ -degenerate, then  $\chi(G) \leq d+1$ .

Proof. Suppose not. Let  $G$  be a counter-example which has minimum order. That is, for each  $v \in V(G)$ ,  $\chi(G-v) \leq d+1$ , but  $\chi(G) = d+2$ . Now, let  $\deg(v) = \delta(G)$  for some  $v \in V(G)$ .

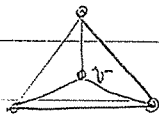
Since  $\chi(G-v) = d+1$  and  $\deg(v) \leq d+1$ ,  $\chi(G) \leq d+1$ .  $\rightarrow \leftarrow$   $\square$

We can also use (\*) to prove the well-known Wagner's theorem.

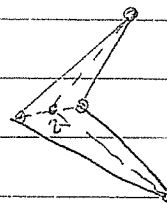
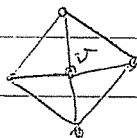
Theorem. If  $G$  is a planar graph, then  $G$  has a drawing on a plane such that each edge is a straight line segment.

Proof. It suffices to consider the case when  $G$  is a maximal planar graph (each region is a triangle). Suppose that  $G$  is such a graph of smallest order in which no straight segment drawing is possible. That is, for each  $v \in V(G)$ ,  $G-v$  is a planar graph and  $G-v$  has a straight segment drawing. Now, let  $\deg(v) = \delta(G) \leq 5$ . Then, the drawing of  $G$  is obtained by considering the following cases.

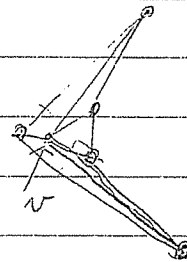
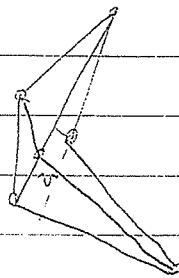
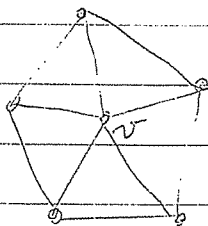
(1)  $\deg(v) = 3$ .



(2)  $\deg(v) = 4$ .



(3)  $\deg(v) = 5$ .



## Theorem (Regular Polyhedra)

There are exactly five regular polyhedra.

Proof. Clearly, a regular polyhedron is a planar graph, moreover, it is a regular graph. Since they <sup>are</sup> polyhedra, the corresponding graphs are either 3-regular, 4-regular or 5-regular. Hence, we have the following equalities:

$$\begin{cases} (1) & p - q + r = 2 \quad \left( k\text{-regular and } \overset{\text{the boundary of}}{\text{each face is of length } l} \right) \\ (2) & kp = 2q = rl \end{cases}$$

$$\Rightarrow q \left( \frac{2}{k} - 1 + \frac{2}{l} \right) = 2 \text{ and } \frac{2}{k} - 1 + \frac{2}{l} > 0$$

$$\Rightarrow (k-2)(l-2) < 4. \text{ Since } k, l \geq 3, k, l \leq 5.$$

This implies that there are at most 9 possibilities for  $(k, l)$ . But,  $(4, 4), (4, 5), (5, 4), (5, 5)$  are not possible. This implies that we have at most five regular polyhedra.

Now, by the existence of tetrahedron, cube, octahedron,

dodecahedron and icosahedron, we conclude the proof.  $\square$

## Graphs which are not planar

Fact If the girth of a graph  $G$  is  $g(G) = g$ , and  $G$  is planar, then  $|G| \leq \frac{g}{g-2}(p-2)$ .

Proof. Since  $g \cdot r \leq 2g$  and  $p - g + r = 2$ , we conclude that  $p - 2 \geq g \cdot \frac{g-2}{g}$ .

Fact  $K_{3,3}$  is not a planar graph.

The materials about topological graph theory are attached for reference in next pages.  
(the following)

## Theorem (Euler-Poincaré)

Let  $G$  be a  $(p, q)$ -graph which has a 2-cell embedding in an orientable surface of genus  $n$ . Then  $p - q + r = 2 - 2n$  where  $r$  is the number of regions.

Note  $K_5$  can be drawn on  $S_1$  with 5 regions.  
genus 1

## Crossing Number

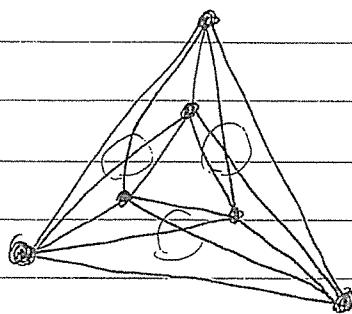
### Definition (Crossing number)

The crossing number of a graph  $G$ , denoted  $cr(G)$ , is (tangling)  
the minimum number of crossings among all drawings of (maximum)  
 $G$  on a plane (or a sphere).

Fact If  $G$  is not a planar graph, then  $cr(G) \geq 1$ .

Fact A drawing of  $G$  on a plane with crossing number  $k$  gives an upper bound of  $cr(G)$ .

Example A drawing of  $K_6$  with 3 crossings

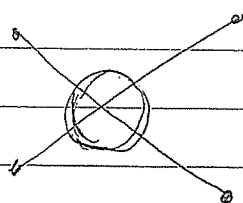


Fact  $cr(K_6) = 3$

Proof. Note that if we change the crossings into vertices, then we have a planar graph.

Let  $cr(K_6) = k$ . Then  $\tilde{K}_6$  (the graph from  $K_6$  after assigning the crossings as vertices) has  $6+k$  vertices,  $15+k$  edges. Since  $\tilde{K}_6$  is a planar graph  $15+k \geq 3(6+k) - 6$ , hence  $k \leq 3$ . ■

(A crossing gives "2" more edges!)



$$\begin{cases} \frac{1}{64} n(n-2)(n-4), \text{ even} \\ \frac{1}{64} (n-1)^2(n-3)^2, \text{ odd} \end{cases}$$

Fact  $cr(K_5) = 1$ ,  $cr(K_{3,3}) = 1$

Problems Determine  $cr(K_n)$  and  $cr(K_{m,n})$ .

(Research problems)

$$? \parallel \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$$

Conjecture If  $x(G) = k$ , Then  $cr(G) \geq cr(K_k)$ .

The conjecture holds for  $k \leq 4$  (trivial!?)

Remark

It is not easy to determine both crossing number and

Ranking number of a graph.

## Chapter 5 Topological Graph Theory

### § 5.1. Basic Notations

Topological graph theory studies the "drawing" of a graph on a surface. A proper drawing on a surface of a graph  $G$  with  $|G| = p$  and  $||G|| = q$  follows the rules :

- (1). There are  $p$  points on the surface which corresponds to the set of vertices in  $G$ ; and
- (2). There are  $q$  curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

- The drawing is on a surface defined on  $\mathbb{R}^3$ .
- A **2-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk defined on  $\mathbb{R}^2$ .
- An **n-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to  $B_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1\}$ .
- A subspace  $M$  of  $\mathbb{R}^3$  is **bounded** if there exists a positive real number  $K$  such that  $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 = K\}$ .
- Let  $M \subseteq \mathbb{R}^3$  be a 2-manifold. Then  $M$  is said to be **closed** if it is bounded and the boundary of  $M$  coincides with  $M$ .
- Let  $M(\subseteq \mathbb{R}^3)$  be a 2-manifold;  $M$  is said to be **orientable** if for every simple closed  $C$  on  $M$ , a clockwise sense of rotation is preserved by traveling once around  $C$ . Otherwise,  $M$  is non-orientable.
- A 2-manifold  $M$  is orientable if and only if it is **two-sided**.

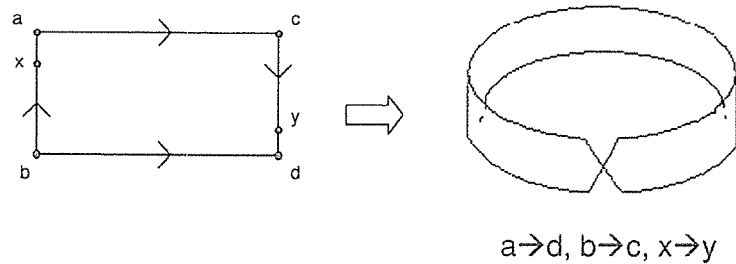
**Definition 5.1.1. (Orientable Surface)**

A surface is a compact orientable 2-manifold that may be thought of as a sphere on which has been placed (inserted) a number of "handles" (holes). A sphere, denoted by  $S_0$ , is the surface of a 3-dimension ball. More precisely,  $S_0 = \{(x, y, z) | x^2 + y^2 + z^2 = r^2, r \in \mathbb{R}^+\}$ .  $S_1$  is known as a torus,  $S_2$  a double torus, and  $S_h$  is a surface obtained by adding  $h$  handles to  $S_0$ .

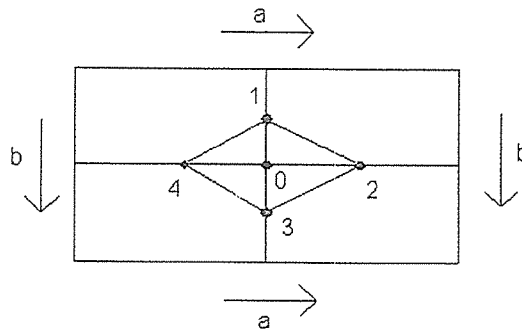
**Definition 5.1.2. (Non-Orientable Surface)**

A surface obtained by adding  $k$  cross-caps to  $S_0$  is known as the non-orientable surface  $N_k$ . (A cross cap is obtained from Möbius band described in what follows.)

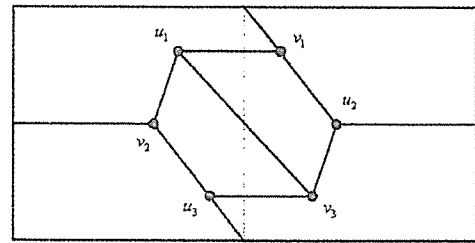
Cross cap: Attach the boundary of a Möbius band to a cycle on  $S_0$  to obtain a cross cap.

**Definition 5.1.3. (Embedding or Imbedding, 2-cell embedding)**

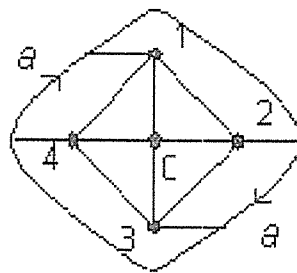
An embedding of a graph in a surface is a continuous 1-1 function from a topological representation of the graph into the **surface**. If every region of the embedding is homeomorphic to a 2-dim open disc, then the embedding is a **2-cell embedding**(圓盤嵌入).



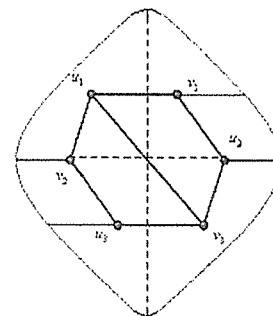
Embedding of  $K_5$  on  $S_1$



Embedding of  $K_{3,3}$  on  $S_1$

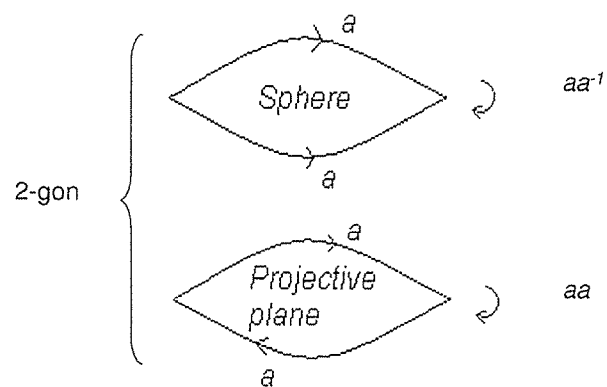


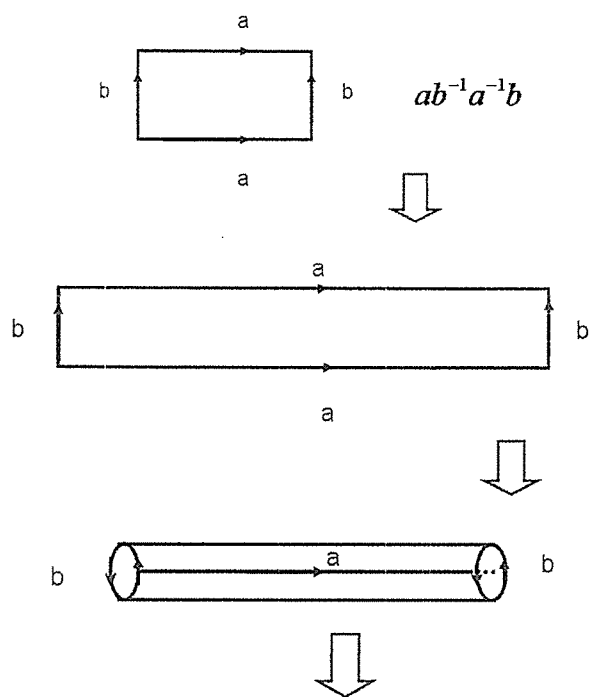
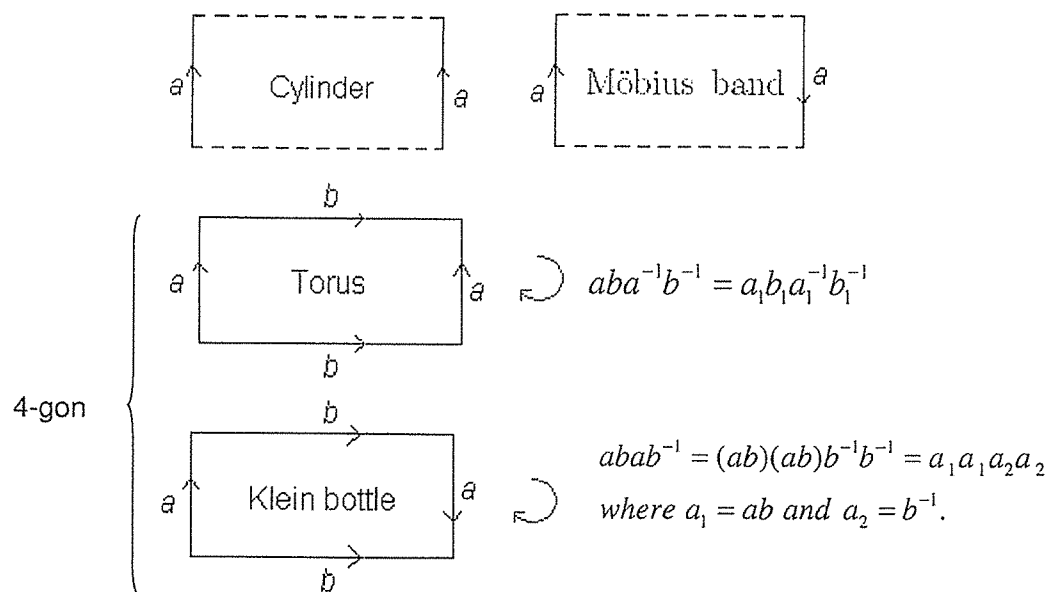
Embedding of  $K_5$  on  $N_1$   
( $N_1$  is also known as a projective plane.)

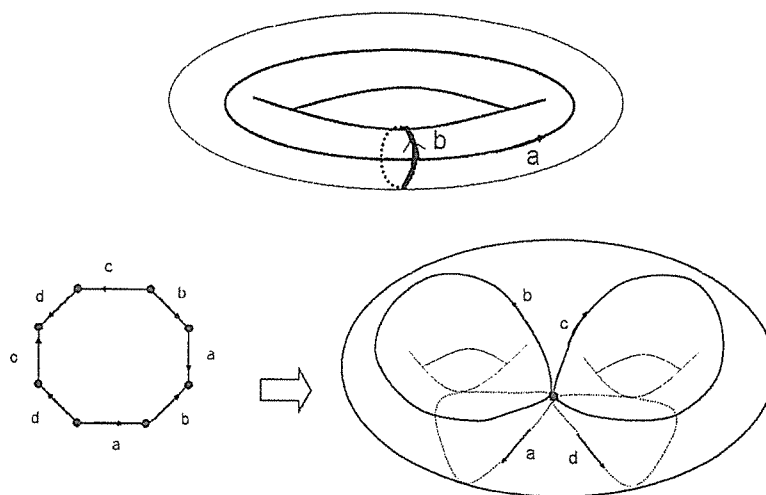


Embedding of  $K_{3,3}$  on  $N_1$

- Surfaces can be represented by a polygon (standard fundamental).







(\*) The standard fundamental polygon for  $S_k$  is  $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}$ .

(\*\*) The standard fundamental polygon for  $N_h$  is  $a_1 a_1 a_2 a_2 \dots a_h b_h$ .

**Definition 5.1.4.** If a graph  $G$  can be embedded in  $S_0$ , then  $G$  is a planar graph.

**Theorem 5.1.1.(Euler's Formula)**

Let  $G$  be a **connected planar** graph which has  $p$  vertices,  $q$  edges and  $r$  regions. Then  $p - q + r = 2$ .

**Proof.** By induction on  $q$ . □

**Corollary 5.1.2.**

Let  $G$  be a planar graph which has  $k$  components,  $p$  vertices,  $q$  edges and  $r$  regions. Then  $p - q + r = 1 + k$ .

**Proof.** By induction on  $k$ .  $k = 1$  is true by Theorem 1. Assume the assertion is true for  $k$ . Let  $G$  be a graph with  $p$  vertices,  $q$  edges and  $r$  regions, and  $G$  have  $k+1 (\geq 2)$  components. Now let  $\tilde{G} = G + e$  ( $e$  connects two components). Then,  $\tilde{G}$  has  $p$  vertices,  $q + 1$  edges,  $r$  regions and  $k$  components. Hence,  $p - (q + 1) + r = 1 + k$ . This implies  $p - q + r = 1 + (k + 1)$ . The proof follows. □