



6th India-Taiwan Conference on Discrete Mathematics

Prime Sum and Prime Difference Graphs

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OUTLINE



Prime Sum
Graphs

G_n



Prime Difference
Graphs

H_n



Summary



CONSEQUENCE OF BERTRAND'S POSTULATE

1998 proved by
L. Greenfield & S. Greenfield

2006 reproduced by
D. Galvin

"For any positive integer n ,
 $\{ 1, 2, \dots, 2n \}$ can be paired
such that the
sum of each pair is a prime."

▲ D. Galvin, Erdos's proof of Bertrand's postulate, April 2006.

▲ L. Greenfield and S. Greenfield, Some problems of combinatorial number theory related to Bertrand's postulate, J. Integer Seq. 1 (1998), Article 98.1.2.



1822 - 1900

JOSEPH LOUIS FRANÇOIS BERTRAND

BERTRAND'S POSTULATE

(Bertrand-Chebyshev Thm.)

For any positive integer $n > 1$
there exists a prime number
 p between n and $2n$

NEW INSIGHTS

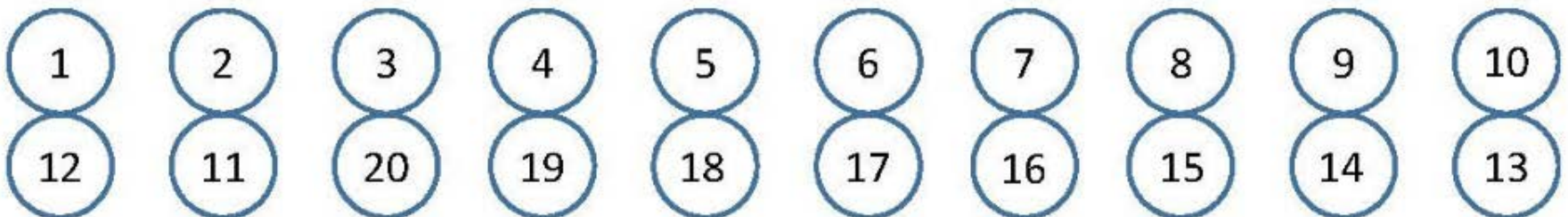


NUMBER
THEORY

PERFECT
MATCHING



GRAPH
THEORY



PRIME SUM GRAPH

For any positive number n ,
define $G_n = (V, E)$ with
 $V = \{1, 2, \dots, n\}$ and
 $E = \{ij : i+j \text{ is prime}\}$

Greenfield & Greenfield
" G_{2n} has a perfect matching."

Is every prime sum graph Hamiltonian?

No!

Obviously, G_n is not when n is odd.

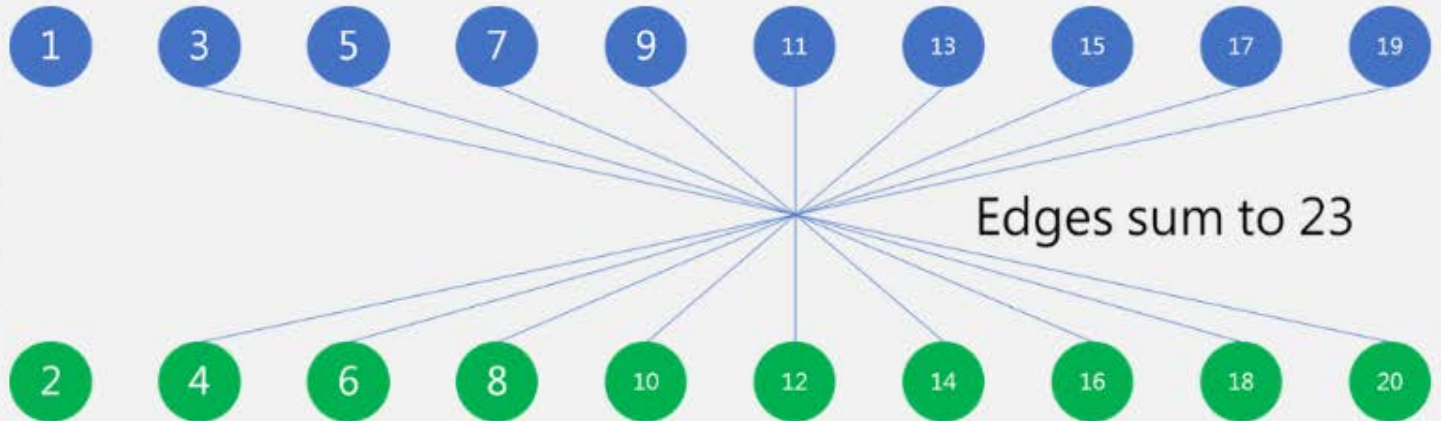


Observation

Bipartite graph

Symmetry

.....





HUNG-LIN FU



HONG-BIN CHEN



JUN-YI GUO

CONJECTURE

" G_{2n} has a Hamilton cycle"

"that is,
for any $2n > 2$, $\{1, 2, \dots, 2n\}$ can
be rearranged into a **cycle**
so that the **sum** of every
two **adjacent numbers** is **prime**"

Richard K. Guy

Unsolved Problems in Number Theory



THIRD EDITION

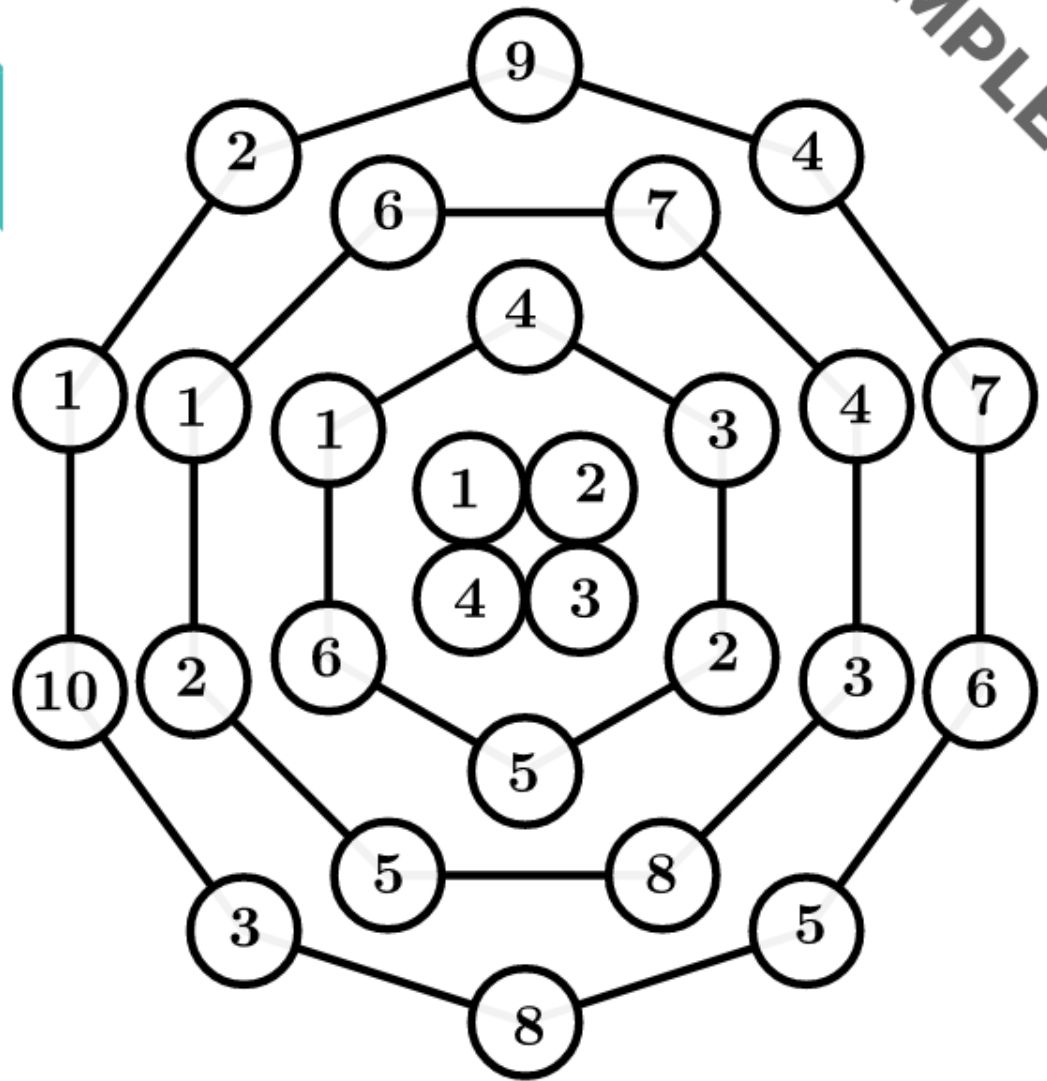
quote from p.105-106

"Antonio Filz (1982) defined a **prime circle** of order $2m$ to be a circular permutation of numbers from 1 to $2m$ with each adjacent pair summing to a prime.

There is essentially only one prime circle for $m=1$, 2, and 3; two for $m=4$ and 48 for $m=5$.

Are there prime circles for all m ? "

EXAMPLES



sum of adjacent numbers = prime



HOW WE DID

SUFFICIENT CONDITION

THEOREM [CFG 2018]

IF
THEN

$\{ p, q, 2n+p, 2n+q \}$ are primes
 $\gcd((q-p)/2, n) = 1$ (p can be 1)

G_{2n} has a Hamilton cycle

4 primes

1 gcd condition

01

$\{ 1, 3, 2n+1, 2n+3 \}$

gcd condition holds directly
at least 3 primes are required

02

If twin prime conjecture is true

then Filz's conjecture is verified
for **infinitely many cases**

03

DIFFICULTY

Need to prove there are infinitely
many prime triples (or quadruples)
satisfying certain conditions

BREAKTHROUGH OF TWIN PRIME CONJECTURE

246 600 70M



Yitang Zhang showed in 2013 that we will never stop finding pairs of primes that are within a bounded distance — within 70 million. Soon after, dozens of outstanding researchers in the world bring it down to 246.

POLYMATH PROJECT

JAMES MAYNARD

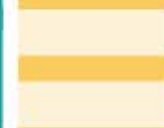
YITANG ZHANG

“
**THANKS TO
BREAKTHROUGH**
”

**OUR
SUFFICIENT
CONDITION**



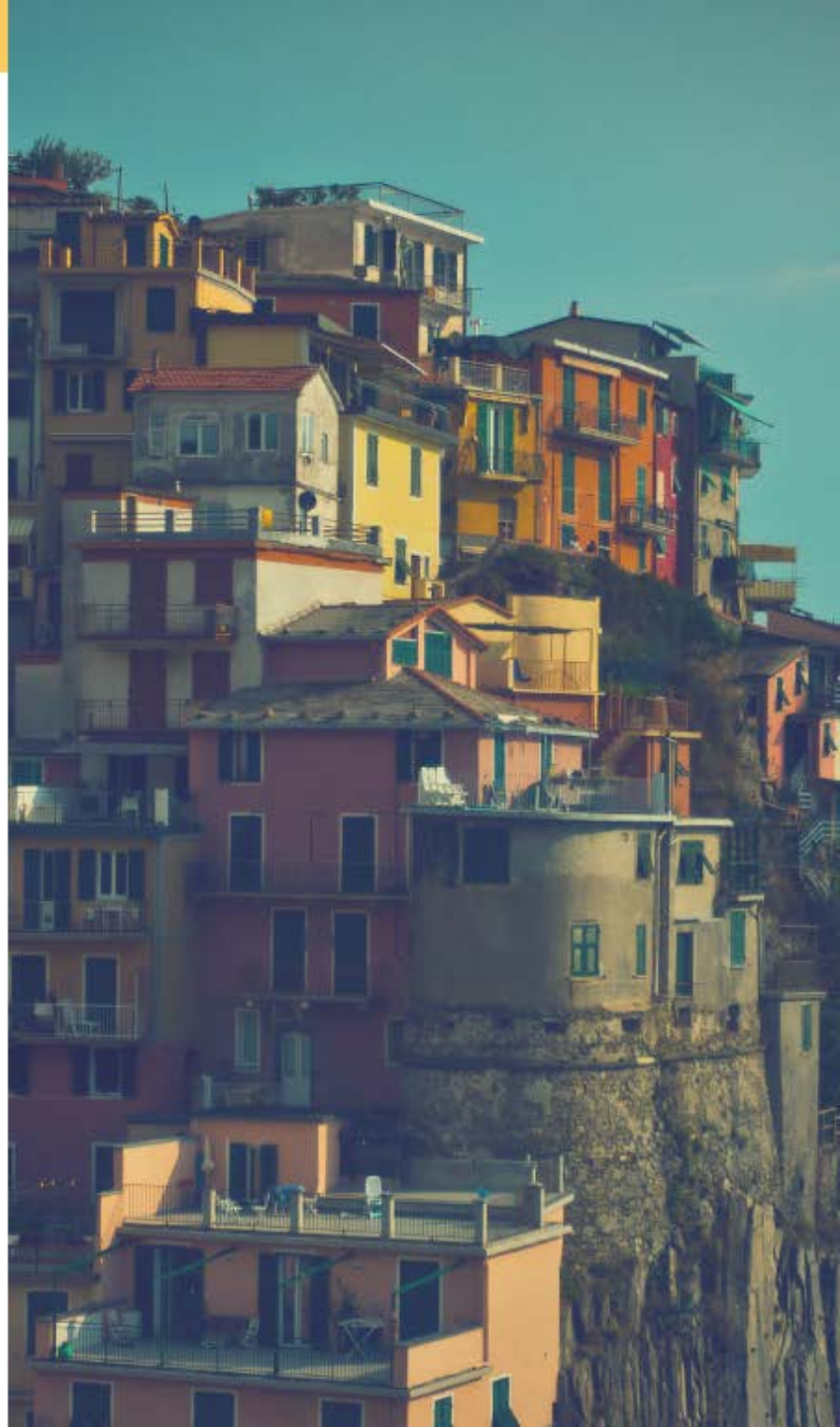
**THE 246
BREAKTHROUGH**



**MAIN
RESULT**

THEOREM [CFG 2018]

"There are **infinitely many** prime sum graphs that have a Hamilton cycle"



Observation

- Since there are infinitely many prime pairs with gap at most 246, **there exists a gap between 2 and 246 which occurs infinitely many times.**
- But, we don't know which one.
- Therefore, we go through all the possible cases and show that for each gap g , there exist two primes satisfying the sufficient condition mentioned above.



IDEA

01

ASSUME there exists g with infinitely many prime pairs satisfying $p' - p = g$.

Take $g=12$ for example.

There exist infinitely many (p, p') with

$$p = 12k' + 1 \quad \text{or} \quad p = 12k' + 5 \quad \text{or}$$

$$p = 12k' + 7 \quad \text{or} \quad p = 12k' + 11$$

02

MATCH prime pairs to gcd condition

Form	(p_1, p_2)	gcd condition
$p = 12k' + 1$	(11,23)	$\gcd(6, 6k + 1) = 1$
$p = 12k' + 5$	(7,19)	$\gcd(6, 6k + 5) = 1$
$p = 12k' + 7$	(5,17)	$\gcd(6, 6k + 1) = 1$
$p = 12k' + 11$	(1,13)	$\gcd(6, 6k + 5) = 1$

CHECK these steps for $g=2, 4, \dots, 246$

03

There are total 6170 cases and this can be done by computers.

A narrow street in a city with tall buildings on both sides, overlaid with a red tint. The street is paved with cobblestones and has double yellow lines on the sides. A few people are walking in the distance, and a car is parked on the right side. The text "WHAT RELEVANT" is overlaid in the center of the image.

WHAT RELEVANT



**Dirac-Ore Type
Condition**



**Prime Sum
Graphs**



**Random
Graphs**



LAJOS POSA

ØYSTEIN ORE (1899 - 1968)

A black and white portrait of Paul Erdős, a Hungarian mathematician, is positioned on the left side of the slide. He is wearing a dark suit, a white shirt, and a dark tie. The background of the slide is white, with a dark grey horizontal band at the bottom. The title 'Dirac-Ore Type Condition' is written in a large, bold, teal font. The text 'Graphs with min. degree $n/2$ (with degree sum of 2 nonadj. vertices at least n) have a Hamilton cycle' is written in a smaller, black font. The text 'Prime sum graph has vertex degree $n/\log n \ll n/2$ ' is written in a white font on the dark grey band. At the bottom of the slide, there are three colored rectangles: red, teal, and yellow.

Dirac-Ore Type Condition

Graphs
with min. degree $n/2$
(with degree sum of 2 nonadj.
vertices at least n)
have a Hamilton cycle

Prime sum graph has vertex degree $n/\log n \ll n/2$



Posa's THM. 1976

The probability
a **random graph** with n vertices
and **$cn \log n$ edges**
for sufficiently large c
contains a Hamilton cycle
tends to 1 as n tends to infinity

Prime sum graph has $O(n^2/\log n) \gg O(n \log n)$ edges

Primes long thought to distribute 'RANDOMLY' in a sense

999999976	13	31	999999989	1000000007
999999978	11	61	999999989	1000000039
999999980	59	101	1000000039	1000000081
999999982	7	67	999999989	1000000049
999999984	5	139	999999989	1000000123
999999986	3	53	999999989	1000000039
999999988	49	61	1000000007	1000000049
999999990	17	223	1000000007	1000000213
999999992	7	89	1000000039	1000000081
999999994	2	43	1000000007	1000000037
999999996	11	241	1000000007	1000000237
999999998	41	233	1000000039	1000000231
1000000000	7	73	1000000007	1000000073

We have verified the sufficient condition
up to
100,000,000

CONTRIBUTION

01

There are infinitely many
prime sum graphs
that are Hamiltonian

CONJECTURE

02

The sufficient condition is
always true?

If so

then we generalize Bertrand's postulate for even case

CHALLENGE

03

Generalized Bertrand's postulate?

" for each $2n$, there is p with $2n < 2n+p < 4n$
such that p and $2n+p$ are prime "

Goldbach's Conjecture

$$2n = p + p$$

Best result by J.R. Chen

$$2n = p + pp$$

**Generalized Bertrand's
postulate** $2n < 2n+p < 4n$

Implies $2n = p + p$, variant of Goldbach's conj.

Best result $2n = p + pp$ by J.R. Chen

Prime Pyramid

Margaret Kenney (1986) and Richard Guy (1993) studied the problem in the context of the term “prime pyramids” :

“ Does G_n has a 1-to- n Hamilton path for all n ?”

Open Question

Prove this can be done for all n ?

Not even known true for infinitely many n ?

“

Theorem [CFG]

For prime triples $(p, 2p-1, 3p-4)$,
there exists a Hamilton path from 1 to $2p-2$ in G_{2p-2} .

Conjecture

There are **infinitely many** such prime triples. (We have checked up to 100 millions and there are 481,784 such triples.)

”

Prime triples

4	p	2p-1	3p-4
5	7	13	17
6	19	37	53
7	31	61	89
8	37	73	107
9	79	157	233
10	157	313	467
11	199	397	593
12	229	457	683
13	271	541	809
14	367	733	1097
15	499	997	1493
16	547	1093	1637
17	661	1321	1979

Prime triples - continued

481775	99997759	199995517	299993273
481776	99998011	199996021	299994029
481777	99998071	199996141	299994209
481778	99998209	199996417	299994623
481779	99998419	199996837	299995253
481780	99998677	199997353	299996027
481781	99999019	199998037	299997053
481782	99999079	199998157	299997233
481783	99999139	199998277	299997413
481784	99999217	199998433	299997647

Another type of triples

prime triples $(p, 2p-3, 3p-4)$

Theorem (CFG)

If there exists a prime triple $(p, 2p-3, 3p-4)$, then we have a Hamilton path from 1 to $2p-2$ in G_{2p-2} .

Proof. We can find the path with a systematic way, see next page for an example.

(31, 59, 89)

⟨1, 30, 29, 2, 57,
32, 27, 4, 55,
34, 25, 6, 53,
36, 23, 8, 51,
...
54, 5, 26, 33,
56, 3, 28, 31,
58⟩.



The number of prime triples

- We also conjecture that there are infinitely many prime triples $(p, 2p-3, 3p-4)$.
- Remind that there are infinitely many prime pairs with gap at most 246.
- So, it is natural to consider the possibility of finding a 1 to n Hamilton path in G_n by using this fact.
- By using a similar idea (but more complicate), we are able to find such paths.

Proof

- Here is the pattern of the sequence:

$\langle 1, 2, 2p-3, p-1, p, 2p-4, 3,$
 $p-3, p+2, 2p-6, 5,$
 $p-5, p+4, 2p-8, 7,$

...

$p-(p-4), p+(p-5), p+1, p-2, 2p-2\rangle.$

- $P = 19$
- Prime triple: $(19, 37, 53)$; $\langle 1, 36 \rangle$ - path.

$\langle 1, 2, 35, 18, 19, 34, 3, 16, 21, 32, 5, 14, 23, 30, 7, 12, 25, 28, 9, 10, 27, 26, 11, 8, 29, 24,$
 $13, 6, 31, 22, 15, 4, 33, 20, 17, 36 \rangle$

SUFFICIENT CONDITION

Theorem [CFG] (?)

If p and $p+2k$ are primes such that $2k + 3 < p$ for some $k < 124$,

then G_{p-1} has a Hamilton $\langle 1, p-1 \rangle$ - path.



Evidence

- For the gap 10, (43, 53).
- Find (1, 11). (Both of them are of gap 10.)
- Now, the Hamilton path from 1 to 42 can found:

$\langle 1, 10, 33, 20, 23, 30, 13, 40, 3,$
 $8, 35, 18, 25, 28, 15, 38, 5,$
 $6, 37, 16, 27, 26, 17, 36, 7,$
 $4, 39, 14, 29, 24, 19, 34, 9,$
 $2, 41, 12, 31, 22, 21, 32, 11, 42 \rangle.$

- For (31, 41), we shall pick 3 and 13 instead.

$\langle 1, 2, 29, 12, 19, 22, 9, 4, 27, 14, 17, 24, 7, 6, 25, 16, 15, 26, 5, 8, 23, 18,$
 $13, 28, 3, 10, 21, 20, 11, 30 \rangle.$

The idea of proof

- It is similar to the case of finding Hamilton cycles in prime sum graph.
- We have to check **all the gaps** with suitable primes.
- Therefore, it takes more effort in settling all the small cases by using computer.
- **Hopefully, this can be done soon.**

Many gaps to check!



Theorem [CFG] (Under checking)

There are **infinitely many** n such that G_n has a Hamilton $\langle 1, n \rangle$ - path.



“

Prime Difference Graphs

Similarly, define a graph H_n on $\{1, 2, \dots, n\}$
by $u \sim v \iff |u-v|$ is prime.

”

“

Prime Difference Graphs

Z.W. Sun (2012) conjectured and proved by Y.G. Chen

“ For $n > 4$, H_n has a Hamilton cycle.”



”

Prime Difference Graphs are Denser

- It is not difficult to see that “**difference**” is more friendly since we have “**2**” to use.
- For example, (1, 3, 5, 8, 10, 7, 9, 2, 4, 6) is a Hamilton cycle in H_{10} .
- As a matter of fact, the order can be odd which is not possible for prime sum graphs in finding Hamilton cycles.
- For example, (1, 3, 5, 2, 4) is a Hamilton cycle in H_5 .

Theorem [CFG]

For $n > 8$, **any** s and t in $[n]$
 H_n has an $\langle s, t \rangle$ - Hamilton
path.

Corollary [CFG]

For $n > 8$, **any edge** e in H_n
there is a **Hamilton cycle**
passing through $e = \{s, t\}$. (The
difference of s and t is a prime.)

An Example

- $n = 30$, $s = 13$ and $t = 20$

$\langle 13, 11, 9, 7, 5, 2, 4, 1, 3, 6, 8, 10, 12, 14, 16, 18, 21, 23, 25, 27, 30, 28, 26, 29, 24, 22, 19, 17, 15, 20 \rangle$.

- For general cases, we start with showing the existence of a $\langle 1, t \rangle$ path for any $t \leq n$ and then consider “s” which is larger than 1.



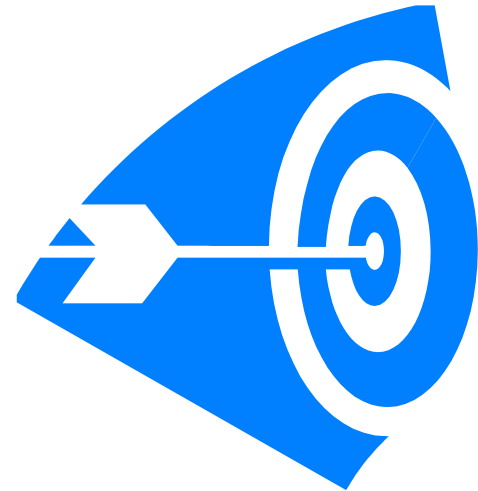
Theorem [CFG]

For $n > 6$

any 2-factor of $K_n \subset H_n$

Obviously

Hamilton cycle is a special case



2-factor Example

- It takes a long while to put up everything since there are so many different 2-factors of order $n \geq 7$.
- The main idea follows from the fact for each $n \geq 5$ there exists a Hamilton cycle in H_n . This fact implies that for any n consecutive integers, there exists a Hamilton cycle of the prime difference graph defined on this set of n integers.
- It is left to put 3-cycles and 4-cycles into the 2-factor with a smart way.
- $n = 30$ and the 2-factor contains one 3-cycle, one 4-cycle, two 5-cycle, one 6-cycle and one 7-cycle. Since $C_3 \cup C_4$ can be $(1, 3, 6) \cup (2, 5, 7, 4)$, we are done.

Theorem [CFG]

H_n has k edge-disjoint Hamilton cycles
whenever $n > N(k)$ (large enough).

Proof. By Green-Tao theorem (2004), there exists an arbitrarily long arithmetic progression of primes. So let $N(k)$ be the integer such that we can find $2k$ primes in arithmetic progression which are greater than $N(k)$: $p_1, p_2, \dots, p_{2k}$. Clearly, $p_i + p_{2k-i} (= n)$ remains the same for $i = 1, 2, \dots, k$. By the difference idea, we are able to obtain k Hamilton cycles in H_n which are edge-disjoint. This concludes the proof. $\square$

“

Prime Sum Graphs

Hamilton
cycle

Filz conjecture

Exist for $2n > 2$
?

[CFG 2018]
Infinitely
many

Hamilton
1-n path

Kenny conjecture

Exist for all n
?

[CFG 2019]
Infinitely
many

”

“

Prime Difference Graphs

For $n > 4$,
 H_n has a Hamilton cycle.

Y.-G. Chen & Z.W. Sun

”

For $n > 8$,
there is a Hamilton cycle
passing through any fixed edge.

01

For $n > 6$,
 H_n contains any 2-factor of K_n .

02

For n large enough,
 H_n has k Hamilton cycles (edge-disjoint)

03

Thanks!

