

選修基礎圖論的注意事項

1. 上課時間：Tuesday 15:30-16:30, Friday 10:10-11:40
2. 上課教室：SA214
3. 為不影響課程的進行以及同學的注意力，上課期間手機請用飛航模式，敬請配合。
4. 上課不點名、但是，我確信：來上課有助於你對這門課的了解；也希望經由這門課的學習能在未來對你(妳)有正面的助益。
5. 兩次考試的成績，不作任何調分；期中考(40%)與期末考(60%)皆按比例算至小數第二位；總成績為兩次相加之後，採無條件進位作為學期成績。
6. 徵答：於上課時公布；答對者適當外加總分每次 1~10 分不等（分數會註記在題目上）。（務必自行回答！）
7. 期中考 11 月 15 日上午 10:00-12:00，
期末考為明年(109)元月 10 日 10:00-12:30（請保留時間，沒有補考。）
8. 助教：何國正（SA 127）
9. Office hours： By appointments. SA 200
email： hlfu@math.nctu.edu.tw
10. 加油！

前言 (FGT)

傅恒舜

這門課主要是補強同學對離散數學的理解；
希望在圖論的部份打好更紮實的基础。

交大應數學在過去的三十多年，除了一般大家所熟悉的數學領域上面都有開課之外，特別在組合數學 (Combinatorial Mathematics) 作更深入的探討，因為它在應用的領域扮演著非常重要的角色，希望你們漸漸地能夠更理解。

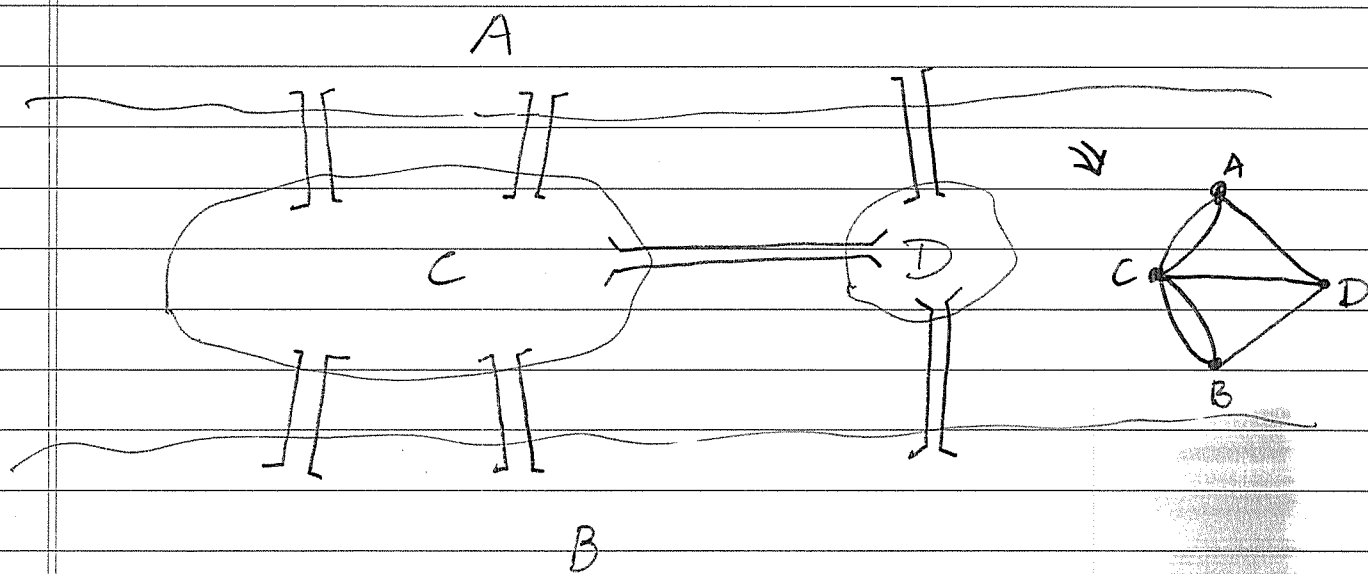
一般的認知中，組合數學的工作主要是在計數 (Counting, Enumeration)，然而，實際上，由於圖論的角色，它涵蓋了幾乎所有與網路相關的研究，包括人際間的社群網路，通訊的連接網路，甚至無法用肉眼看到的神經網路，幾凡離散的模型都有圖的影子。

歡迎各位同學加入探索的行列！

2019, 9, 10

動機 (Motivation)

一般都相信七橋問題 (Seven-Bridges Problem) 是引發尤拉 (歐拉, Euler) 完成第一篇圖論相關論文的主要動機。



問題：可否從 A, B, C, D 中的一地出發，走過七座橋各一次，然後回到原地？

Euler 將這個問題以“位置几何 (Position Geometry)”的概念轉換成後來熟悉的圖，如上，並且完整地刻劃 (充要條件)

那一類型的圖可以完成上述的“Tour”。該論文於 1736 年

完成，1941 在期刊中發表。(Euler 成了圖論之父！)

Definition (General graph)

A graph $G = (V, E)$ where V ($V(G)$) is the vertex-set and E is the edge set and E is a collection of 1-subsets or 2-subsets of V .

Example $V = \{1, 2, 3, 4, 5\}$

$$E = \{\{1\}, \{3\}, \{1, 2\}, \{2, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{3, 5\}, \{1, 5\}\}$$

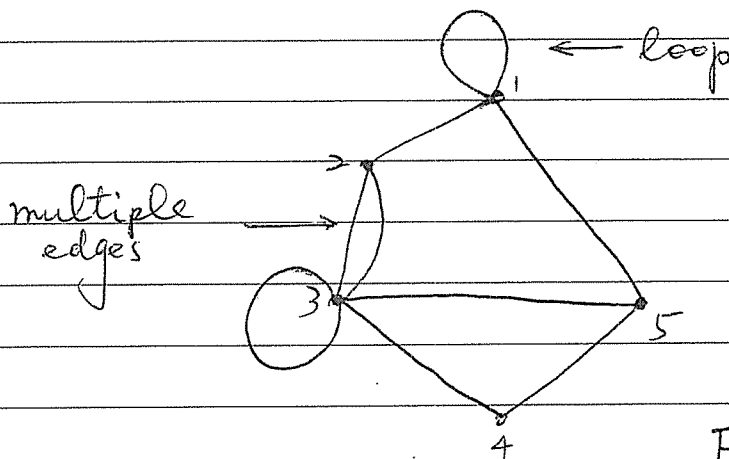


Figure 1 : G

Definition

Simple graph : No loops, No multi-edges

Multigraph : No loops

Pseudo-graph : General graph

Definition A graph G is called a hypergraph if the edge of the ^{edge} set contains more than "two" vertices of $V(G)$.

In this course, all graphs considered are mainly simple graphs. If we need a more general graph, we shall mention them.

G : Simple graph , $G = (V(G), E(G))$

$|V(G)| \stackrel{\text{def}}{=} |G|$ is the order of G .

$|E(G)| \stackrel{\text{def}}{=} \|G\|$ is the size of G .

$$N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$$

$$N_G[v] = N_G(v) \cup \{v\}$$

$|N_G(v)| \stackrel{\text{def}}{=} \deg_G(v)$ is the degree of v .

 $\swarrow$ odd vertices (degree odd)
 $\searrow$ even vertices (degree even)

Theorem 1 The number of vertices ^{in G} with odd degree is even.

Proof. Since each edge contributes degrees "2", $\sum_{v \in V(G)} \deg_G(v)$ is equal to $2 \cdot \|G\|$. By the fact that the degree sum is even, there are (even number) of vertices whose degree is odd.

(This is also known as the "shake hands principle".)

Basic properties of graphs

NO. 1-3

DATE

About "Degrees" in G

$\Delta(G)$: maximum degree

$\delta(G)$: minimum degree

$d(G)$: average degree ($\delta(G) \leq d(G) \leq \Delta(G)$)

$\varepsilon(G)$: average size (of one-vertex) $\varepsilon(G) = \frac{1}{2}d(G)$

$\|G\| = \frac{1}{2} \sum_{v \in V(G)} \deg_G(v)$ (Total number of edges)

$\varepsilon(G) = \|G\|/|G|$ (Density of G !).

(*) $\sum_{v \in V(G)} \deg_G(v)$ is even. (The 1st theorem of Graph Theory.)

(By shake-hands principle.)

Corollary The number of odd vertices in a graph is "even".

→ For some graph G , we do know its average degree but not $\Delta(G)$ or $\delta(G)$. (?)

1-dimension Sperner's Lemma (?)

2-dimension Sperner's Lemma

NO. 2 1-4
DATE
1 or 2
(The # of $\langle 1, 2 \rangle$ edges is odd.)

Example

As in Figure 1, ABC is a triangle, A, B and C are labelled 1, 2 and 3 respectively. All the vertices on AB, BC , and CA are labelled with numbers in $\{1, 2\}, \{2, 3\}$ and $\{1, 3\}$ respectively. And all the vertices inside ABC can be labelled with 1, 2 or 3 arbitrarily. Show that there exists a triangle inside ABC such that its vertices are labelled with 1, 2 and 3 respectively.

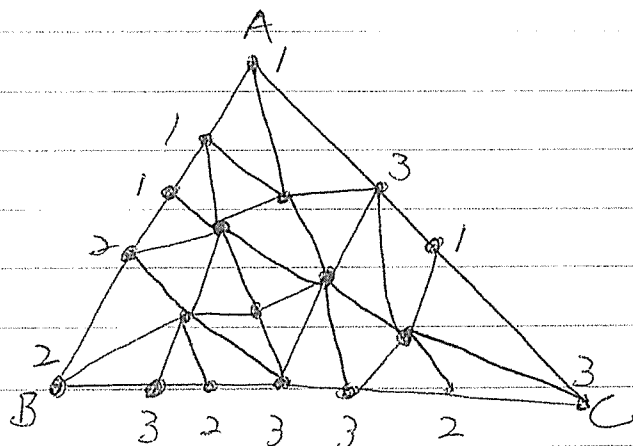


Figure 1, Triangulation of ABC

Hint: Construct a graph and then prove that there exists a vertex of odd degree inside ABC .

Problem Extend 2-dim to high-dim.

Sperner's lemma

In mathematics, **Sperner's lemma** is a combinatorial analog of the Brouwer fixed point theorem, which is equivalent to it.

Sperner's lemma states that every **Sperner coloring** (described below) of a triangulation of an *n*-dimensional simplex contains a cell colored with a complete set of colors.

The initial result of this kind was proved by Emanuel Sperner, in relation with proofs of invariance of domain. Sperner colorings have been used for effective computation of fixed points and in root-finding algorithms, and are applied in fair division (cake cutting) algorithms. It is now believed to be an intractable computational problem to find a Brouwer fixed point or equivalently a Sperner coloring, even in the plane, in the general case. The problem is PPAD-complete, a complexity class invented by Christos Papadimitriou.

According to the Soviet *Mathematical Encyclopaedia* (ed. I.M. Vinogradov), a related 1929 theorem (of Knaster, Borsuk and Mazurkiewicz) had also become known as the *Sperner lemma* – this point is discussed in the English translation (ed. M. Hazewinkel). It is now commonly known as the Knaster–Kuratowski–Mazurkiewicz lemma.

Contents

Statement

- One-dimensional case
- Two-dimensional case
- Multidimensional case

Proof

- Commentary

Generalizations

- Polytopes
- Permutations
- Degrees
- Trees and cycles
- Cubic Sperner lemma

Applications

Equivalent results

See also

References

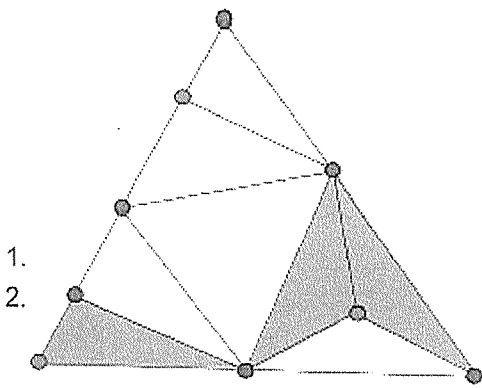
External links

Statement

One-dimensional case

In one dimension, Sperner's Lemma can be regarded as a discrete version of the intermediate value theorem. In this case, it essentially says that if a discrete function takes only the values 0 and 1, begins at the value 0 and ends at the value 1, then it must switch values an odd number of times.

Two-dimensional case



Two-dimensional case example

The two-dimensional case is the one referred to most frequently. It is stated as follows:

Given a triangle ABC, and a triangulation T of the triangle, the set S of vertices of T is colored with three colors in such a way that

A, B, and C are colored 1, 2, and 3 respectively

Each vertex on an edge of ABC is to be colored only with one of the two colors of the ends of its edge. For example, each vertex on AC must have a color either 1 or 3.

Then there exists a triangle from T , whose vertices are colored with the three different colors. More precisely,

there must be an odd number of such triangles.



One-dimensional case example

Multidimensional case

In the general case the lemma refers to a n -dimensional simplex

$$\mathcal{A} = A_1 A_2 \dots A_{n+1}.$$

We consider a triangulation T which is a disjoint division of $\mathcal{A}$ into smaller n -dimensional simplices. Denote the coloring function as $f: S \rightarrow \{1, 2, 3, \dots, n, n+1\}$, where S is again the set of vertices of T . The rules of coloring are:

1. The vertices of the large simplex are colored with different colors, i. e. $f(A_i) = i$ for $1 \leq i \leq n+1$.
2. Vertices of T located on any k -dimensional subspace of the large simplex

$$A_{i_1} A_{i_2} \dots A_{i_{k+1}}$$

are colored only with the colors

$$i_1, i_2, \dots, i_{k+1}.$$

Then there exists an odd number of simplices from T , whose vertices are colored with all $n+1$ colors. In particular, there must be at least one.

Proof

We shall first address the two-dimensional case. Consider a graph G built from the triangulation T as follows:

The vertices of G are the members of T plus the area outside the triangle. Two vertices are connected with an edge if their corresponding areas share a common border with one endpoint colored 1 and the other colored 2.

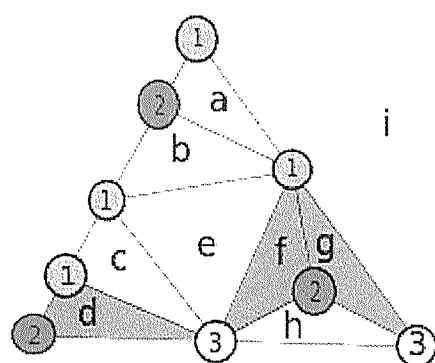
Note that on the interval AB there is an odd number of borders colored 1-2 (simply because A is colored 1, B is colored 2; and as we move along AB, there must be an odd number of color changes in order to get different colors at the beginning and at the end). Therefore, the vertex of G corresponding to the outer area has an odd degree. But it is known (the handshaking lemma) that in a finite graph there is an even number of vertices with odd degree. Therefore, the remaining graph, excluding the outer area, has an odd number of vertices with odd degree corresponding to members of T .

It can be easily seen that the only possible degree of a triangle from T is 0, 1, or 2, and that the degree 1 corresponds to a triangle colored with the three colors 1, 2, and 3.

Thus we have obtained a slightly stronger conclusion, which says that in a triangulation T there is an odd number (and at least one) of full-colored triangles.

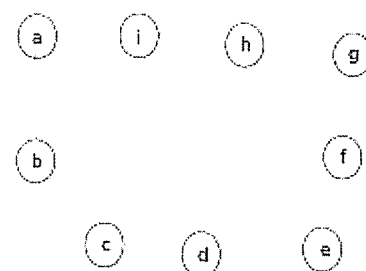
A multidimensional case can be proved by induction on the dimension of a simplex. We apply the same reasoning, as in the two-dimensional case, to conclude that in a n -dimensional triangulation there is an odd number of full-colored simplices.

Commentary



A simple two-dimensional triangulation of the example figure, colored and named in accordance with the assumptions of Sperner's Lemma

Here is an elaboration of the proof given previously, for a reader new to graph theory: This diagram numbers the colors of the vertices of the example given previously. The small triangles whose vertices all have different numbers are shaded in the graph. Each small triangle becomes a node in the new graph derived from the triangulation. The small letters identify the areas, eight inside the figure, and area i designates the space outside of it. As described previously, those nodes that share an edge whose endpoints are numbered 1 and 2 are joined in the derived graph. For example, node d shares an edge with the outer area i , and its vertices all have different



The graph derived from the example figure

numbers, so it is also shaded. Node b is not shaded because two vertices have the same number, but it is joined to the outer area. One could add a new full-numbered triangle, say by inserting a node numbered 3 into the edge between 1 and 1 of node a , and joining that node to the other vertex of a . Doing so would have to create a pair of new nodes, like the situation with nodes f and g .

Generalizations

Polytopes

Suppose that, instead of an $n - 1$ -dimensional simplex, we have a d -dimensional polytope with n vertices.

Then there are at least $n - d$ fully labeled simplices, where "fully labeled" indicates that every label on the simplex has a different color. For example, if a (two-dimensional) polygon with n vertices is triangulated and colored according to the Sperner criterion, then there are at least $n - 2$ fully labeled triangles.

The general statement was conjectured by Atanassov in 1996, who proved it for the case $d = 2$.^[1] The proof of the general case was first given by de Loera, Peterson, and Su in 2002.^[2]

Permutations

Suppose that, instead of a single labeling, we have n different Sperner labelings.

We consider pairs (simplex, permutation) such that, the label of each vertex of the simplex is chosen from a different labeling (so for each simplex, there are $n!$ different pairs).

Then there are at least $n!$ fully labeled pairs. This was proved by Ravindra Bapat.^[3]

Another way to state this lemma is as follows. Suppose there are n people, each of whom produces a different Sperner labeling of the same triangulation. Then, there exists a simplex, and a matching of the people to its vertices, such that each vertex is labeled by its owner differently (one person labels its vertex by 1, another person labels its vertex by 2, etc.). Moreover, there are at least $n!$ such matchings. This can be used to find an envy-free cake-cutting with connected pieces.

Degrees

Suppose a triangle is triangulated and labeled with {1,2,3}. Consider the cyclic sequence of labels on the boundary of the triangle. Define the *degree* of the labeling as the difference between the number of switches from 1 to 2, and the number of switches from 2 to 1. See examples in the table at the right. Note that the degree is the same if we count switches from 2 to 3 minus 3 to 2, or from 3 to 1 minus 1 to 3.

Sequence	Degree
123	1 (one 1-2 switch and no 2-1 switch)
12321	0 (one 1-2 switch minus one 2-1 switch)
1232	0 (as above; recall sequence is cyclic)
1231231	2 (two 1-2 switches and no 2-1 switch)

Musin proved that *the number of fully labeled triangles is at least the degree of the labeling*.^[4] In particular, if the degree is nonzero, then there exists at least one fully labeled triangle.

If a labeling satisfies the Sperner condition, then its degree is exactly 1: there are 1-2 and 2-1 switches only in the side between vertices 1 and 2, and the number of 1-2 switches must be one more than the number of 2-1 switches (when walking from vertex 1 to vertex 2). Therefore, the original Sperner lemma follows from Musin's theorem.

Trees and cycles

There is a similar lemma about finite and infinite trees and cycles.^[5]

Cubic Sperner lemma

A variant of Sperner's lemma on a cube (instead of a simplex) was proved by Harold W. Kuhn.^[6] It is related to the Poincaré–Miranda theorem.^[7]

Applications

Sperner colorings have been used for effective computation of fixed points. A Sperner coloring can be constructed such that fully labeled simplices correspond to fixed points of a given function. By making a triangulation smaller and smaller, one can show that the limit of the fully labeled simplices is exactly the fixed point. Hence, the technique provides a way to approximate fixed points.

For this reason, Sperner's lemma can also be used in root-finding algorithms and fair division algorithms; see Simmons–Su protocols.

Sperner's lemma is one of the key ingredients of the proof of Monsky's theorem, that a square cannot be cut into an odd number of equal-area triangles.^[8]

Fifty years after first publishing it, Sperner presented a survey on the development, influence and applications of his combinatorial lemma.^[9]

Equivalent results

There are several fixed-point theorems which come in three equivalent variants: an algebraic topology variant, a combinatorial variant and a set-covering variant. Each variant can be proved separately using totally different arguments, but each variant can also be reduced to the other variants in its row. Additionally, each result in the top row can be deduced from the one below it in the same column.^[10]

Algebraic topology	Combinatorics	Set covering
Brouwer fixed-point theorem	Sperner's lemma	Knaster–Kuratowski–Mazurkiewicz lemma
Borsuk–Ulam theorem	Tucker's lemma	Lusternik–Schnirelmann theorem

See also

- [Topological combinatorics](#)

References

1. Atanassov, K. T. (1996), "On Sperner's lemma", *Studia Scientiarum Mathematicarum Hungarica*, **32** (1–2): 71–74, MR 1405126 (<https://www.ams.org/mathscinet-getitem?mr=1405126>)
2. De Loera, Jesus A.; Peterson, Elisha; Su, Francis Edward (2002), "A polytopal generalization of Sperner's lemma", *Journal of Combinatorial Theory, Series A*, **100** (1): 1–26, doi:10.1006/jcta.2002.3274 (<https://doi.org/10.1006/jcta.2002.3274>), MR 1932067 (<https://www.ams.org/mathscinet-getitem?mr=1932067>)
3. Bapat, R. B. (1989). "A constructive proof of a permutation-based generalization of Sperner's lemma". *Mathematical Programming*. **44**: 113. doi:10.1007/BF01587081 (<https://doi.org/10.1007/BF01587081>).
4. Oleg R Musin (2014). "Around Sperner's lemma". arXiv:1405.7513 (<https://arxiv.org/abs/1405.7513>)  [math.CO] (<https://arxiv.org/archive/math.CO>).
5. Niedermaier, Andrew; Rizzolo, Douglas; Su, Francis Edward (2014), "A tree Sperner lemma", in Barg, Alexander; Musin, Oleg R., *Discrete Geometry and Algebraic Combinatorics*, Contemporary Mathematics, **625**, Providence, RI: American Mathematical Society, pp. 77–92, arXiv:0909.0339 (<https://arxiv.org/abs/0909.0339>) , doi:10.1090/conm/625/12492 (<https://doi.org/10.1090/conm/625/12492>), MR 3289406 (<https://www.ams.org/mathscinet-getitem?mr=3289406>)
6. Kuhn, H. W. (1960), "Some Combinatorial Lemmas in Topology", *IBM Journal of Research and Development*, **4** (5): 518, doi:10.1147/rd.45.0518 (<https://doi.org/10.1147/rd.45.0518>)
7. Michael Mürger (2016), *Topology for the working mathematician* (<http://www.math.ru.nl/~mueger/topology.pdf>) (PDF), Draft
8. Aigner, Martin; Ziegler, Günter M. (2010), "One square and an odd number of triangles", *Proofs from The Book* (4th ed.), Berlin: Springer-Verlag, pp. 131–138, doi:10.1007/978-3-642-00856-6_20 (https://doi.org/10.1007/978-3-642-00856-6_20), ISBN 978-3-642-00855-9
9. Sperner, Emanuel (1980), "Fifty years of further development of a combinatorial lemma", *Numerical solution of highly nonlinear problems (Sympos. Fixed Point Algorithms and Complementarity Problems, Univ. Southampton, Southampton, 1979)*, North-Holland, Amsterdam-New York, pp. 183–197, 199–217, MR 0559121 (<https://www.ams.org/mathscinet-getitem?mr=0559121>)
10. Nyman, Kathryn L.; Su, Francis Edward (2013), "A Borsuk–Ulam equivalent that directly implies Sperner's lemma", *American Mathematical Monthly*, **120** (4): 346–354, doi:10.4169/amer.math.monthly.120.04.346 (<https://doi.org/10.4169/amer.math.monthly.120.04.346>), MR 3035127 (<https://www.ams.org/mathscinet-getitem?mr=3035127>)

External links

- [Proof of Sperner's Lemma](http://www.cut-the-knot.org/Curriculum/Geometry/SpernerLemma.shtml) (<http://www.cut-the-knot.org/Curriculum/Geometry/SpernerLemma.shtml>) at cut-the-knot
- [Sperner's lemma and the Triangle Game](http://nrich.maths.org/1383) (<http://nrich.maths.org/1383>), at the n-rich site.

Retrieved from "https://en.wikipedia.org/w/index.php?title=Sperner%27s_lemma&oldid=843217709"

This page was last edited on 27 May 2018, at 18:04 (UTC).

Text is available under the [Creative Commons Attribution-ShareAlike License](#); additional terms may apply. By using this site, you agree to the [Terms of Use](#) and [Privacy Policy](#). Wikipedia® is a registered trademark of the [Wikimedia Foundation, Inc.](#), a non-profit organization.