

Graph Theory - An Introduction

In this lecture note of Graph Theory we select 100 Theorems to introduce the basic properties of graphs.

Theorem 1 (Veblen, 1912)

The edge set of a graph can be partitioned into cycles if and only if every vertex has even degree.

Proof.

($\Rightarrow$) A vertex contained in t cycles has degree $2t$.

($\Leftarrow$) The cycles can be obtained recursively. We start with finding the first cycle. Let $\langle x_0, x_1, \dots, x_l \rangle$ be a path of *maximal* length l in G .

Since $x_0x_1 \in E(G)$, $\deg_G(x_0) \geq 2$. Let $y (\neq x_1)$ be a neighbor of x_0 , i.e., $x_0y \in E(G)$. Now, $y \in \{x_2, x_3, \dots, x_l\}$. For otherwise, we have a longer path. So, if $y = x_i$, then we have a cycle $C = (x_0, x_1, \dots, x_i)$. The process continues in $G - E(C)$. (Each vertex is of even degree in the graph $G - E(C)$.) ■

Theorem 2 (Mantel, 1907)

Every graph of order n and size greater than $\left\lfloor \frac{n^2}{4} \right\rfloor$ contains a triangle (C_3 or K_3).

Proof.

(1st)

Since $K_3 \not\leq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \emptyset$. This implies that $\deg_G(x) + \deg_G(y) \leq |G| = n$. (Figure 1) Now, consider

$$\begin{aligned} \sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) &= \sum_{x \in V(G)} (\deg_G(x))^2 \quad (\text{Two-way counting}) \\ &\leq n \cdot \|G\| \\ &= n \cdot e(G). \end{aligned}$$

By Cauchy's inequality,

$$\begin{aligned} (2e(G))^2 &= (\sum_{x \in V(G)} \deg_G(x))^2 \\ &\leq n \cdot \sum_{x \in V(G)} (\deg_G(x))^2 \\ &\leq (n)^2 \cdot e(G). \end{aligned}$$

Hence, $e(G) \leq \frac{n^2}{4}$. ■

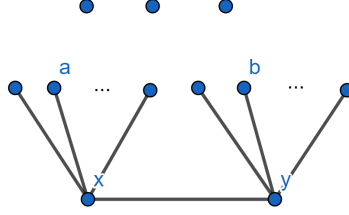


Figure 1. $\deg_G(x) = a + 1$, $\deg_G(y) = b + 1$

(2nd)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2)

Since $K_3 \not\leq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies that

$$\|G\| \leq \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G)).$$

$\|G\|$ will take a maximum when $\Delta(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Hence, we have the proof. ■

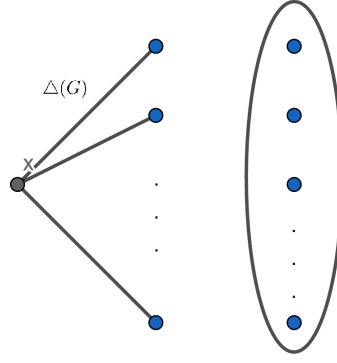


Figure 2.

Theorem 3

A graph is bipartite if and only if it does not contain an odd cycle.

Proof.

($\Rightarrow$) Let $G = (A, B)$ where A and B are its partite sets. If $(x_0, x_1, \dots, x_l)$ is a cycle of G , then x_0 and x_l must be in different partite sets. Hence, the index l must be odd, thus the cycle is of even length.

($\Leftarrow$) W.L.O.G, let G be a connected graph. Let $x \in V(G)$ and $V_1 = \{y \mid y \in V(G) \text{ and } d(x, y) \text{ is even}\}$. Hence, $x \in V_1$. Let $V_2 = V(G) \setminus V_1$. It suffices to claim that both V_1 and V_2 are independent sets. First, consider V_2 . Clearly, for each $z \in V_2$, $d(x, z)$ is odd. Suppose that $z_1, z_2 \in V_2$ and $z_1 \sim z_2$ ($z_1 z_2 \in E(G)$). (Figure 3) Let P_1 and P_2 be the two paths such that $P_1 = \langle x, \dots, z_1 \rangle$ and $P_2 = \langle x, \dots, z_2 \rangle$, moreover they are the shortest paths connecting x to z_1 and x to z_2 respectively.

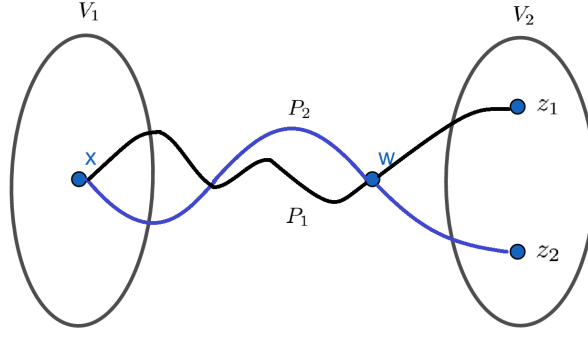


Figure 3.

Let w be the last vertex in which P_1 and P_2 intersect. Also, let $\|P_1\| = 2s + 1$ and $\|P_2\| = 2t + 1$. (Note that if $V(P_1) \cap V(P_2) = \{x\}$ (Figure 4), then we have an odd cycle (x, P_1, z_1, z_2, P_2) (length $2s + 2t + 3$).) Now, if w does exist, then $\langle x, P_1, w \rangle$ and $\langle x, P_2, w \rangle$ are of the same length, let the length be h .(?) So, the cycle $(w, \dots, z_1, z_2, \dots)$ is of length $(2s + 1 - h) + (2t + 1 - h) = 2s + 2t - 2h + 3$, an odd integer. Thus, an odd cycle exists, a contradiction. Hence, V_2 is an independent set. A similar argument can be applied to show that V_1 is also an independent set. (x is not adjacent to any vertex of $V_1 \setminus \{x\}$.)

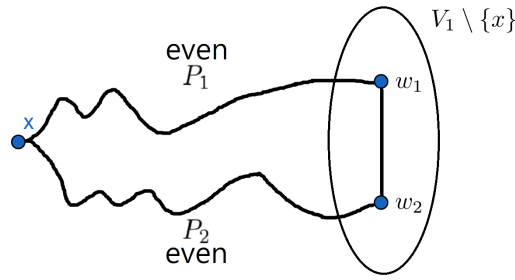


Figure 4.

Open problem

How many edges can a graph G of order n have such that $G \not\cong C_4$?

Theorem 4

The following statements are equivalent for a graph G .

- (a) G is a tree. (G is connected and acyclic.)
- (b) G is connected and every edge of G is a bridge.
- (c) G is a maximal acyclic graph. (If x and y are not adjacent, then $G + xy$ contains a cycle.)

Proof.

(a) $\Rightarrow$ (b)

Let xy be an edge of G and $G - xy$ is connected. Then, there exists a path P connecting x and y in $G - xy$. Clearly, G contains a cycle (x, P, y) in G , a contradiction.

(b) $\Rightarrow$ (c)

If G is not a maximal acyclic graph, then there exists a pair of vertices z_1 and z_2 in G such that $G + z_1z_2$ is also acyclic. Since G is connected, there exists a path connecting z_1 and z_2 , say P . This implies that (z_1, P, z_2) is a cycle in $G + z_1z_2$, a contradiction.

(c) $\Rightarrow$ (a)

If G is not connected, then there exists a pair of vertices x_1 and x_2 (in different components) such that $G + x_1x_2$ is also acyclic. $\rightarrow\leftarrow$ ■

Theorem 5 (Euler, 1741)

A nontrivial connected graph (multigraph) has an eulerian circuit

(Euler circuit) if and only if each vertex has even degree. Moreover, a connected graph has an eulerian trail from a vertex x to a vertex $y \neq x$ if and only if x and y are the only two vertices of odd degree.

Proof.

The second statement follows directly from the first one. We prove the first statement.

($\Rightarrow$) If a circuit passes a vertex x h times, then $\deg_G(x) = 2h$.

($\Leftarrow$) By induction on $\|G\|$. Since $\|G\| \geq 1$, $\delta(G) \geq 2$ (G is not a tree!) and thus G contains a cycle. Let Z be a circuit in G with the maximum number of edges. If Z is an eulerian circuit, then we are done. Suppose not. Let H be a nontrivial component of $G - E(Z)$. Since G is connected, $V(H) \cap V(Z) \neq \emptyset$. Now, H is a nontrivial connected graph (even graph). Hence, H contains an eulerian circuit Y . By using x , we can attach Z and Y together to obtain a larger circuit. (Figure 5) This contradicts to the maximality of $|E(Z)|$. Hence, Z must be an eulerian circuit of G . ■

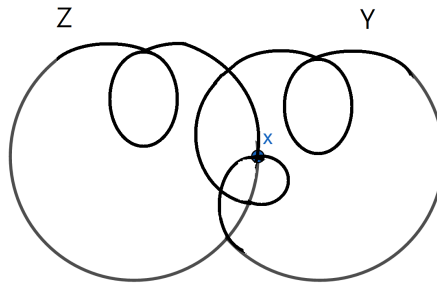
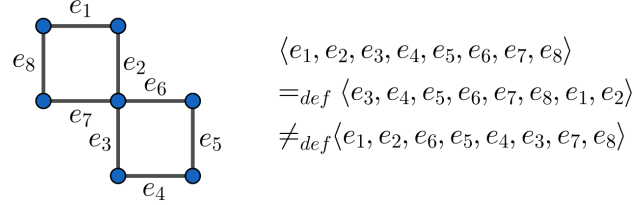


Figure 5.

Open problem

Find the number of distinct eulerian circuits of an eulerian graph G .

(Two circuits are the same if they can be obtained each other by a cyclic shift of edges.)



Theorem 5.5 (*BEST Theorem*)

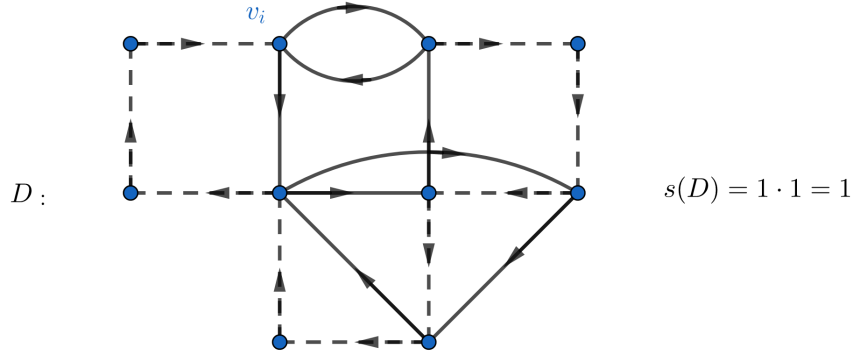
A digraph D has an eulerian (directed) circuit if and only if D is strongly connected and for each vertex $v \in V(D)$, $\deg_D^+(v) = \deg_D^-(v)$. Moreover, if D is an eulerian graph, $s(D)$ is the number of distinct eulerian circuits, then

$s(D) = t_i(D) \cdot \prod_{j=1}^n (\deg_D^+(v_j) - 1)!$ for every $i \in \{1, 2, \dots, n\}$ where $t_i(D)$ is the number of spanning trees oriented toward v_i .

(Note: The counting part of the theorem was proved by "de Bruijn and van Aardenne-Ehrenfest", Smith and Tutte. (two independent groups))

Proof.

The existence part can be obtained by a similar argument as the "multigraph" version. (We omit the proof of 2nd part.)

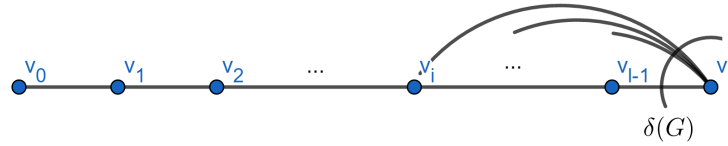
Figure 6. Spanning tree oriented toward v_i

Theorem 6

Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$ (provided $\delta(G) \geq 2$).

Proof.

Let $\langle v_0, v_1, \dots, v_l \rangle$ be a longest path. Then, $N_G(v_l) \subseteq \{v_0, v_1, \dots, v_{l-1}\}$. For otherwise, we have a longer path. Since v_l has at least $\delta(G)$ neighbors $l \geq \deg_G(v_l) \geq \delta(G)$. This concludes the first part. Now, let i be the smallest index in $\{0, 1, 2, \dots, l-1\}$ such that $v_i v_l \in E(G)$. Hence, $(v_i, v_{i+1}, \dots, v_l)$ is a cycle of length at least $\delta(G) + 1$. ■

Figure 7. $l - i \geq \delta(G)$

Exercise A-1

Every connected graph G contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$.

Theorem 7

For each connected graph G , $rad(G) \leq diam(G) \leq 2rad(G)$.

Proof.

It suffices to consider the second inequality. Let u and v be two vertices in G such that $d(u, v) = diam(G)$. Let w be a vertex in the center of G , i.e., $ecc(w) = rad(G)$. By the fact that " d " is a metric, $d(u, v) + d(w, v) \geq d(u, w)$. This implies that

$$ecc(w) + ecc(w) = 2rad(G) \geq d(u, w) + d(w, v) \geq d(u, v) = diam(G). \quad \blacksquare$$

(Note. The eccentricity of $w \in V(G)$ is $\max\{d(x, w) \mid x \in V(G)\}$.)

Theorem 8

A graph of minimum degree δ and girth (shortest cycle) g has at least $n_0(\delta, g)$ vertices where

$$n_0(\delta, g) = \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i, & g =_{def} 2r + 1, \text{ and} \\ 2 \cdot \sum_{i=0}^{r-1} (\delta - 1)^i, & g = 2r. \end{cases}$$

Proof.

Case 1. $g = 2r + 1, r \geq 1$.

Let v_0 be a fixed vertex in G , see Figure 9.

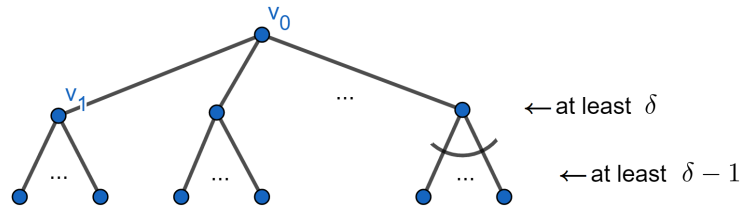


Figure 9.

Then, there are at least δ neighbors of v_0 , and for each neighbor, say v_1 , v_1 has at least $\delta - 1$ neighbors. Since $g = 2r + 1$, G contains at least $1 + \delta + \delta(\delta - 1) + \dots + \delta(\delta - 1)^{r-1}$ vertices. This concludes the proof of the Case 1.

Case 2. $g = 2r$

In this case, we start with an edge u_0v_0 , see Figure 10. By a similar argument, G contains at least $2 \cdot [(\delta - 1) + (\delta - 1)^2 + \dots + (\delta - 1)^{r-1}]$ vertices. ■

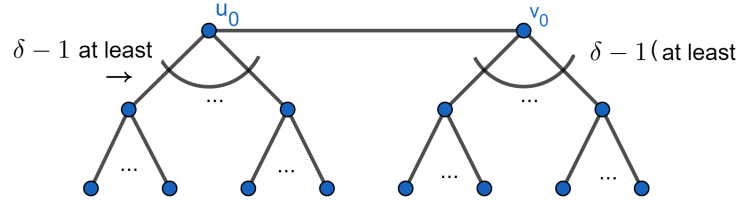


Figure 10.

Theorem 9

If $\delta(G) \geq 3$, then $g(G) < 2 \log_2 |G|$.

Proof.

Note that if $\delta_1 \geq \delta_2 \geq 3$, then $n_0(\delta_1, g) \geq n_0(\delta_2, g)$. It suffices to consider $n_0(3, g)$. By Theorem 8,

$$|G| \geq n_0(3, g) = 2^r + 2^r - 2 > 2^r \quad (g = 2r) \text{ and}$$

$$|G| \geq n_0(3, g) = 1 + 3 \cdot \frac{2^r - 1}{2 - 1} = \frac{3}{\sqrt{2}} 2^{\frac{g}{2}} - 2 > 2^{\frac{g}{2}} \quad (g = 2r + 1).$$

This implies that $r < \log_2 |G|$ and thus $g < 2 \log_2 |G|$. ■

Theorem 10

A (d, g) -cage is a d -regular graph with girth g and minimum number of vertices. Prove that a (d, g) -cage is 2-connected. ($g \geq 3$)

Proof.

First, we claim that if $g_1 > g_2$, then a (d, g_1) -cage contains more vertices than the order of a (d, g_2) -cage. Suppose not. Let G_1 and G_2 be two cages respectively and $|G_1| < |G_2|$. It suffices to consider the case $g_1 = g_2 + 1$. Let $\|G_1\| = f(d; g_1)$ and $\|G_2\| = f(d; g_2)$.

(a) d is even

Let C be a cycle (in G_1) of length g_1 and $uv_1 \in E(C)$ where $N_{G_1}(u) = \{v_1, v_2, \dots, v_d\}$. Let $E' = \{v_1v_2, v_3v_4, \dots, v_{d-1}v_d\}$. Now, consider $G_1 - u + E'$ and denote the component contains v_1 by G'_1 . Clearly, G' is a simple graph and G'_1 contains a cycle of length $g_2 : C - u + v_1v_2$. Further, if C' is a cycle of G'_1 and $E(C') \cap E' = \emptyset$, then C' is a cycle of G_1 and thus of length at least g_1 . On the other hand, if $E(C') \cap E' \neq \emptyset$, then let v_iv_j be one of the edges. Let P be a $\langle v_i, \dots, v_j \rangle$ path on C' satisfying $E(P) \cap E' = \emptyset$. So, $P + \{uv_i, uv_j\}$ is a cycle of G_1 . This implies that $|E(C')| \geq g_1 - 1 = g_2$. This concludes that G'_1 is a d -regular graph with girth at least g_2 and $|G_2| \leq |G'_1| = f(d; g_1) - 1 \leq |G_1|$.

(b) d is odd

Let C be a cycle (in G_1) of length g_1 and $uv_1, uv_2 \in E(C)$. Let $N_{G_1}(u) = \{u_1, u_2, \dots, u_{d-1}, w\}$. Clearly, $w \notin V(C)$. For otherwise, we have a cycle of length less than g_1 . (See Figure 11.)

Now, let $N_{G_1}(w) = \{u, x_1, x_2, \dots, x_{d-1}\}$ and G'_1 be the component (contains v_1) of $G - uw + \{v_{2i-1}v_{2i}, x_{2i-1}x_{2i} | 1 \leq i \leq (d-1)/2\}$. Again, G'_1 is simple and G'_1 is a (d, g) -graph with at most $f(d; g_1) - 2$ vertices. Hence, $f(d; g_2) < f(d; g_1)$.

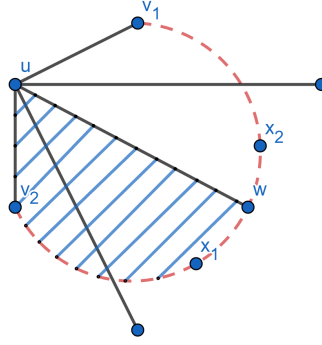
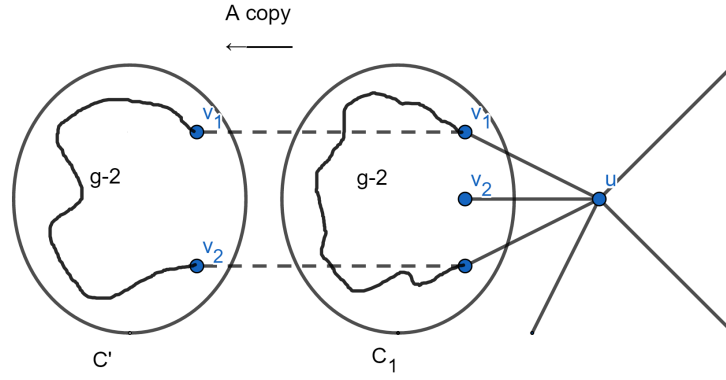


Figure 11. Shorter cycle

Proof of the theorem (A (d, g) -cage is 2-connected.)

Suppose not. Let u be a cut-vertex. Let $C_1, C_2, \dots, C_w$ be the components of $G - u$, with $|V(C_i)| \leq |V(C_{i+1})|$, $i = 1, 2, \dots, w - 1$. Consider C_1 . In C_1 , $\forall v_1, v_2 \in V(C_1) \cap N_G(u)$, $d(v_1, v_2) \geq g - 2$. (Figure 11) Let C' be an isomorphic copy of C_1 with isomorphism φ . Now, construct a new graph H where $V(H) = V(C') \cup V(C_1)$ and $E(H) = E(C') \cup E(C_1) \cup \{v\varphi(v) | v \in V(C_1) \cap N_G(u)\}$. By observation, $|H| < |G|$, H is d -regular and H has girth at least $\min\{g, 2g - 2\} = g$.

Figure 12. Construction of H

This implies that G is not a (d, g) -cage, a contradiction. ■

Facts

1. It has been proved that a $(3, g)$ -cage is 3-connected.
2. It is conjectured that a (d, g) -cage is d -connected.

Reference

H.L. Fu, K.C. Huang and C.A. Rodger, Connectivity of cages, JGT, Vol. 24, No. 2, 187-191(1997).

Theorem 11

$$(V(G) = \{v_1, v_2, \dots, v_n\})$$

The vertices of a connected graph G can be enumerated so that $G_i =_{def} \langle v_1, v_2, \dots, v_i \rangle_G$ is connected for every $i = 1, 2, \dots, n$.

Proof.

We construct all G_i 's recursively. Define $\langle \{v_1\} \rangle_G$ be G_1 where v_1 is an arbitrary vertex of G . Assume that $G_1, G_2, \dots, G_i$ have been

constructed for $i \geq 1$ and $i < n$. Now, let $v \in V(G) \setminus V(G_i)$. Since G is connected, there exists a path connecting v and v_1 . Let v_{i+1} be the last vertex on the path not in G_i . (See Figure 13.) Then, $\{v_1, v_2, \dots, v_i, v_{i+1}\}$ induces a connected graph, G_{i+1} . ■

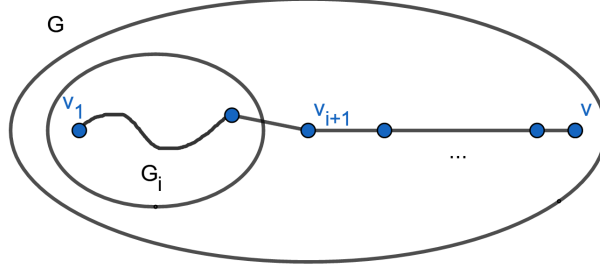


Figure 13. Graph enumeration

- (•) The vertices of a tree T can be enumerated such that every v_{i+1} has a unique neighbor in $\{v_1, v_2, \dots, v_i\}_T$. ($v \sim v_1$ is unique.)

Theorem 12

If T is a tree and G is any graph with $\delta(G) \geq \|T\|$, then $T \leq G$.

Proof.

(1st)

By induction on $\|T\|$. Clearly, the assertion is true for $\|T\| = 1$. Assume that the assertion is true for $\|T\| = k \geq 2$. Consider a tree T with $\|T\| = k + 1$. Since $\delta(G) \geq k + 1 > k$, G contains a tree (arbitrarily given) of size k , T' . (Let $T' = T - v_0$ where v_0 is a pendant vertex of T , moreover $v_0 v' \in E(T)$.) Now, consider T' in G . v' corresponds to a vertex u in $V(G)$. Since $\delta(G) \geq k + 1$, u has a neighbor u_0 which is not

a vertex of T' . Note here that T' has k vertices not including u . By attaching u_0 to T' , we have a subgraph T .

(2nd)

Direct construction

Let T be a tree defined on $\{v_1, v_2, \dots, v_n\}$ such that for each i , every vertex v_{i+1} has a unique neighbor in $\{v_1, v_2, \dots, v_i\}_T$. Now, we can construct directly on G the tree T by following the enumerating order. (Pick the vertices and edges recursively.) Since $\delta(G) \geq \|T\|$, there is always a vertex in G available for the next choice. ■

Theorem 13

Every graph G with at least one edge has an (induced) graph with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.

Proof.

We construct a sequence $G = G_0 \geq G_1 \geq G_2 \geq \dots$ of induced subgraphs of G as follows. If G_i has a vertex v_i of degree $\deg_{G_i}(v_i) \leq \varepsilon(G_i)$, let $G_{i+1} =_{\text{def}} G_i - v_i$; if not, we terminate our sequence and set $H =_{\text{def}} G_i$.

Now, by the selection process, $\delta(H) > \varepsilon(H)$. Furthermore,

$$\varepsilon(G_{i+1}) \geq \frac{\|G_{i+1}\|}{\|G_i\| - 1} = \frac{\|G_i\| - \deg_{G_i}(v_i)}{|G_i| - 1} \geq \frac{\|G_i\| - \varepsilon(G_i)}{|G_i| - 1} =$$

$$\frac{\|G_i\| - \|G_i\|/|G_i|}{|G_i| - 1} = \frac{\|G_i\| \left(1 - \frac{1}{|G_i|}\right)}{|G_i| - 1} = \frac{\|G_i\|}{|G_i|} = \varepsilon(G_i).$$

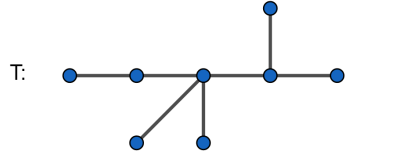
Hence, $\varepsilon(H) \geq \varepsilon(G)$. ■

(•) If there exists no v_i to delete, then G is the subgraph we need.

Tree Packing problem

Given a tree T of size $\|T\| = n$. Can we pack $2n + 1$ copies of T into the complete graph of order $2n + 1$?

For example,



Fact 1

K_{2n+1} has $(2n + 1) \cdot n$ edges.

Fact 2

If we can pack T ($\|T\| = n$) into K_{2n+1} , then it is a *decomposition* of K_{2n+1} into T 's.

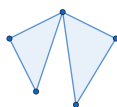
Review

An $\mathcal{H}$ -decomposition of G is a partition of $E(G)$ such that each part induces a graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$, then we simply say G has an H -decomposition.

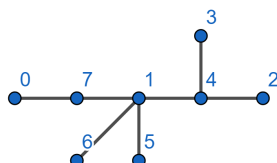
Review

A packing or $\mathcal{H}$ -packing of G is a collection of edge-disjoint subsets of $E(G)$ such that induces a graph in $\mathcal{H}$.

e.g. A K_3 -packing of K_5 .

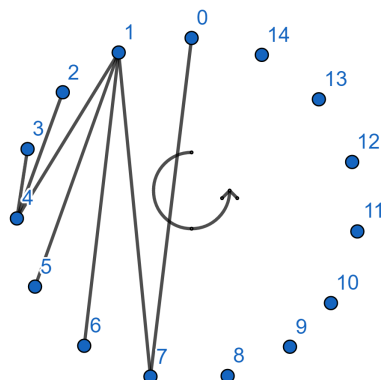


An idea of packing.



(Find a good labeling!)

[Graceful labeling or β -labeling]



Definition (Graceful labeling) (or a β -labeling)

A vertex-labeling $\varphi : V(G) \xrightarrow{1-1} \{0, 1, 2, \dots, \|G\|\}$ is a graceful labeling if all the weights of uv , $|\varphi(u) - \varphi(v)|$ are distinct.

Graceful labeling tree conjecture

:Trees are graceful!

$k(G)$: Connectivity of G , $k'(G)$: edge-connectivity of G

Theorem 14

For each connected graph G , $k(G) \underset{\textcircled{1}}{\leq} k'(G) \underset{\textcircled{2}}{\leq} \delta(G)$.

Proof.

② Let v be a vertex of G such that $\deg_G(v) = \delta(G)$. Since the deletion of all edges incident v will provide a disconnected graph, $\delta(G) \geq k'(G)$.

① Let F (Figure 14) be an edge cut of G which has $k'(G)$ edges.

Case 1. $\exists v \in V(G)$, v is not incident any edge of F . Let C be the component in $G - F$ which contains v . Then, C contains exactly one vertex of each edge of F . (?) Hence, deleting all such vertices results in a disconnected graph. Hence $k(G) \leq k'(G)$.

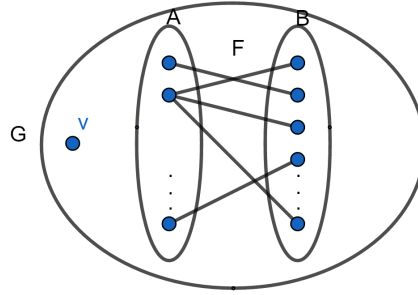


Figure 14. Edge - cut F

Case 2. $\forall u \in V(G)$, u is incident to an edge of F .

Let C be the component of $G - F$ which contains an arbitrary vertex v . Now, if $wv \notin F$, then w is incident to an edge of F . Then,

$\deg_G(v) \leq |F|$. This implies (again) $k(G) \leq |F| \leq k'(G)$. On the other hand all, $wv \in F$, $w \in N_G(v)$, then (W.L.O.G.) we have a complete graph of order $|F| + 1 = |G|$. $k(G) = |G| - 1 = |F| = k'(G)$. ■

(•) It is interesting to know the graphs G satisfying $k(G) = \delta(G)$.

Theorem 15 (Mader, 1972)

Let $k \in \mathbb{N}$. Every graph G with $d(G) \geq 4k$ has a $(k+1)$ -connected subgraph H such that $\varepsilon(H) > \varepsilon(G) - k$.

Proof.

For convenience, let $\varepsilon = \varepsilon(G)$ and consider $G' \leq G$ s.t. $|G'| \geq 2k$ and $\|G'\| > \varepsilon \cdot (|G'| - k)$. Since $\|G\| = \varepsilon \cdot |G| > \varepsilon(|G| - k)$, such graphs G' do exist. Let H be the one with minimum order.

Clearly, $|H| > 2k$, for otherwise, $\|H\| > \varepsilon \cdot k \geq 2k^2 > \binom{|H|}{2} = k \cdot (2k - 1)$.
 $\rightarrow \leftarrow$ The minimality of H implies that $\delta(H) > \varepsilon$, by Theorem 13 (we can choose a proper induced subgraph). Hence, $|H| \geq \varepsilon$. By the choice of H , we have $\|H\| > \varepsilon(|H| - k)$ mentioned above, thus $\varepsilon(H) > \varepsilon - k$.
 $\left(\frac{\|H\|}{|H|} > \frac{\varepsilon(|H|) - \varepsilon \cdot k}{|H|} = \varepsilon - \frac{\varepsilon \cdot k}{|H|} \geq \varepsilon - \frac{|H| \cdot k}{|H|} = \varepsilon - k \right)$ Therefore, H satisfies the condition we need. It's left to check that H is

$(k+1)$ -connected. Suppose not. Let K be a set of k vertices in H such that $H - K$ is disconnected, see Figure 15. Now, let $V(H) = U_1 \cup U_2$ such that $U_1 \cap U_2 = K$, $H_1 = \langle U_1 \rangle_H$, $H_2 = \langle U_2 \rangle_H$ and no vertices of $U_1 \setminus U_2$ is adjacent to vertices in $U_2 \setminus U_1$.

Let $v \in U_1 \setminus U_2$. Since $\deg_H(v) \geq \delta(H) > \varepsilon$, this implies that $|H| > \varepsilon \geq 2k$, so is $|H_2| \geq 2k$. Because of the fact that H is the choice of

G' with minimum order, $\|H_1\| \leq \varepsilon(|H_1 - k|)$ and $\|H_2\| \leq \varepsilon(|H_2 - k|)$.

Hence $\|H\| \leq \|H_1\| + \|H_2\| \leq \varepsilon(|H_1| + |H_2| - 2k) = \varepsilon(|H| - k)$. A

contradiction to the choice of H . ■

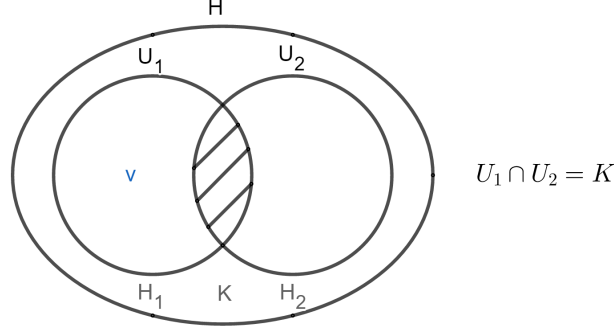


Figure 15. $H_1 = \langle U_1 \rangle_H$ and $H_2 = \langle U_2 \rangle_H$

- A maximal connected subgraph without a "cut vertex" is call a "block".

↓

increase the # of components

- (••) Every block of a graph G is either a maximal 2-connected subgraph or a bridge (with its endvertices), or an isolated vertex.
(See Figure 16.)

- A block graph of G is a bipartite graph $bc(G) = (A, B)$ where A is the set of cutvertices of G and B is the set of blocks, and $a \in A$ is incident to $B_i \in B$ if $a \in V(B_i)$.

Theorem 16

The block graph of a *connected* graph is a tree.

Proof.

Observe that each block is an induced subgraph and any two blocks have at most one cutvertex in common.

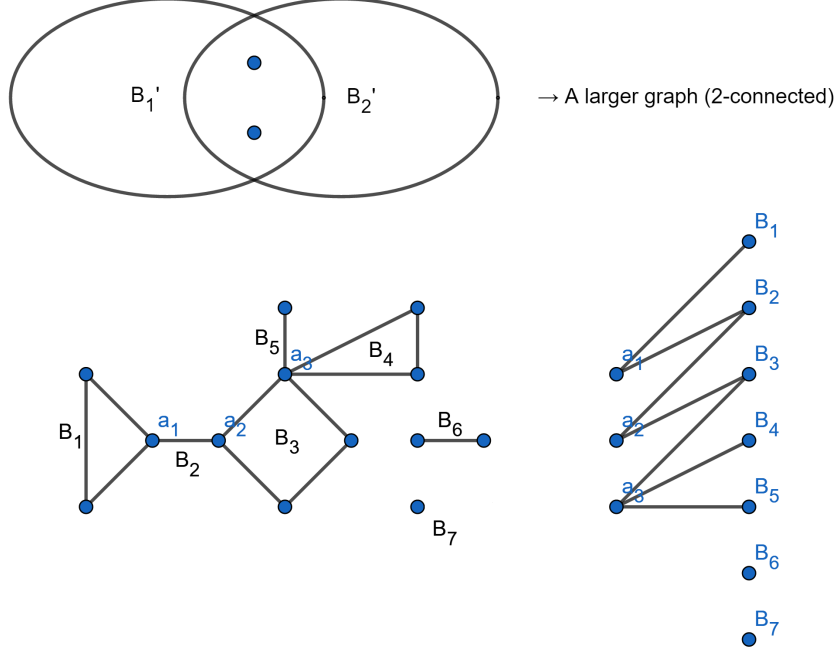


Figure 16. A block graph

Since G is connected, if G has only one block and thus contains no cutvertex, $bc(G)$ is a single vertex. The proof is trivial. Assume that G contains more than one block. Then, each vertex is either in a block or a cutvertex itself. Now, consider a cutvertex v and a block B_i . Let $u \in B_i \setminus \{v\}$. ($B_i \setminus \{v\}$ is non-empty since G is connected.) Then, we have a path P connecting v and u . Clearly, v is going to connect to a block which contains some vertices of P . If this block B_i , then vB_i is an edge of $bc(G)$, done. Otherwise, this path will contain a cutvertex following the block and travels to another block and finally to B_i . The

other two cases, cutvertex to cutvertex and block to block can be verified similarly.

Now, for the acyclic part, a cycle in $bc(G)$ will produce a cycle in G which passes all cutvertices involved. But, in that case, none of these cutvertices are cutvertices anymore, a contradiction. This completes the proof. ■

Theorem 17

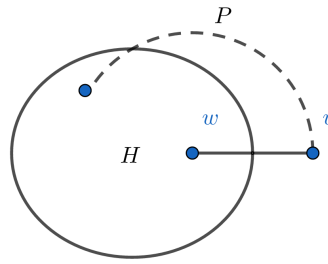
A graph G is 2-connected if and only if can be constructed from a cycle by successively adding H -paths to graphs H already constructed.

Proof.

($\Leftarrow$) From the construction, it is clear that G contains no cutvertices. Hence, G is 2-connected.

($\Rightarrow$) Assume that G is 2-connected and H is a maximum (size) subgraph following the construction. This is possible, since G contains a cycle. Infact, H is an *induced* subgraph, since for any two vertices x and y in $V(H)$ and $xy \in E(G) \setminus E(H)$, we have an H -path and a larger subgraph will be obtained.

Now, assume that $H \neq G$. $\exists v \in V(G) \setminus V(H)$, $w \in V(H)$ and $vw \in E(G) \setminus E(H)$. Since G is 2-connected, $G - w$ contains a $v - H$ path P . (See Figure 17) This implies that $\langle w, v, P \rangle$ is an H -path. Hence, a larger subgraph is obtained, a contradiction. ■

Figure 17. An extra H -path

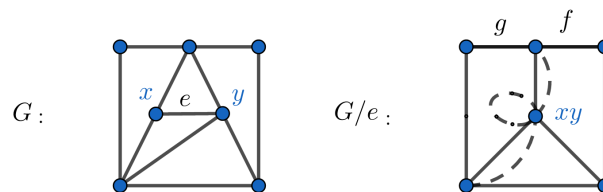
Definition (Graph minors)

A graph M is called a minor of G if M can be obtained from G by contracting edges, deleting vertices and edges.

(*) Determine whether a graph G contains a minor H is considered as the most important problem in the study of graph structure.

Review (Edge-contraction)

Given an edge $xy = e$ of a graph G , the graph G/e is obtained from G by contracting e ; that is to identify the vertices x and y and deleting resulting loops and duplicate edges.

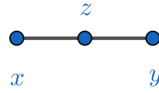


Example

K_4 is a minor of the above G . (Contracting e, f and g .)

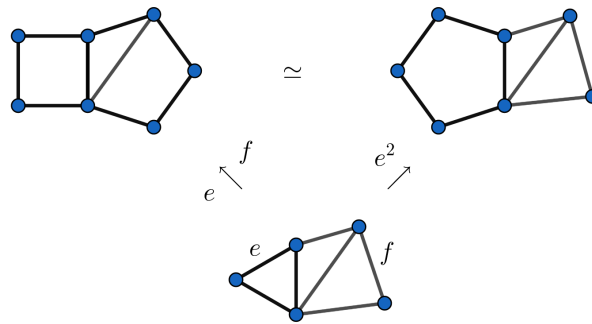
Definition (Subdivision)

A subdivision of an edge xy is obtained by adding a new vertex z such that we have edges xz and zy .



Definition (Homeomorphic Graphs)

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

Remark

Two cycles are homeomorphic.

(•) The reverse of subdivision can be "*considered*" as *contraction*.

Theorem 18

If G is 3-connected and $|G| > 4$, then G has an edge such that G/e is 3-connected.

Proof.

Suppose not. Then, for each $xy \in E(G)$, the graph G/xy contains a separator S with $|S| \leq 2$. ($G - S$ is disconnected.) Since $k(G) \geq 3$, the contracted vertex $v_{xy} \in S$ and $|S| = 2$, i.e., $\exists z \in V(G)$, s.t. $S = \{v_{xy}, z\}$. Therefore, $T = \{x, y, z\}$ is a separator set of G . Since one proper subset of T can separate G , each vertex of T is incident to every component of $G - S$. (See Figure 18)

Among all edges of G , we choose an edge xy and its corresponding vertex z such that the component C has minimum size. (*) Let $v \in V(C)$ and $vz \in E(G)$. By assumption, G/vz is again not 3-connected and there exists a corresponding vertex w such that $\{v, z, w\}$ separates G .

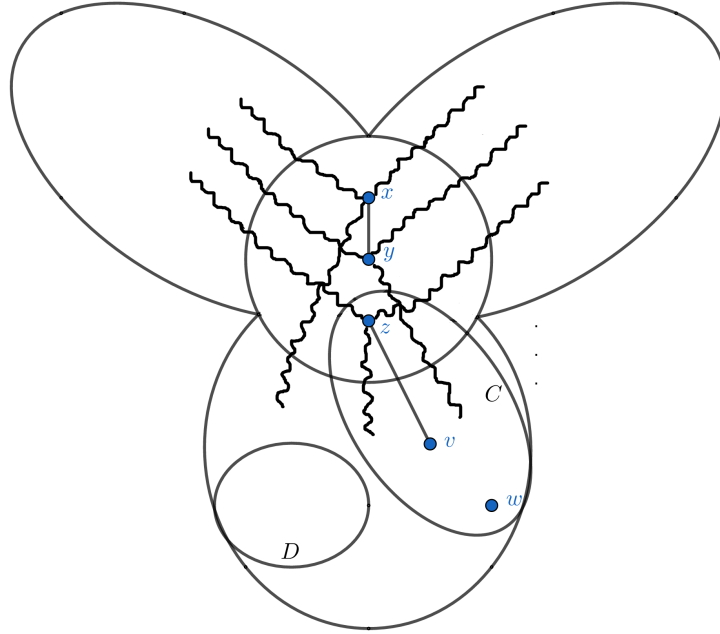


Figure 18. 3-connected graph

Moreover, each vertex of $\{v, z, w\}$ is incident to every component of $G - \{v, z, w\}$. Since $xy \in E(G)$, $G - \{v, z, w\}$ has a component D s.t.

$D \cap \{x, y\} = \emptyset$. By the fact $v \in V(C)$, the neighbor of v in D is also in C . Hence, $D \cap C \neq \emptyset$. This implies that D is a proper subset of C , i.e., $|D| < |C|$, a contradiction to the choice of C . ($|C|$ is minimum.) ■

Theorem 19 (Tutte, 1961)

A graph G is 3-connected if and only if there exists a sequence $G_0, G_1, \dots, G_n$ of graphs satisfying :

- (a) $G_0 = K_4$ and $G_n = G$; and
- (b) G_{i+1} has an edge xy (with degree of x, y at least 3) such that $G_i = G_{i+1}/xy, 1 \leq i < n$.

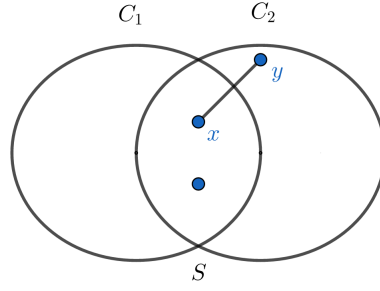
Proof.

($\Rightarrow$) By Theorem 18, we start with G as G_n and end at $K_4 = G_0$.

($\Leftarrow$) Let $G_0, G_1, \dots, G_n$ be a sequence of graphs satisfying (a) and (b).

It suffices to show that if $G_i = G_{i+1}/xy$ is 3-connected, then G_{i+1} is also 3-connected, for all $1 \leq i < n$.

Suppose not. Let S be a separator with $|S| \leq 2$. Also, let C_1 and C_2 be two components of $G_{i+1} - S$. Since $xy \in E(G_{i+1})$, let $\{x, y\} \cap V(C_1) = \emptyset$. (Figure 19) Now, if $\{x, y\} \subseteq C_2$, then $G_i - S$ is disconnected, a contradiction. Hence, at most one of $\{x, y\}$ is in C_2 , either x or y , but not both. Furthermore, if $v \notin \{x, y\}$ and $v \in V(G)$, then $G_i - S$ is also disconnected, a contradiction to the fact G_i is 3-connected. Hence, C_2 contains exactly one vertex of degree at most 2, $\rightarrow \leftarrow$. ■

Figure 19. G_{i+1}

Theorem 20

Every non-trivial graph G contains at least two vertices which are not cutvertices.

Proof.

Review that if v is a cutvertex of G , then the number of components of G , $c(G)$ is smaller than that of $c(G - v)$. Now, consider u and v such that $d(u, v) = \text{diam}(G)$. We show both u and v are not cutvertices. Suppose not. Let u be a cutvertex, then $G - u$ is disconnected. Let w be a vertex which is in a component different from v belongs. Since u is a cutvertex and v, w are in different components, all $v - w$ path must pass through u . This implies that $d(v, w) > d(v, u) = \text{diam}(G)$. Hence, u can not be a cutvertex. Similarly, v is not a cutvertex either. ■

Definition (Network)

A network N is a digraph D with two distinguished vertices u and v , called the *source* and *sink* of N , respectively, and a non-negative integer-valued function c on $E(D)$. The digraph is the underling

digraph of N and the function c is the capacity function on N . For convenience, $c(\vec{a}) = c((x, y)) = c(x, y)$ for each arc $\vec{a} = (x, y)$, is the capacity of $\vec{a}$.

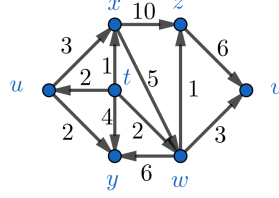


Figure 20. A network

$$N^+(x) = \{y \mid y \in V(D) \text{ and } (x, y) \in E(D)\}$$

$$N^-(x) = \{y \mid y \in V(D) \text{ and } (y, x) \in E(D)\}$$

(•) A flow in a network N , f , is a function on $E(D)$, s.t.

- ① $0 \leq f(\vec{a}) \leq c(\vec{a})$ for every $\vec{a} \in E(D)$ (capacity bound) and
- ② $\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x)$ for every $x \in V(D) \setminus \{u, v\}$.
(Conservation law)

- The net flow into x is equal to $\sum_{y \in N^-(x)} f(y, x) - \sum_{y \in N^+(x)} f(x, y)$
which is zero except $x \in \{u, v\}$.

- $(V_1, V_2) = \{(x, y) \in E(D) \mid x \in V_1 \text{ and } y \in V_2\}$ (digraph version!)

(••) A cut in N is $(X, V(D) \setminus X)$ such that $u \in X$ and $v \in V(D) \setminus X$.

||
 X'

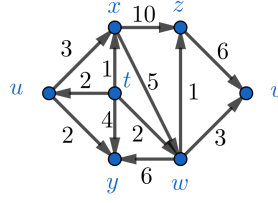


Figure 20. A network

Definition

Let $K = (X, X')$ be a cut in N . Then, the capacity of K , $\text{cap } K = c(X, X') = \sum_{(x,y) \in K} c(x, y)$. e.g. In Figure 20, let $X = \{u, x, t, y\}$, then $c(X, X') = 17$.

(*) Definition

The value of a flow f in N is defined as the net value flow out the source and therefore the net value flow into the sink. (Denoted by $\text{val } f$.)

Theorem 21

Let f be a flow in a network N and $K = (X, X')$ be a cut in N . Then, $\text{val } f \leq \text{cap } K$.

Proof.

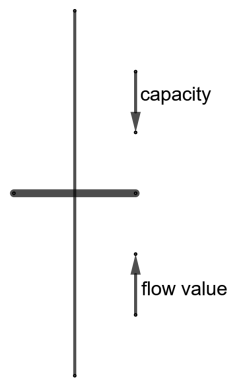
Note that $\text{val } f = f(u, \{u\}') - f(\{u\}', u)$ and $f(x, \{x\}') - f(\{x\}', x) = 0, \forall x \in X \setminus \{u\}$. ($u \in X, v \in X'$) This implies that $\sum_{x \in X} [f(x, \{x\}') - f(\{x\}', x)] = \text{val } f$

||

$f(X, X') - f(X', X)$ (Calculation).

Now, $f(X, X') \leq \text{cap}(X, X')$ and $f(X', X) \geq 0$. Hence, $\text{val } f \leq \text{cap } K$. ■

- A minimum cut is a cut K in N such that for every cut K' in N , $\text{cap } K \leq \text{cap } K'$.
- A minimum flow is a flow f in N such that for every flow f' in N , $\text{val } f \geq \text{val } f'$.



min-max problem

$\Rightarrow$ If there exists a K and an f s.t. $\text{cap } K = \text{val } f$, then K is a minimum cut and f is a maximum flow.

Theorem 22 (Ford and Fulkson, 1956-1962)

In any network N defined on D , the value of a maximum flow equals the capacity of a minimum cut.

Proof.

Clearly, if there exist no cuts such that its capacity of the cut is $\text{val } f$, then f does not exist. (Theorem 21) So, it suffices to claim that

if the value of a maximum flow f is v , then there exists a cut K , such that $\text{cap } K = \text{val } f = v$.

Define a subset $S \subseteq V(D)$ recursively as follows. Let $s \in S$. If $x \in S$, and $c(x, y) > f(x, y)$ or $f(y, x) > 0$, then let $y \in S$. We shall prove that (S, S') is a cut with capacity v . First, we claim $t \notin S$. Suppose not, i.e., $t \in S$. Hence, we can find a sequence of vertices in N such that $s = x_0, x_1, \dots, x_l = t$. Moreover, if we let $\varepsilon_i = \max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\}$, $i = 0, 1, \dots, l-1$, then $\varepsilon_i > 0$. Let $\varepsilon = \min\{\varepsilon_i\}$. Now, let $f^*(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \varepsilon$ if $c(x_i, x_{i+1}) - f(x_i, x_{i+1}) = \varepsilon_i > 0$ and $f^*(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \varepsilon$ if $f(x_{i+1}, x_i) = \varepsilon_i > 0$. As a consequence, f^* is a flow from s into t with value $\text{val } f^* = v + \varepsilon$, a contradiction. (See Figure 21) By the definition of a flow,

$$\text{val } f = v = \sum_{x \in S, y \in S'} f(x, y) - \sum_{x \in S', y \in S} f(x, y). \quad \text{---(1)}$$

Again, by the definition of S , if $x \in S$ and $y \in S'$, then $c(x, y) = f(x, y)$ and $f(y, x) = 0$. This implies that (1) =

$\sum_{x \in S, y \in S'} c(x, y) = \text{cap}(S, S') = v$, the proof follows. ■

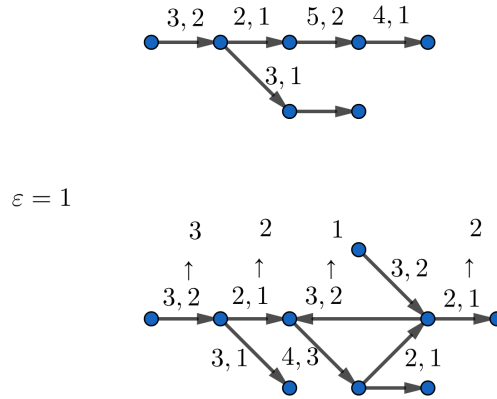


Figure 21. augmenting flow f^*

Theorem 23 (Menger, 1927)

Let s and t be two nonadjacent vertices of a graph G . Then, the minimal number of vertices separating s from t is equal to the maximal number of vertex-disjoint (Internally) $s - t$ paths.

Proof.**(1st)**

Let the number of vertices separating s and t be k . Then, it is easy to see that there are at most k independent (vertex-disjoint) paths connecting s and t . Also, if $k = 1$, then we have a path joining s and t . Now, suppose the assertion is not true, i.e., we can find less than k independent $s - t$ paths for certain k . Now, take the minimal k in which we have a counterexample. Then, among all such examples, let G be the one with minimum size (number of edge).

First, we notice that s and t have at most $k - 1$ independent paths and no common neighbors. For otherwise, let sx and xt be edges of G . Then, $G - x$ will be a counterexample for " $k - 1$ " (smaller than k).

Let W be a separating set of s and t and $|W| = k$. Suppose, neither $N_G(s) = W$ nor $N_G(t) = W$. (Figure 22-1)

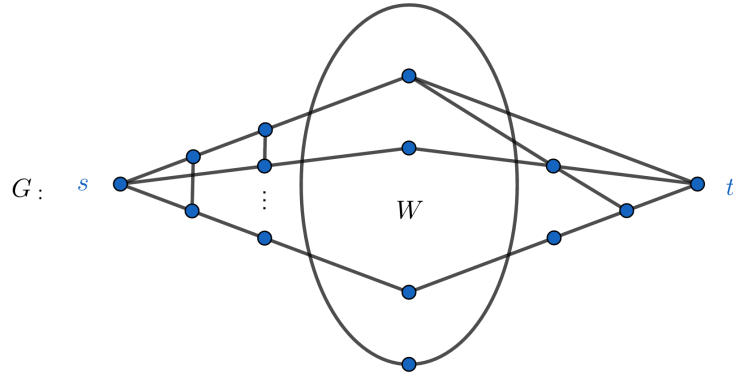


Figure 22-1.

Let G_s be obtained by deleting all the vertices to the left of G in Figure 22-1 and adding a replacing s' with edges joining to W , see Figure 22-2. Now, G_s has fewer edges than G and thus there are k independent $s' - t$ paths. Hence, we have k $W - t$ independent paths. With the same technique, we derive k $s - W$ independent paths (by changing s to t).

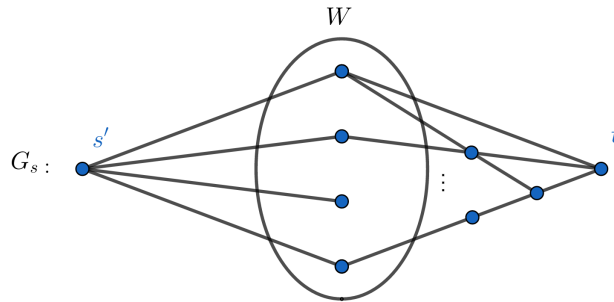


Figure 22-2.

So, as a conclusion, either s or t must have their neighbors W . Let $N_G(s) = W$ and $P = \langle s, x_1, x_2, \dots, x_l, t \rangle$ be a shortest $s - t$ path. Then, $l \geq 2$. Consider $G - x_1x_2$.

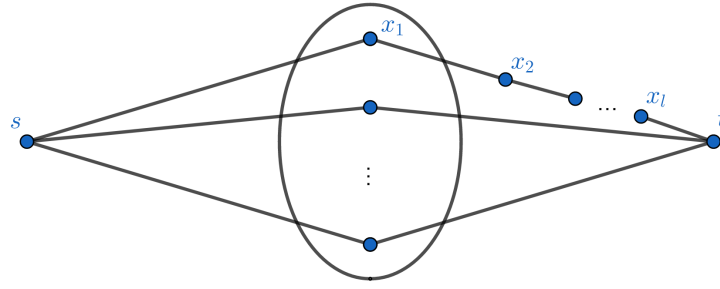


Figure 22-3.

In $G - x_1x_2$, there exists an $s - t$ separating set W_0 of size $k - 1$. Then, both $W_1 = W_0 \cup \{x_1\}$ and $W_2 = W_0 \cup \{x_2\}$ are $s - t$ separating sets of G . By the fact that P is a shortest $s - t$ path, s is not adjacent to x_2 and t is not adjacent to x_1 . This implies that that $N_G(s) = W_1$ since t is not adjacent to a vertex of the separating set W_1 . Similarly, $N_G(t) = W_2$. Hence, $N_G(s) \cap N_G(t) = W_0$ (s and t have common neighbors), a contradiction. ($|W_0| = k - 1 \geq 1$.) ■

(2nd)

Apply Theorem 22.

(•) The "Edge" version of Menger's Theorem can be started as follows:

Let s and t be two vertices of G . Then, the minimal number of edges separating s from t is equal to the maximal number of edge-disjoint $s - t$ paths. (We can prove this part by using Theorem 22. Replace each edge xy of G by (x, y) and (y, x) and assign capacity "1" to each arcs.)

Theorem 24

If S and T are arbitrary subsets of $V(G)$, then the maximal number of vertex-disjoint (including endvertices) $S - T$ paths is $\min\{|W| \mid W \subseteq V(G) \text{ and } G - W \text{ has no } S - T \text{ paths}\}$.

Proof.

By adding to new vertices s and t as in Figure 23, we have a new graph $\tilde{G}$. Now, by Menger's Theorem, the maximal number of $S - T$ paths is the same as that of $s - t$ paths in $\tilde{G}$. Hence, we have the proof. ■

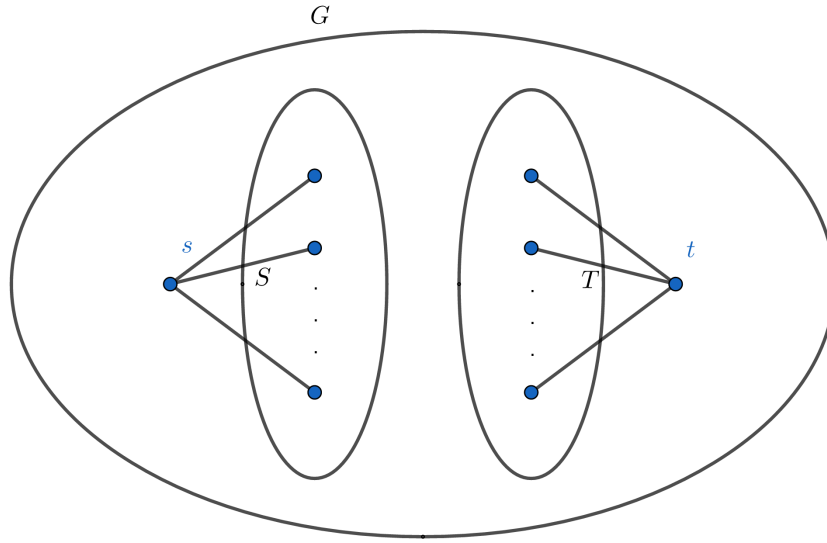


Figure 23. graph $\tilde{G}$

Definition (System of Distinct Representatives, SDR)

Let $A = \{A_1, A_2, \dots, A_n\}$ be a collection of subsets of a given set X . Then, an ordered n -tuple $(a_1, a_2, \dots, a_n)$ is called an SDR of A if $a_i \in A_i$, $i = 1, 2, \dots, n$ and all elements a_i 's are distinct.

Hall's Theorem(1935)

$A = \{A_1, A_2, \dots, A_n\}$ has an SDR if and only if for each $1 \leq k \leq n$, the union of any k subsets in A contains at least k distinct elements, i.e., $|\bigcup_{j=1}^k A_{i_j}| \geq k$. (Hall's condition)

We can use a bipartite graph to depict the above idea.

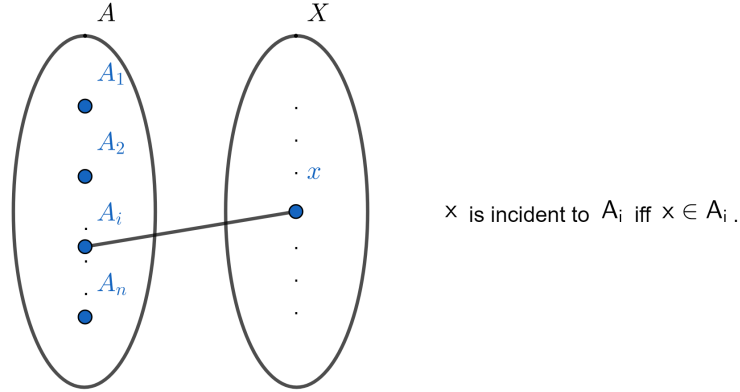


Figure 24. (Marriage problem)

Theorem 25

A bipartite graph $G = (A, B)$ contains a matching saturates A if and only if for every $S \subseteq A$, $\Gamma(S) = \bigcup_{x \in S} N_G(x)$ contains at least $|S|$ elements of B , i.e., $|\Gamma(S)| \geq |S|$.

Proof.

(1st)

($\Rightarrow$) By the existence of a matching saturates A .

($\Leftarrow$) By Theorem 24, it suffices to prove that there are $|A|$ vertex-disjoint $A - B$ paths (and thus a matching saturates A).

Suppose not. Then, there exists a subset A_1 of A and a subset B_1 of B such that there is no edge between $A \setminus A_1$ and $B \setminus B_1$, see Figure 24, and $|A_1| + |B_1| < |A|$. (The number of $A - B$ paths is less than $|A|$.)

Hence, there are no edges between $A \setminus A_1$ and $B \setminus B_1$, equivalently $\Gamma(A \setminus A_1) \subseteq B_1$. Then, $|\Gamma(A \setminus A_1)| \leq |B_1| < |A| - |A_1| = |A \setminus A_1|$. $\rightarrow\leftarrow$ ■

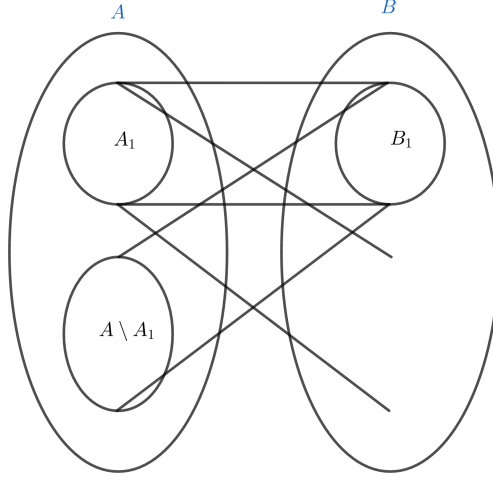


Figure 24.

(2nd)

By induction on $|A|$. Clearly, it's true for $|A| = 1$.

First, if for each $S \subsetneq A$, $|\Gamma(S)| \geq |S| + 1$, then set $a_1 \in A$ and $a_1 b_1 \in E(G)$ where $b_1 \in N_G(a_1)$. Now, consider the bipartite graph $G' = (A_1 \setminus \{a_1\}, B_1 \setminus \{b_1\})$. Since for each $S' \subseteq A_1 \setminus \{a_1\}$, $|\Gamma(S')| \geq |S'|$, there exists a matching saturates $A_1 \setminus \{a_1\}$. Combining with $a_1 b_1$, we have a matching needed.

Second, if there exists a proper subset S of A such that $|\Gamma(S)| = |S|$. By induction, we have a matching M_S saturates S . Now, consider $(A - S, B - T)$ where T is the set of vertices used in M_S . If there exists an $S' \subseteq A - S$ such that $|\Gamma(S')| < |S'|$, then $|\Gamma(S \cup S')| < |S \cup S'|$ a contradiction. Hence, the Hall's condition holds for the graph $(A - S, B - T)$. By induction, we have a matching saturates $A - S$. As a consequence, G has a matching saturates A .

(3rd) (Rado)

Let $G = (A, B)$ be a minimal (size) graph satisfying the condition. It suffices to claim that G contains $|A|$ independent edges (matching of size $|A|$). Suppose not. There exist two vertices a_1 and a_2 in A and b in B such that a_1b and a_2b are edges of G . Since both $G - a_1b$ and $G - a_2b$ violate Hall's condition, there exist two subsets A_1 and A_2 of A such that $|\Gamma(A_1)| = |A_1|$, $|\Gamma(A_2)| = |A_2|$ and a_i ($i = 1, 2$) is the only vertex of A_i which is adjacent to b .

$$\begin{aligned} \text{Hence, } |\Gamma(A_1) \cap \Gamma(A_2)| &\geq |\Gamma(A_1 - a_1) \cap \Gamma(A_2 - a_2)| + 1 \\ &\geq |\Gamma(A_1 \cap A_2)| + 1 \\ &\geq |A_1 \cap A_2| + 1. \end{aligned}$$

$$\begin{aligned} \text{On the other hand, } |\Gamma(A_1 \cup A_2)| &= |\Gamma(A_1) \cup \Gamma(A_2)| \\ &= |\Gamma(A_1)| + |\Gamma(A_2)| - |\Gamma(A_1 \cap A_2)| \\ &\leq |A_1| + |A_2| - |A_1 \cap A_2| - 1 \\ &= |A_1 \cup A_2| - 1. \quad (\rightarrow \leftarrow). \end{aligned}$$

■

(•) Can you find another proof?

Theorem 26 (König)

Every r -regular bipartite graph contains r edge-disjoint perfect matching.

Proof.

By induction on r . Clearly, it is true for $r = 1$. Let $r \geq 2$ and G be the r -regular bipartite graph where $G = (A, B)$. Then $|A| = |B|$. So, it

suffices to find a matching saturates A . Now for any subset S of A , $\Gamma(S) = \bigcup_{x \in S} N_G(x)$. If $|S| = k$, then S is incident to $k \cdot r$ edges. Since each vertex of B is of degree r , it takes at least k vertices of B to join with these $r \cdot k$ edges. This implies that $|\Gamma(S)| \geq |S|$. So, by Hall's Theorem, a matching saturates A can be obtained. Following the same process we conclude proof. ■

Theorem 27

Let $G = (A, B)$ be a bipartite graph such that for each $S \subseteq A$, $|\Gamma(S)| \geq |S| - d$, $d < |A|$. Then, G contains a matching with $|A| - d$ edges.

Proof.

Clearly, if $d = 0$, then we have a matching with $|A|$ edges. Now, let $d > 0$ and $B' = B \cup D$ where $D = \{y_1, y_2, \dots, y_t\}$ and $D \cap B = \emptyset$. Let $G' = (A, B')$ such that $E(G') = E(G) \cup \{y_i a_j | i = 1, 2, \dots, d; j = 1, 2, \dots, |A|\}$. (Join each vertex in D to every vertex of A .)

Now, for each $S \subseteq A$, $|\Gamma(S)| \geq |S|$ (in G'). Hence G' has a matching saturates A . This implies G has a matching of size at least $|A| - d$. ■

Remark

The following results can be obtained by applying Hall's Theorem.

1. A Latin rectangle can be extended to a Latin square.
2. An $n \times n$ matrix (non-negative) $A = (a_{ij})$ is said to be doubly stochastic if $\sum_{i=1}^n a_{ij} = 1$ for every j and $\sum_{j=1}^n a_{ij} = 1$ for every i .

Then, there exist $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda_k = 1$, and permutation matrices

$P_1, P_2, \dots, P_m$ such that $A = \sum_{k=1}^m \lambda_k P_k$.

3. More ...

Appendix

Use max-flow min-cut theorem to prove Hall's Theorem.

Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, $m \leq n$.

Claim: There exists a matching in $G = (X, Y)$ saturates X if and only if for each subset $A \subseteq X$, $|\Gamma(A)| \geq |A|$.

(Note) We shall prove the theorem for both $(\Rightarrow)$ and $(\Leftarrow)$ by using network argument.

First, we construct a network N by (1) adding s and t such that (s, x_i) and (y_j, t) are arcs in N , $1 \leq j \leq n$, (2) orienting $x_i y_j \in E(G)$ with (x_i, y_j) , and (3) $c(s, x_i) = c(y_j, t) = 1$ and $c(x_i, y_j) = M > |X| = m$. (See Figure 1 below.)

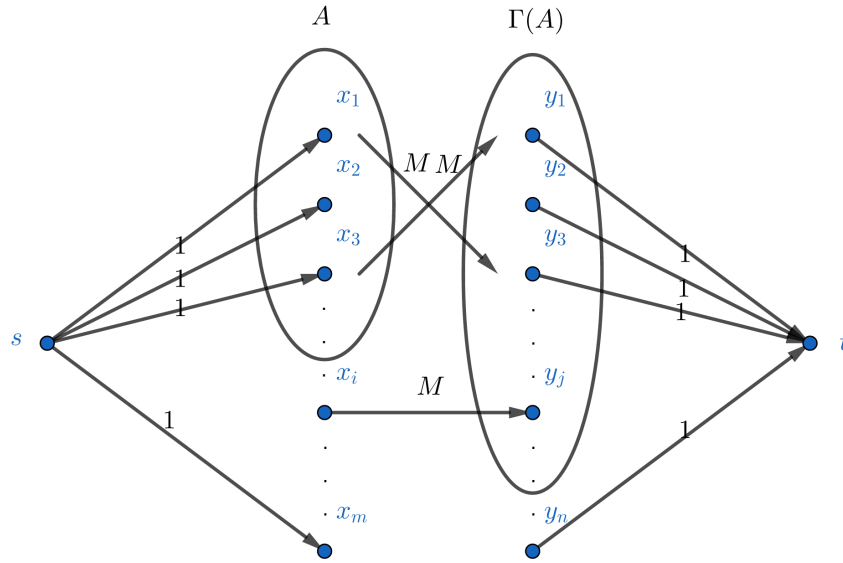


Figure 1.

Now, it is easy to see that a flow of value $|X| = m$ will provide a matching saturates X since each (y_j, t) can take at most one flow value through the flow.

($\Rightarrow$) Suppose that there exists a subset A of X such that $|A| > |\Gamma(A)|$. Let $S = \{s\} \cup A \cup \Gamma(A)$. Then, $\langle S, \bar{S} \rangle$ is a cut with capacity $(|X| - |A|) + |\Gamma(A)|$ since there are no arcs from the vertices of A to the $(s \rightarrow \bar{S}) \quad (S \rightarrow t)$

vertices of $Y \setminus \Gamma(A)$. This capacity of cut is less than $|X|$. Hence, there exists no flow with value $|X|$ and thus no matching saturates X . ($\Leftarrow$) Assume that $|\Gamma(A)| \geq |A|$ for each $A \subseteq X$. It suffices to claim that all cuts have capacity at least $|X|$. Let $S = \{s\} \cup A \cup B$, see Figure 2. ($A \subseteq X$ and $B \subseteq Y$.) $\leftarrow$ They can be empty. (B is not necessary be $\Gamma(A)$).

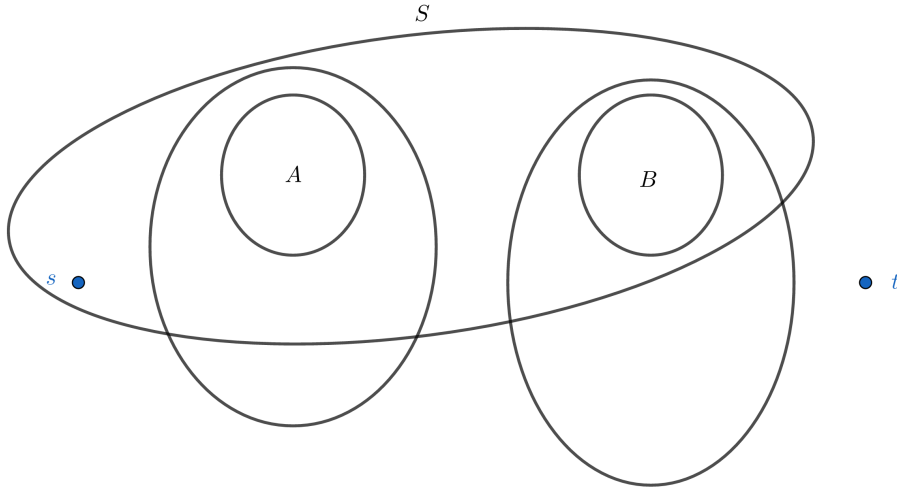


Figure 2.

Now, if there exists an arc from the vertices of A into $Y \setminus B$, then $c(S, \bar{S}) \geq M > |X|$. On the other hand, if $\Gamma(A) \subseteq B$, then $c(S, \bar{S}) = (|X| - |A|) + |B| \geq |X| - |A| + |\Gamma(A)| \geq |X|$.

$$s \rightarrow \bar{S}$$

Since $c(S, \bar{S}) = |X|$ in the case $S = \{s\}$, we obtain a min-cut with capacity $|X|$ and thus there exists a flow with maximum value $|\bar{X}|$.

The proof follows. ■

Use max-flow min-cut theorem to prove Menger's Theorem.

Proof.

First, we proof a directed version of Menger's Theorem.

- (•) If s and t are distinct vertices of a digraph D such that $s \not\sim_D t$, then the maximum number of internally disjoint $u - v$ directed paths in D equals the maximum number of vertices in a $u - v$ separating set in D .
- (••) For the undirected version, we replace each edge uv by a pair of arcs (u, v) and (v, u) .

Proof of (•).

Let $\tilde{D}$ be the digraph obtained as follows:

- (1) $\forall x \in V(D) \setminus \{s, t\}$, split x into two vertices x' and x'' , also let $(x', x'') \in A(\tilde{D})$;
- (2) $\forall (x, y) \in A(D)$ such that $\{x, y\} \cap \{s, t\} = \emptyset$, replace (x, y) with (x'', y'') ;
- (3) For $x \neq t$, $(s, x) \in A(D)$, $(x, s) \in A(D)$, replace them with (s, x') and (x'', s) respectively, and

- (4) Replace (t, x) with (t, x') and (x, t) with (x'', t) if $x \neq s$. As a consequence, we have a network N defined on D by assigning each arc a capacity "1".

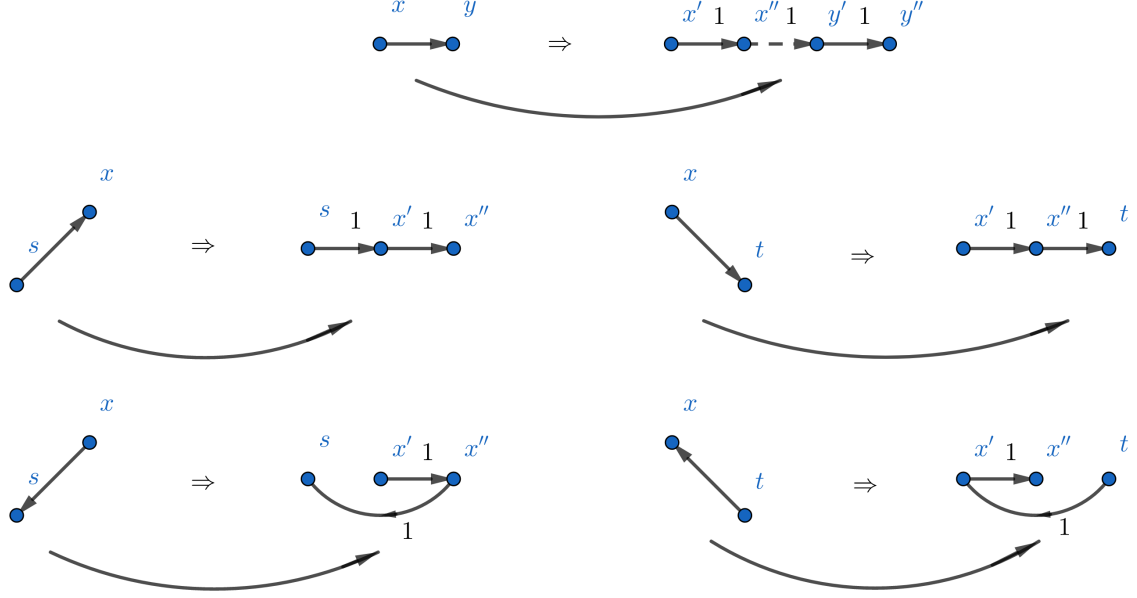


Figure 3.

Let m be the maximum number of internally disjoint $s - t$ paths in D and n be the minimum number of vertices in an $s - t$ separating set in D . Let A be a $u - v$ separating set of arcs in $\tilde{D}$ and $|A| = n'$. First, we observe that if $(S, \bar{S})$ is a cut, then $\text{cap}(S, \bar{S}) \leq n'$. Here, S contains s if $(s, x') \in A$, S contains x if $(x', x'') \in A$ and $(x'', t) \in A$. Moreover, $\text{cap}(S, \bar{S}) \geq n$. As a matter of fact, the min-cut is of capacity n and thus N has a maximum flow n . By the way, the construction of network shows that a flow value 1 will give a path (directed) from s to t . Since each arc has capacity one, these paths are internally disjoint. The proof follows. ■

- (•) A 1-factor of a graph G is a 1-regular spanning subgraph of G .

Theorem 28 (Tutte's 1-factor theorem)

A nontrivial graph G has a 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G - S$, $o(G - S) \leq |S|$.

Proof.

($\Rightarrow$) Assume that F is a 1-factor of G and there exists a proper subset W of $V(G)$ such that $o(G - W) > |W|$. Since an odd component H has an odd number of vertices, one of the vertices in H incident to F must be joining a vertex of W . But, we have more odd components than $|W|$. One of the vertices in W will be incident to at least two edges in F , a contradiction.

($\Leftarrow$) Since $o(G - \emptyset) \leq 0$, G contains only even components. Hence, $|G|$ is even. Furthermore, if $|S|$ is odd (even), $o(G - S)$ must be odd (even). So, $|S|$ and $o(G - S)$ are of the same parity. We shall prove the sufficiency by induction on $|G| = n$. Clearly, if $n = 2$, then $G \cong K_2$.

Assume for all graphs H of even order less than n that if

$o(H - W) \leq |W|$ for every proper subset of $V(H)$, then H has a 1-factor. Let G be a graph of order n and $o(G - S) \leq |S|$ for each proper subset S of $V(G)$. We claim that G has a 1-factor.

Case 1. $\forall S \subsetneq V(G)$, $|S| \geq 2$ and $o(G - S) < |S|$. —(*)

The fact of parity shows that $o(G - S) \leq |S| - 2$ for all S . Let $e = uv$ be an edge of G and consider $G - \{u, v\}$. By the fact that for each proper subset T of $V(G - \{u, v\})$, $o(G - \{u, v\} - T) \leq |T|$ and induction

hypothesis, $G - \{u, v\}$ has a 1-factor, so is G . (If

$o(G - \{u, v\} - T) > |T| = |T \cup \{u, v\}| - 2$, then

$o(G - (\{u, v\} \cup T)) \geq |T \cup \{u, v\}|$, a contradiction to $(*)$.)

Case 2. $\exists R \subseteq V(G)$, s.t. $o(G - R) = |R|$ where $2 \leq |R| \leq n$.

Among all such R 's, let S be the one of maximum cardinality $|S| = h$. Now, let $G_1, G_2, \dots, G_h$ denote the odd components of $G - S$. Note that the h odd components are the only components in $G - S$. For otherwise, let G_0 be an even component of $G - S$ and $v_0 \in V(G_0)$. Then, $o(G - S \cup \{v_0\}) \geq h + 1 = |S \cup \{v_0\}|$. In fact, $o(G - S \cup \{v_0\}) = |S \cup \{v_0\}|$ by the assumption. Now, we have a larger "R" for S , a contradiction.

For $i = 1, 2, \dots, h$, let S_i be the set of vertices in S which are adjacent with vertices in G_i . None of S_i 's will be empty. For otherwise, G_i is an odd component of G and it is not possible. (G has only even components.)

Now, for $1 \leq k \leq h$, consider the union T of "any" k sets in $\{S_1, S_2, \dots, S_h\}$. Suppose that $|T| < k$. Since $o(G - T)$ is at least k , $o(G - T) \geq k > |T|$ which violates the assumption $o(G - S) \leq |S|$. So, $\{S_1, S_2, \dots, S_h\}$ has an SDR $(v_1, v_2, \dots, v_h)$ where $v_i \in S_i$. Moreover, in G_i , let $u_i \underset{G}{\sim} v_i$.

For showing that G has a 1-factor, it's left to show that for each $i = 1, 2, \dots, h$, $G_i - u_i$ has a 1-factor. Therefore, let W be a proper subset of $V(G_i - u_i)$ and we claim $o(G - u_i - W) \leq |W|$. (This will imply the existence by induction.)

Suppose not. Let $o(G - u_i - W) > |W|$. Again, since $o(G - u_i - W)$ and $|W|$ are of the same parity, we have $o(G - u_i - W) \geq |W| + 2$. Now, combining with S , $o(G - u_i - W - S) = o(G - S) + o(G - u_i - W) - 1$.

$$\begin{aligned}
 o(G - u_i - W - S) &= o(G - S) + o(G - u_i - W) - 1 \\
 &\geq |S| + |W| + 2 - 1 \\
 &= |S| + |W| + 1 \\
 &= |\{u_0\} \cup W \cup S|.
 \end{aligned}$$

Hence, we conclude that $o(G - (\{u_0\} \cup W \cup S)) = |\{u_0\} \cup W \cup S|$. Since $\{u_0\} \cup W \cup S$ is larger than S (in size), this contradicts to the choice of S . As a consequence, we have the fact: $G_i - u_i$ contains a 1-factor and thus G has a 1-factor. ■

Theorem 29 (Petersen)

Every 2-edge-connected (bridge less) cubic graph G has a 1-factor F and $G - F$ is a 2-factor.

Proof.

Let $S \subseteq V(G)$ and consider an odd component in $G - S$. (Notice that if $o(G - S) = 0$, then $o(G - S) \leq |S|$.) Since G is cubic, the number of edges between S and C must be odd. (Otherwise, the degree sum of $V(C)$ in $G - S$ is odd.) By the assumption that G is 2-edge-connected, there are at least three edges in $\langle S, C \rangle$. This implies that the number of edges between S and $G - S$ is at least $3 \cdot o(G - S)$. By the fact that G is cubic, such edges are at most $3 \cdot |S|$. Hence, $3 \cdot o(G - S) \leq 3 \cdot |S|$.

By Tutte's 1-factor theorem, G has a 1-factor F and $G - F$ is clearly a 2-factor. ■

Theorem 30 (Petersen's 2-factor theorem)

Let k be an even integer. Then, a k -regular graph contains $\frac{k}{2}$ edge-disjoint 2-factors.

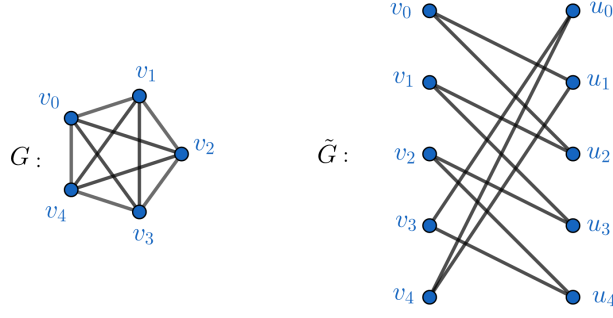
Proof.

It suffices to consider a connected k -regular graph G . Let $k = 2h$. By Euler's circuit theorem, G has an eulerian circuit $Z : ((v_0, v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_t, v_0))$.

Now, we defined a bipartite graph $\tilde{G} = (A, B)$ such that $|A| = |B| = |G|$, (See Figure 26.) and $v_i \sim_{\tilde{G}} v_j$ if v_i, v_j are two consecutive vertices in Z . Since G is $2h$ -regular, $\tilde{G}$ is h -regular. By König's Theorem, $\tilde{G}$ contains h edge-disjoint perfect matchings. It is not difficult to see that a perfect matching in $\tilde{G}$ gives a 2-factor in G . This concludes the proof. ■

(•) Unfortunately, we are not able to control the type of 2-factors we are going to obtain.

(••) A perfect matching in $\tilde{G}$ can be represented as a permutation.



$((v_0, v_1, v_2, v_3, v_4, v_0, v_2, v_4, v_1, v_3))$ is an eulerian circuit of G .

- A graph F (or a class $\mathcal{F}$) is said to be forbidden in a class of graphs $\mathcal{G}$ if for each $G \in \mathcal{G}$, $G \not\supseteq F$ (or $G \not\supseteq F$ for each $F \in \mathcal{F}$).
- $ex(n; F) = \max\{\|G\| \mid G \text{ is a graph of order } n \text{ such that } G \not\supseteq F\}$.
 $ex(n; \mathcal{F})$ can be defined accordingly.
- The graph G of order n with $\|G\| = ex(n; F)$ is called an extremal graph of order n with forbidden graph F .
- The class of bipartite graphs with partite sets of sizes m and n respectively is denoted by $G_2(m, n)$.
- The extremal size of graphs in $G_2(m, n)$ which do not contain $K_{s,t}$ is denoted by $z(m, n; s, t)$. (The notation is in honor of Zarankiewicz.)
- Notice that $ex(n; K_{s,t})$ is different from $z(m, n; s, t)$.
- $z(m, n; s, t) \geq 2ex(n; K_{s,t})$. (?)
- $T_r(n) =_{def} K_{\lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor, \dots, \lfloor \frac{n}{r} \rfloor}$ and $\|T_r(n)\| = t_r(n)$.

Theorem 31 (Turán, 1941)

$ex(n; K_{r+1}) = t_r(n)$ and $T_r(n)$ is the unique extremal graph.

Proof.

(1st)

By induction on n . (To show $ex(n; K_{r+1}) = t_r(n)$.)

Since $T_r(n)$ does not contain K_{r+1} , $ex(n; K_{r+1}) \geq t_r(n)$. We claim $ex(n; K_{r+1}) \leq t_r(n)$. Let G be a graph such that $G \not\supseteq K_{r+1}$ and G is of maximum size. Then, $G \geq K_r$. For otherwise, we may add more edges to G . Let $W \subseteq V(G)$ and $\langle W \rangle_G \cong K_r$. Let $U = V(G) \setminus W$.

Now, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + \|\langle U \rangle_G\|$. The term $(r-1)(n-r)$ comes from the fact that each vertex of U is incident to at most $r-1$ vertices of W . By induction hypothesis, $\|\langle U \rangle_G\| \leq t_r(n-r)$. Hence, $\|G\| \leq \binom{r}{2} + (r-1)(n-r) + t_r(n-r) = t_r(n)$. This is a direct consequence of adding one vertex of W to one partite set of $T_r(n-r)$ and $\left\lfloor \frac{n-r}{r} \right\rfloor + 1 = \left\lfloor \frac{n}{r} \right\rfloor \left(\left\lfloor \frac{n-r}{r} \right\rfloor + 1 = \left\lfloor \frac{n}{r} \right\rfloor \right)$.

Next, we claim the uniqueness. The proof is also by induction on n . Let $y \in V(G)$ such that $\deg_G(y) = \delta(G)$. Clearly, $G - y$ does not contain K_{r+1} and thus $\|G - y\| = \|G\| - \delta(G) \geq t_r(n-1)$ by the proof of the first part. By induction, $T_r(n-1)$ is the unique graph which is isomorphic to $G - y$. This implies that in $G - y$ the smallest partite set is of size $\left\lfloor \frac{n-1}{r} \right\rfloor$. Since $T_r(n-1)$ contains a K_r from r -partite sets, y is incident to at most $r-1$ partite sets of $T_r(n-1)$. Therefore, y can be recognized as a vertex in one of the partite sets, and thus the number of edges between y and $G - y$ is $(n-1) - \left\lfloor \frac{n-1}{r} \right\rfloor = n - \left\lfloor \frac{n}{r} \right\rfloor$. This implies that $G \cong T_r(n)$. ■

(2nd) (Zykov)

Only $ex(n; K_{r+1}) = t_r(n)$.

Let $v_1 \in V(G)$ such that $\deg_G(v_1) = \Delta(G)$ and let $W = N(v_1)$. Let $G_1 = G - \langle N(v_1) \rangle_G + T_{r-1}(\Delta(G))$, and $U_1 = V(G_1) \setminus (W \cup \{v_1\})$. See Figure 26.

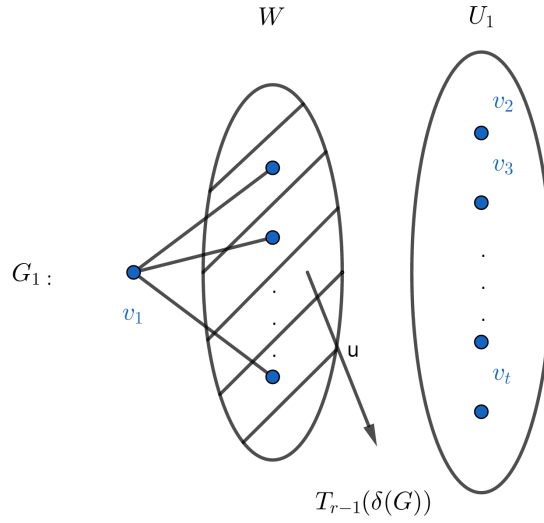


Figure 26. G_1

If U_1 is an empty set, then we stop and evaluate $\|G_1\|$. Otherwise, if $U_1 \neq \emptyset$, let $v_2 \in U_1$. Now, delete all edges with $U_2 = V(G) \setminus (W \cup \{v_1, v_2\})$ in G_1 which are incident to v_2 , E_2 and add v_2u for each $u \in W$ to $G_1 - E_2$. The new graph is defined as G_2 . Since $T_{r-1}(\Delta(G))$ defined on W does not contain K_r , G_2 does not contain K_{r+1} . By continuing this process (until U_t is empty), we shall obtain a complete r -partite graph H such that $\|H\| \geq \dots \geq \|G_2\| \geq \|G_1\| \geq \|G\|$. (Notice that $v_1, v_2, \dots, v_t$ is a new partite set.)

(3rd)

We can replace all the vertices of U_1 at the same time by deleting all the edges incident to U_1 and add $\langle W, U_1 \rangle$ to obtain a complete r -partite graph. ■

Theorem 32 (Erdős, 1970)

Let $G \not\geq K_{r+1}$. Then, there exists an H satisfying (1) H is an r -partite graph, (2) $V(H) = V(G)$, and (3) $\forall x \in V(G)$, $\deg_G(x) \leq \deg_H(x)$. Moreover, if G is not a complete r -partite graph, then there exists a vertex $z \in V(G)$, s.t. $\deg_G(z) < \deg_H(z)$.

Proof.

By induction on r for the whole statement, and $r = 1$ is true. Let the assertion be true for $r' < r$.

Let $x \in V(G)$ s.t. $\deg_G(x) = \Delta(G)$, $N(x) = W$ and $\langle W \rangle_G = G_0$. Clearly, $G_0 \not\geq K_r$. By induction, there exists an $(r-1)$ -partite graph H_0 , s.t., $V(H_0) = W$, $\forall y \in W$, $\deg_{G_0}(y) \leq \deg_{H_0}(y)$, moreover, if G_0 is not a complete $(r-1)$ partite graph, then there exists a $y' \in W$, s.t. $\deg_{G_0}(y') < \deg_{H_0}(y')$.

Now, let $H = H_0 \vee (V \setminus W)$ (join), see Figure 27. So, H is an r -partite graph. For $z \in V \setminus W$, $\deg_G(z) \leq \Delta(G) = |W| = \deg_H(z)$, and if $z \in W$, $\deg_G(z) \leq \deg_{G_0}(z) + n - |W| \leq \deg_{H_0}(z) + n - |W| = \deg_H(z)$. This concludes the first part. For the second part, assume that $\deg_G(x) = \deg_H(x)$ for all $x \in V(G)$. Hence, $\|G\| = \|H\|$ and thus $\|G_0\| = \|H_0\|$. (For otherwise, $\|G\| > \|H\|$.) Moreover, $\deg_{G_0}(x) = \deg_{H_0}(x)$ for each $x \in W$. Suppose not. Let

$\deg_{G_0}(x') < \deg_{H_0}(x')$ for some vertex $x' \in W$. This implies that $\deg_G(x') < \deg_H(x') = \deg_{H_0}(x) + n - |W|$, a contradiction. As a consequence, G_0 (H_0) is a complete $(r-1)$ -partite graph and G is a complete r -partite graph as well. ■

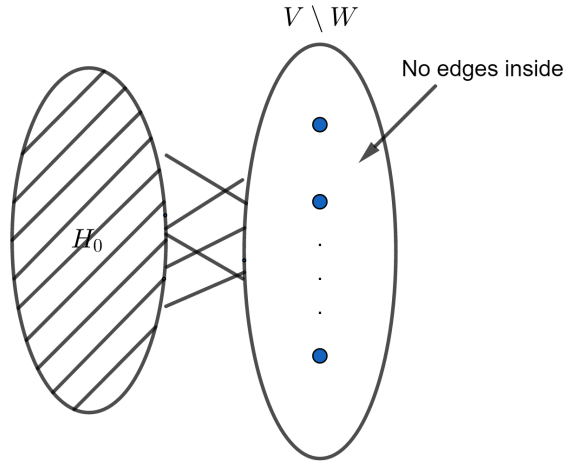


Figure 27.

(••) Try to estimate $z(m, n; s, t)$

Theorem 33 (Important Lemma)

Let $2 \leq s \leq m$, $2 \leq t \leq n$, $0 \leq r \leq m$, $z = km + r$ and $z = my$. $y \in \mathbb{R}^+$

Let G be a bipartite graph, $G \in G_2(m, n)$ and $G \not\supseteq K_{s,t}$. Then,

$$m \cdot \binom{y}{t} \leq (m-r) \binom{k}{t} + r \binom{k+1}{t} \leq (s-1) \binom{n}{t}.$$

(Remark. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is convex if

$$tf(x) + (1-t)f(y) \geq f(xt + y \cdot (1-t)), \quad 0 \leq t \leq 1.)$$

Proof.

Let $G = (A, B)$ where $|A| = m$ and $|B| = n$. Define a graph

$H = \left(A, \binom{B}{t} \right)$. $\binom{B}{t}$ is the collection of all t -subset of B . And $x \sim_G y$ for each $y \in T$. Figure 28 is an example $|A| = 5$, $|B| = 6$ and $t = 3$.

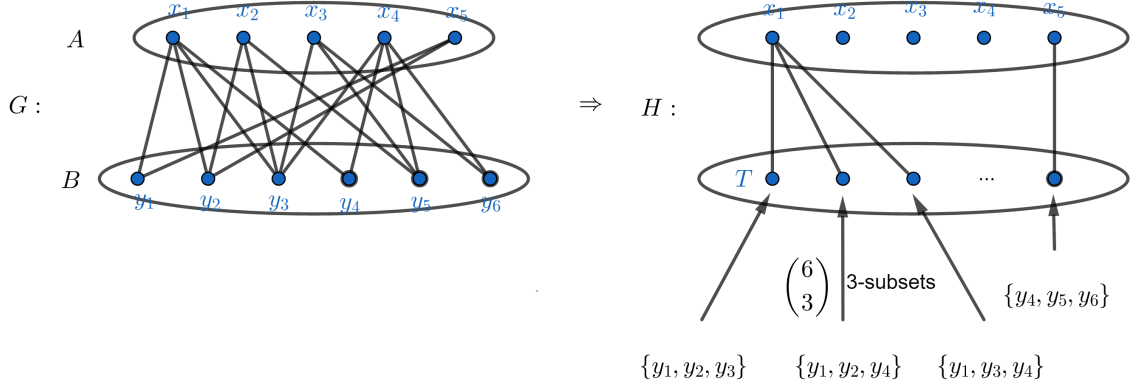


Figure 28. H induced by G

Hence, we have

- (1) $\|H\| = \sum_{x \in A} \binom{\deg_G(x)}{t}$, (For example, in Figure 28, $\|H\| = 10$.)
- (2) $\forall T \in \binom{B}{t}$, $\deg_H(T) \leq s - 1$, ($G \not\prec K_{s,t}$)
- (3) $\|H\| \leq (s - 1) \binom{n}{t}$. (From (2).)

Now, since $z = km + r = my = \sum_{x \in A} \deg_G(x)$, (*)
 $m \binom{y}{t} \leq (m - r) \binom{k}{t} + r \binom{k+1}{t} \leq \sum_{x \in A} \binom{\deg_G(x)}{t} \leq (s - 1) \binom{n}{t}$. ■

(*) comes from the property of combination number. For example,

$$z = 16, k = 3, m = 5, r = 1 \text{ and } t = 2. \text{ Then,}$$

$$5 \cdot \binom{3.2}{2} \leq 4 \binom{3}{2} + \binom{4}{2} \leq \binom{a}{2} + \binom{b}{2} + \dots + \binom{c}{2}.$$

Theorem 34

$$z(m, n; s, t) \leq (s - 1)^{\frac{1}{t}} \cdot (n - t + 1) \cdot m^{1 - \frac{1}{t}} + (t - 1)m.$$

Proof.

By Theorem 33, $m \binom{y}{t} \leq (s-1) \binom{n}{t}$, $\frac{\binom{y}{t}}{\binom{n}{t}} \leq \frac{s-1}{m}$. Hence,
 $\frac{y(y-1)\cdots(y-t+1)}{n(n-1)\cdots(n-t+1)} \leq \frac{s-1}{m}$. By the fact $\frac{y-i}{n-i} \geq \frac{y-t+1}{n-t+1}$ for each
 $0 \leq i \leq t-1$, we have $\left(\frac{y-t+1}{n-t+1}\right)^t \leq \frac{s-1}{m}$, i.e.,
 $(y-t+1)^t \leq (s-1) \cdot (n-t+1)^t \cdot m^{-1}$. This implies that
 $y-t+1 \leq (s-1)^{\frac{1}{t}} \cdot (n-t+1) \cdot m^{-\frac{1}{t}}$ and
 $y \leq (s-1)^{\frac{1}{t}} \cdot (n-t+1) \cdot m^{-\frac{1}{t}} + (t-1)$.
Hence, $z = my \leq (s-1)^{\frac{1}{t}} \cdot (n-t+1) \cdot m^{1-\frac{1}{t}} + (t-1)m$. ■

Theorem 35-1

$$z(m, n; 2, 2) \leq \frac{n}{2} [1 + 4n - 3]^{\frac{1}{2}}.$$

Proof.

By Theorem 33, $n \cdot \binom{y}{2} \leq \binom{n}{2}$. Hence, $n \cdot y(y-1) \leq n(n-1)$ and we
have $y^2 - y - (n-1) \leq 0$. A direct calculation shows that
 $y \leq \frac{1 + \sqrt{4n-3}}{2}$. This implies that $z = m \cdot y = n \cdot y \leq \frac{n(1 + \sqrt{4n-3})}{2}$. ■

Theorem 35-2

If $n = q^2 + q + 1$ and q is a prime power, then
 $z(m, n; 2, 2) = \frac{n}{2} [1 + 4n - 3]^{\frac{1}{2}}$. (Proof. By the existence of a
projective of order q .)

Theorem 36

$$ex(n; K_{s,t}) \leq \frac{1}{2}(s-1)^{\frac{1}{t}} \cdot n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n, \text{ i.e.,}$$

$$ex(n; K_{s,t}) \leq \frac{1}{2}z(n, n; s, t).$$

Proof.**(1st)**

Let G be an extremal graph such that $\|G\| = ex(n; K_{s,t})$. Define a bipartite graph $H = (A, B)$ based on G . Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$ and $a_i \sim_H b_j$ if and only if $v_i \sim_G v_j$, see Figure 29 for an example. Now, clearly, $\|H\| = 2\|G\|$ and $a_i \not\sim_H b_j$ for $i = 1, 2, \dots, n$. Moreover, if $G \not\geq K_{s,t}$, then $H \not\geq K_{s,t}$. (?) This concludes that $\|G\| = \frac{1}{2}\|H\| \leq \frac{1}{2}z(n, n; s, t)$. ■

(2nd) (Two-way counting)

Consider the number of stars $K_{1,t}$. Since $G \not\geq K_{s,t}$, every set of t vertices of $V(G)$ has at most $s-1$ centers of stars whose pendent vertices are these vertices. So, there are at most $(s-1)\binom{n}{t}$ t -stars. The number of stars $K_{1,t}$ can be obtained by $\sum_{i=1}^n \binom{d_i}{t}$ where $d_i = \deg_G(v_i)$. Now, let $m = \|G\|$. $\sum_{i=1}^n d_i = 2m$. By a similar technique as Theorem 34, we conclude the proof ■

$$(*) \quad n \cdot \binom{2m/n}{t} \leq (s-1)\binom{n}{t}.$$

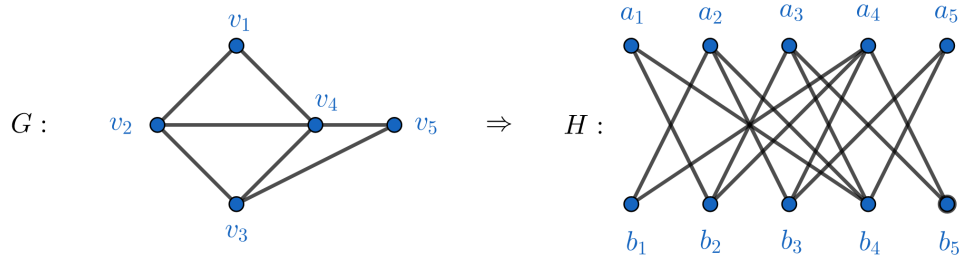


Figure 29. Bipartite version

Ramsey Theory

- (•) The Ramsey number $R(s, t)$ is the smallest value "n" for which either a graph G of order n contains K_s or $\bar{G} \geq K_t$.
- (•) Edge-coloring version of Ramsey number. The Ramsey number $R(s, t)$ is the smallest value "n" for which any 2-edge-colored K_n (red and blue), either there exists a red K_s or a blue K_t . (A red K_s is a complete graph of order s such that all its edges are colored red.)
- (•) $R(3, 3) = 6$ (Do you know this fact?)

Theorem 37

The following statements are true:

- (1) $R(s, 2) = s$ and $R(2, t) = t$,
- (2) $R(s, t) = R(t, s)$,
- (3) For $s > 2$, $t > 2$, $R(s, t) \leq R(s, t-1) + R(s-1, t)$, and
- (4) $R(s, t) \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$.

Proof.

(1) and (2) are easy to see.

Claim of (3)

Let $n = R(s, t - 1) + R(s - 1, t)$. Then, in K_n , each vertex is of degree $R(s, t - 1) + R(s - 1, t) - 1$. Therefore, if K_n is 2-edge-colored by red and blue, then the edges incident to a fixed vertex $x \in V(K_n)$ are either red edges or blue edges. By Pigeon-hole principle, either there are $R(s, t - 1)$ blue edges or $R(s - 1, t)$ red edges. If the first case holds, then in $\langle N_{K_n}(x) \rangle_{K_n}$ (a complete graph of order $R(s, t - 1)$), either there exists a red K_s or blue K_{t-1} . Hence, we have a red K_s or a blue K_t in K_n . The other case can be obtained by a similar argument.

Claim of (4)

By *inductive* argument. (Or induction)

$$\begin{aligned}
 R(s, t) &\leq R(s, t - 1) + R(s - 1, t) \\
 &\leq \binom{s + t - 1 - 2}{s - 1} + \binom{s - 1 + t - 2}{t - 1} \\
 &= \binom{s + t - 3}{s - 1} + \binom{s + t - 3}{s - 2} \\
 &= \binom{s + t - 3 + 1}{s - 1} \\
 &= \binom{s + t - 2}{s - 1}.
 \end{aligned}$$

■

Theorem 38 (Erdős and Szekeres, 1935)

For each $s \geq 2$, $R(s) \leq \frac{2^{2s-2}}{s^{1/2}}$. ($R(s) =_{def} R(s, s)$.)

Proof.

$R(s, s) \leq \binom{2s-2}{s-1}$. We claim $\binom{2s-2}{s-1} \leq \frac{2^{2s-2}}{s^{1/2}}$ by induction on s .

First, if $s = 2$, $2 \leq \frac{4}{\sqrt{2}}$, the assertion is true. Assume that the assertion

is true for $s = k$, thus $\binom{2k-2}{k-1} \leq \frac{2^{2k-2}}{k^{1/2}}$. Now, we calculate

$$\begin{aligned} \binom{2k}{k} &= \frac{(2k)!}{k!k!} = \frac{2k \cdot (2k-1) \cdot (2k-2)!}{k^2(k-1)! \cdot (k-1)!} = \frac{2k(2k-1)}{k^2} \cdot \binom{2k-2}{k-1} \leq \\ &= \frac{4k^2 - 2k}{k^2} \cdot \frac{2^{2k-2}}{k^{1/2}} = \frac{4k-2}{k} \cdot \frac{2^{2k-2}}{k^{1/2}}. \quad (1) \end{aligned}$$

Since $(k+1)^{1/2} \leq \frac{4k \cdot k^{1/2}}{4k-2}$, we conclude that $\binom{2k}{k} \leq \frac{2^{2k}}{(k+1)^{1/2}}$. ■

(•) The result has been there for almost 50 years before the improvement due to Thomason in 1988:

$$R(s) \leq 2^{2s}/s.$$

(••) The original proof by Ramsey shows that

$$R(s) \leq 2^{2s} - 3 = \frac{2^{2s-2}}{2}. \quad (1930)$$

Theorem 39

$$R(k) \geq \lceil 2^{\frac{k}{2}} \rceil. \quad (k \geq 3)$$

Proof. (Probabilistic method)

Consider a random red-blue coloring of the edges of K_n . For a fixed set T of k vertices, let A_T be the event that $\langle T \rangle_{K_n}$ is monochromatic.

Hence $P(A_T) = \left(\frac{1}{2}\right)^{\binom{k}{2}} \cdot 2(\text{red or blue}) = 2^{1-\binom{k}{2}}$. Since there are $\binom{n}{k}$

possible sets for T , the probability that at least one of A_T occurs is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$. Now, if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then no event A_T occurs is of positive probability, i.e., there exists a coloring of edges such that no monochromatic K_k occurs. Therefore, for such n , $R(k) > n$.

Let $n = \lfloor 2^{\frac{k}{2}} \rfloor$. (It suffices to show that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.)
 $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \leq \frac{(2^{\frac{k}{2}})^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \leq \frac{2^{1+\frac{k}{2}}}{k!} < 1 \quad (k \geq 3)$.
 $(1 - \binom{k}{2}) = 1 - \frac{k^2}{2} + \frac{k}{2}$ Hence, $R(k) \geq \lceil 2^{\frac{k}{2}} \rceil$. ■

(•) Combining Theorems obtained above

$$2^{\frac{s}{2}} \leq R(s) \leq 2^{2s-3} \text{ for } s \geq 2.$$

(••) Open problem: $R(s) = 2^{(c+o(1))s}$, (c may be equal to 1).

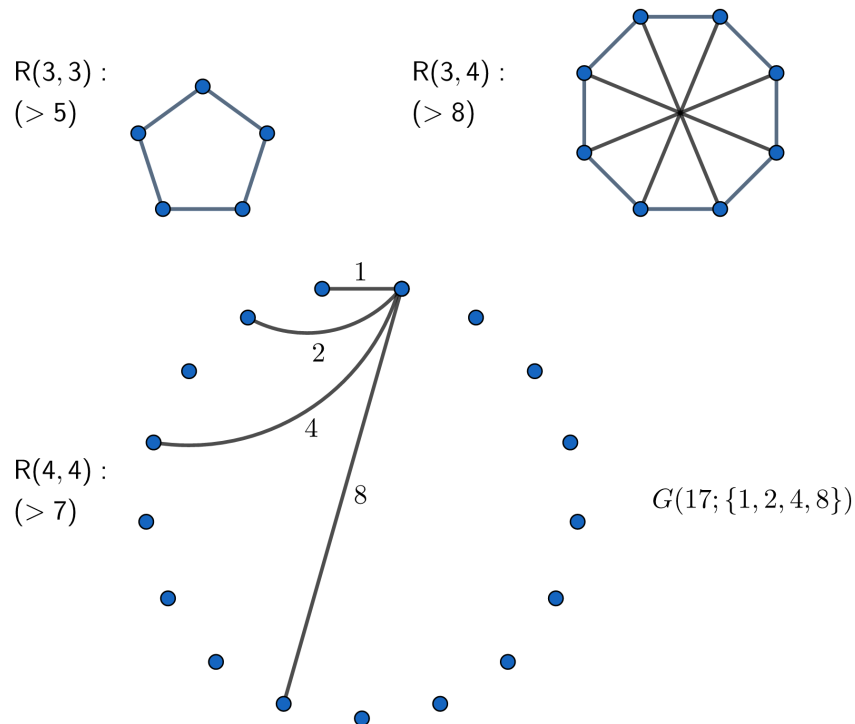
Theorem 40

Known results of $R(s, t)$.

t \ s	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	36-41	49-61	59-84	73-115
5			43-48	58-87	80-143	101-216	133-316
6				102-165	115-298	134-495	183-780
7					205-540	217-1031	252-1713
8						282-1870	329-3583
9							565-6588

$$R(s, t) = R(t, s)$$

- (•) The results of lower bounds are obtained by "a special edge-coloring" with two colors. Corresponding to the coloring we have G and $\bar{G}$ of order (prescribed).



Bonus

Find as many vertices (n) as possible such that a graph G of order n satisfying $G \not\supseteq K_5$ and $\bar{G} \not\supseteq K_5$. (Try 43!)

Exercise C-1

Find a better upper bound for $R(s)$. (Do your best!)

Theorem 41

$$R(p_1, p_2, \dots, p_t) \leq R(p_1 - 1, p_2, \dots, p_t) + R(p_1, p_2, p_3, \dots, p_t) + \dots + R(p_1, p_2 - 1, \dots, p_t - 1) - t + 2.$$

Proof.

By a similar argument as the proof

$$R(s, t) \leq R(s-1, t) + R(s, t-1) - 1. \quad \blacksquare$$

$$(\bullet) \quad R(3, 3, 3) \leq 6 + 6 + 6 - 3 + 2 = 17 \text{ (Theorem 41).}$$

($\bullet\bullet$) There exists a 3-edge-coloring of K_{16} such that no mono-chromatic triangles occur.

Theorem 42

$$R(3, 3, \dots, 3) =_{def} R_k(3) \leq \lfloor e \cdot k! \rfloor + 1.$$

k -tuples

Proof.

Since $R(3, 3) = 6$, $R(3, 3, 3) = 17$, the assertion is true for $k = 2$ and

3. Assume that it holds for $k-1$ when $k > 3$. Hence,

$$R_{k-1}(3) \leq \lfloor e(k-1)! \rfloor + 1. \text{ By Theorem 41,}$$

$$\begin{aligned} R_{k-1}(3) &\leq k(\lfloor e(k-1)! \rfloor + 1) - k + 2 \\ &= k\lfloor e(k-1)! \rfloor + 2. \end{aligned}$$

$$\begin{aligned} \text{Now, } k\lfloor e(k-1)! \rfloor &= k \left[(k-1)! \cdot \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(k-1)!} + \frac{1}{k!} + \dots \right) \right] = \\ &= k \left[M + \frac{1}{k} + \frac{1}{(k+1)k} + \frac{1}{(k+2) \cdot (k+1)k} + \dots \right] = \dots = \lfloor (k!) \cdot e \rfloor - 1. \quad (?) \quad \blacksquare \end{aligned}$$

($\bullet$) Instead of $R(s, t)$, we use $R(H_1, H_2)$ to denote the smallest integer n such that any 2-edge-coloring (red, blue) of K_n , either there exists a red H_1 or a blue H_2 .

Theorem 43

$$R(C_4, C_4) =_{\text{def}} R(C_4) \leq 8.$$

Proof.

Consider a graph $G \not\geq C_4$ and $|G| = 8$. By Theorem 35-1, $\|G\| \leq \frac{1}{2}z(8, 8; 2, 2) \leq \frac{n \cdot (1 + \sqrt{4n - 3})}{4} = 2(1 + \sqrt{29}) < 14$. That is, if a graph G is of order 8 and size 14, then $G \geq C_4$. Now, in a 2-edge-colored K_8 , either red or blue edges induce such a graph, the proof follows. ■

Theorem 44

$$R_k(C_4) \leq k^2 + k + 2.$$

Proof.

Let $n = k^2 + k + 2$ and consider a k -edge-colored K_n . Since $ex(n; C_4) \leq \frac{n}{4}(1 + \sqrt{4n - 3}) = \frac{k^2 + k + 2}{4} \cdot (1 + \sqrt{4k^2 + 4k + 5}) \leq \frac{k^2 + k + 2}{2} \cdot \frac{1 + \sqrt{4k^2 + 4k + 5}}{2}$. Now, compare $(k^2 + k + 2)$ and $\frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2}$. By direct calculation, we have $(k^2 + k + 2) > \frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2}$. This implies that $k \cdot ex(n; C_4) \leq \frac{k^2 + k + 2}{2} \cdot \frac{k(1 + \sqrt{4k^2 + 4k + 5})}{2} < \frac{(k^2 + k + 2)(k^2 + k + 1)}{2} = \binom{n}{2}$. ■

$$(\bullet) R(H_1, H_2) = \min\{n \mid |G| = n, G \geq H_1 \text{ or } \bar{G} \geq H_2\}.$$

Theorem 45

For $l \geq 1$ and $p \geq 2$, $R(lK_2, K_p) = 2l + p - 2$.

Proof.

Let O_k (stable set) denote the graph of order k which contains no edges. Let H be a graph of order $2l + p - 3$ such that $H = O_{p-2} \cup K_{2l-1}$ (see Figure 30).

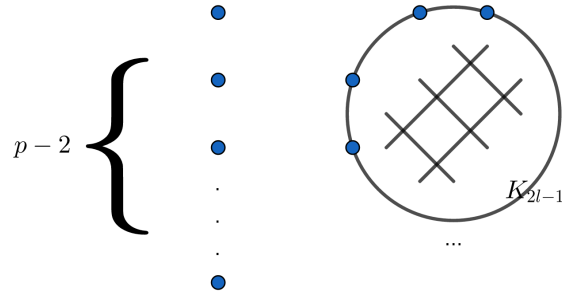


Figure 30.

So, it is not difficult to see that H does not contain a matching of size l . As a consequence, $\bar{H} = O_{2l-1} \vee K_{p-2}$ (join) does not contain a K_p . Hence, $R(lK_2, K_p) \geq 2l + p - 2$.

On the other direction, let G be a graph of order n which does not contain a matching of size l . Let $s < l$ be the maximum number of independent edges in G .

Now, consider $\bar{G}$. Since in G , the set of vertices not incident to these s edges induces a graph of order $n - s$ which has no edges, $\bar{G}$ contains a complete graph of order $n - 2s \geq n - 2(l - 1) = p$. This concludes the proof. ■

- (•) Topological Graph Theory studies the "drawing" of a graph on a surface.

(••) A proper drawing on a surface of a graph G with p vertices and q edges follows the rules:

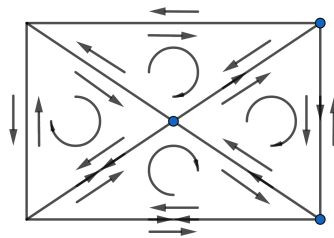
- (1) There are p points on the surface which corresponds to the set of vertices in G ; and
- (2) There are q curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

2-manifold: A connected topological space in which every point has a neighborhood homomorphic to the open unit disk defined on $\mathbb{R}^2$.

Bound subspace: A subspace M of $\mathbb{R}^3$ is bounded if $\exists K \in \mathbb{R}^+$ such that $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 = K\}$.

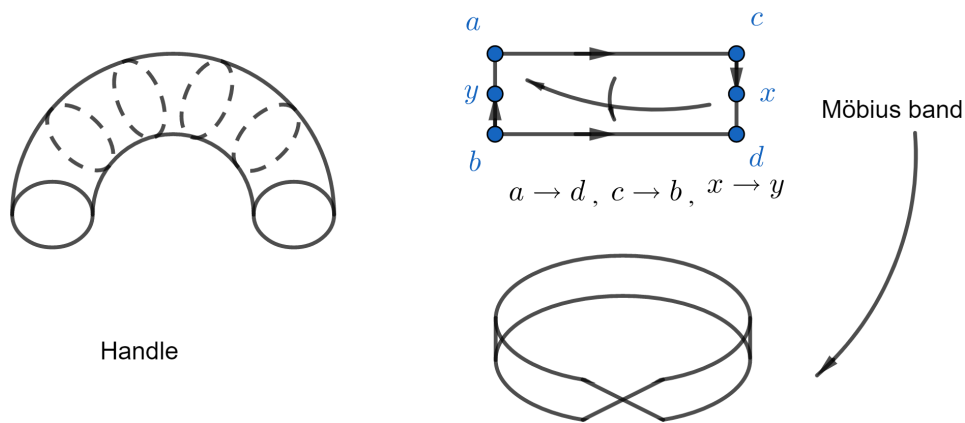
Closed: M is closed if its boundary ∂M coincides with M .

Orientable: M is orientable if for every simple closed curve C on M , a clockwise sense of rotation is preserved once around C . Otherwise, M is non-orientable.

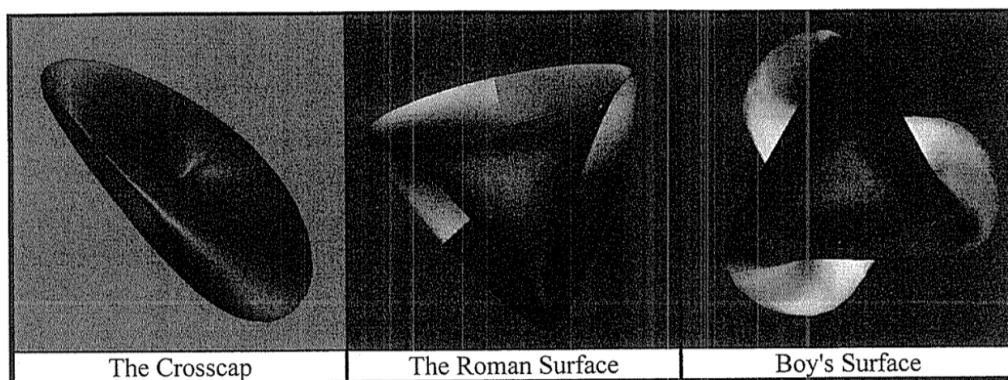


Orientable Surface: A surface S_k is a compact "orientable" 2-manifold that may be thought of as a sphere on which has been placed (inserted) a number k of "handles" (holes).

non-orientable Surface: A surface obtained by adding k cross-caps to a sphere (S_0) is a non-orientable surface N_k .



- (•) Adding a crosscap: Attach the boundary of a Möbius band to a cycle on S_0 .



Definition (Embeddable)

A (p, q) -graph G is said to be embeddable on a surface if it is possible to draw G *properly* (drawing without crossings) on the surface.

Definition (Planar graph)

A graph is planar if it can be embedded in the plane, equivalently, embedded on the sphere.

Definition (2-cell embedding)

A region is called a 2-cell if any simple closed curve in that region can be continuously deformed or contracted in that region to a single point, equivalently, a 2-cell is topologically homeomorphic to $\mathbb{R}^2$. An embedding of G on a surface is a 2-cell embedding of G if all the regions so determined are 2-cells.

The following figure shows embeddings of $K_{3,3}$ on S_1 and S_2 respectively. S_k is a surface obtained by attaching k handles.

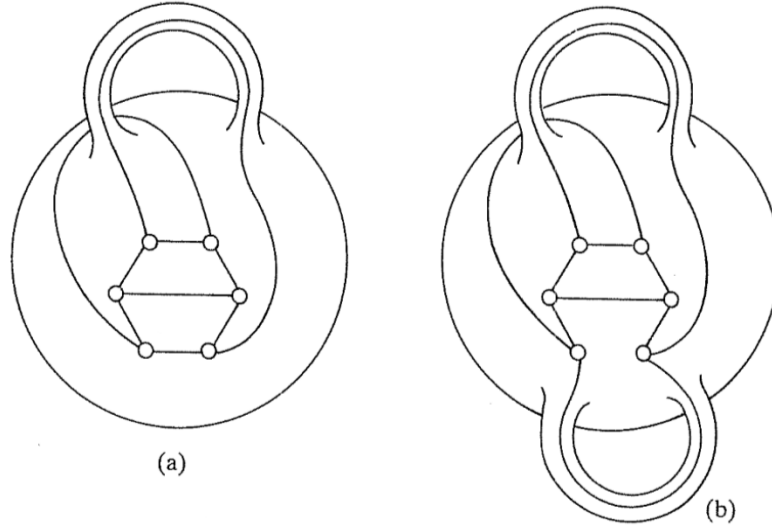


Figure 31. Embeddings of $K_{3,3}$ on surfaces S_1, S_2

- (•) S_0 : Sphere
- (•) $N_0 \simeq S_0$ (Homeomorphic), N_h : Attach h crosscaps to $N_0(S_0)$.

Definition (Genus)

The number of handles (resp. crosscaps) (on a surface) is referred to as genus of the orientable surface (resp. non-orientable surface). We use $\gamma(G)$ (respectively $\tilde{r}(G)$) to denote the smallest genus of all

orientable surfaces (resp. non-orientable surfaces) on which G can be embedded.

(*) If G is a planar graph, then $\gamma(G)$ (so is $\tilde{r}(G)$) is equal to zero.

But, G can also be embedded on a surface with genus larger than "0".

(**) Given a graph G , determining $\gamma(G)$ is a difficult problem.

Theorem 46 (Euler)

Let G be a connected planar graph with p vertices, q edges and f faces (regions). Then $p - q + f = 2$.

Proof.

By induction on q . Since G is connected, G has at least $p - 1$ edges.
 (?) If G has $p - 1$ edges and G is connected, then G is a tree which contains no cycles. This implies that $f = 1$ and thus $p - (p - 1) + 1 = 2$.
 The assertion is true for "minimal" graphs. Let the hypothesis is true for $k = \|G\| \geq p - 1$. Now, consider G with $k + 1$ edges. Clearly, G contains a cycle. Let e be a cycle edge. Since G is a connected planar graph (with q faces), $G - e$ is also a connected planar graph. Moreover, $\|G - e\| = k$ and $G - e$ has k edges and $q - 1$ faces. By induction $p - k + q - 1 = 2$ and thus $p - (k + 1) + q = 2$. This concludes the proof. ■

Theorem 47

If G is a planar graph with largest size, then $\|G\| = 3|G| - 6$.

Proof.

By observation, if G has maximum size, then each region of G is a triangle. Since each edge of G is in the boundary of exact two regions, $3 \cdot f = 2 \cdot g$ where f is the number of regions and g is the size of G , i.e., $q = \|G\|$. Now, by Euler's formula $p - q + f = 2$ equivalently $|G| - \|G\| + \frac{2}{3}\|G\| = 2 \Rightarrow 3 \cdot |G| - 6 = \|G\|$. (G is a *maximal planar graph!*) ■

Corollary

If G is a planar graph, then $\|G\| \leq 3 \cdot |G| - 6$.

Corollary

In any planar graph, there exists at least one vertex of degree smaller than 6.

This corollary is very useful.

Corollary

The degree sum of a planar graph is at most $6 \cdot |G| - 12$.

(•) We can give a more accurate estimation of the above corollary

Theorem 48

Let G be a maximal planar graph (triangulated) of order p , and let

p_i denote the number of vertices of degree i in G for $i = 3, 4, \dots, \Delta(G) = d$. Then

$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \dots + (d-6)p_d + 12.$$

Proof.

Since $p = \sum_{i=3}^d p_i$ and $2q = \sum_{i=3}^d i \cdot p_i$, we have

$$\sum_{i=3}^d i \cdot p_i = 2(3p - 6) = 6 \cdot \sum_{i=3}^d p_i - 12.$$

This implies the conclusion. ■

Theorem 49

There are exactly five regular polyhydra.

Proof.

Notice that a regular polyhydron is a polyhydron whose faces (regions) are bounded by congruent (全等) regular polygons and whose polyhydral angles are congruent (相等).

First, we convert a polyhydron into a regular planar graph. (See Figure 32.) Let the number of vertices, edges and faces be p, q and f respectively.

By Theorem 46, $p - q + f = 2$. Hence,

$$\begin{aligned} -8 &= 4q - 4p - 4f \\ &= 2q + 2q - 4p - 4f \\ &= \sum_{i \geq 3} i f_i + \sum_{i \geq 3} i p_i - 4 \sum_{i \geq 3} p_i - 4 \sum_{i \geq 3} f_i \quad (f_i: \# \text{ of } i\text{-face}) \\ &= \sum_{i \geq 3} (i-4) f_i + \sum_{i \geq 3} (i-4) p_i. \end{aligned}$$

Since the polyhedron is regular, all degrees and face sizes are the same, let them be k and h respectively. Therefore,

$$-8 = (h - 4)f_h + (k - 4)p_k.$$

By the fact that every planar graph contains a vertex of degree less than six, we only have nine cases to consider: $3 \leq h \leq 5$ and $3 \leq k \leq 5$.

From direct checking, only 5 cases are possible, namely,

- (1) $f_3 = p_3 = 4$ (Tetrahedron) 四面體
- (2) $f_3 = 8$ and $p_4 = 6$ (Octahedron) 八面體
- (3) $f_3 = 20$ and $p_5 = 12$ (Icosahedron) 二十面體
- (4) $f_4 = 6$ and $p_3 = 8$ (Cube) 六面體
- (5) $f_5 = 12$ and $p_3 = 20$ (Dodecahedron) 十二面體. ■

See Figure 32 for regular polyhedra.

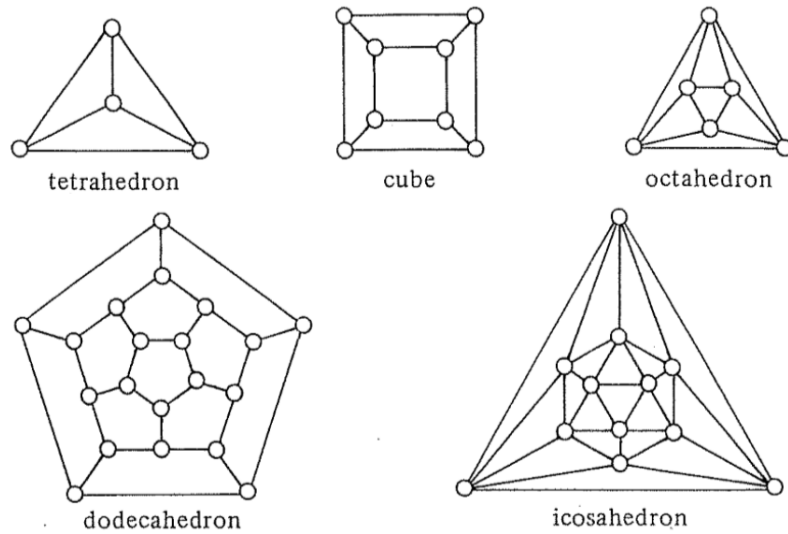


Figure 32. The graphs of the regular polyhedra

Theorem 50 (Fáry (1948), Wagner(1936))

A planar graph G can be embedded in the plane so that each edge is a straight line segment.

Proof.

The proof is by induction on the order of G . It suffices to prove the case when G is a connected maximal planar graph. Clearly, it is true for small orders. Assume the hypothesis is true for order k and let G be a connected maximal planar graph of order $k + 1$.

Since G is maximal, $3 \leq \delta(G) \leq 5$.

Case 1. $\delta(G) = 3$

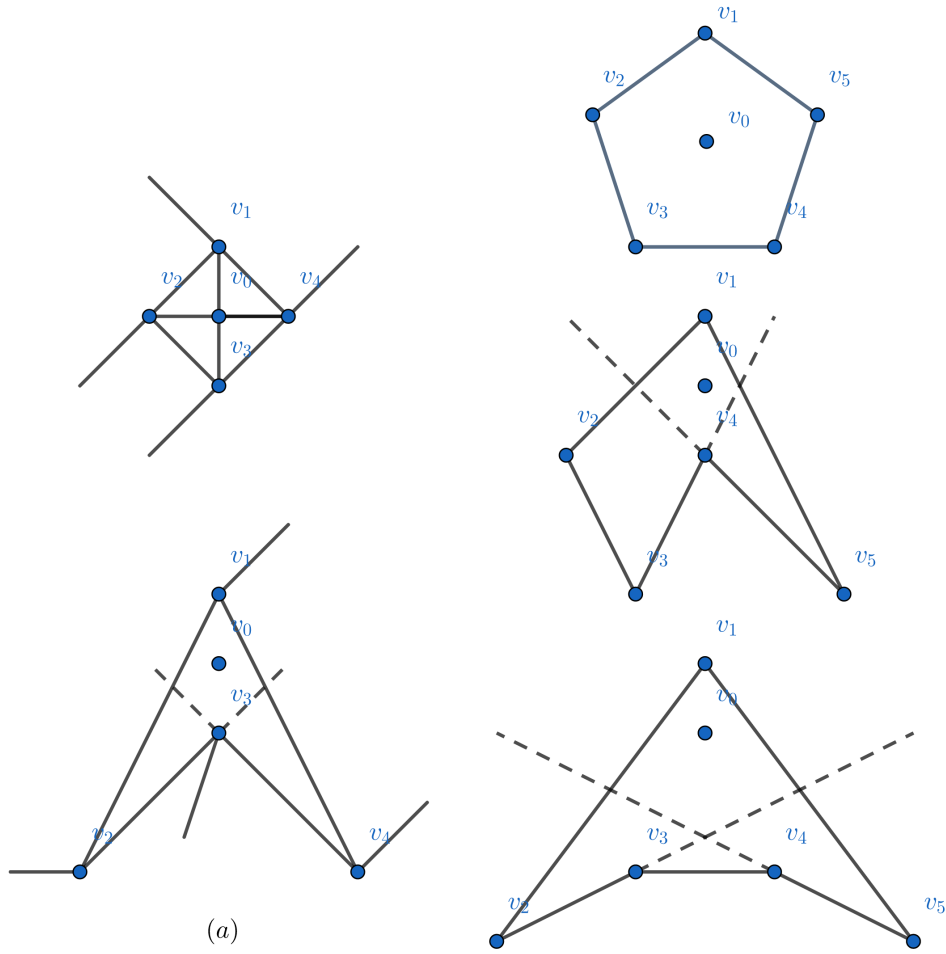
Let $v_0 \in V(G)$ such that $\deg_G(v_0) = 3$ and v is adjacent to v_1, v_2 and v_3 . Since G is maximal, $\langle \{v_1, v_2, v_3\} \rangle_G \cong K_3$. This implies that $G - v_0$ is also a maximal planar graph. By induction $G - v_0$ has a straight line segment embedding. Now, put v_0 back to the graph $G - v_0$ such that v_0 is inside the region bounded by $\langle \{v_1, v_2, v_3\} \rangle$ and connect v_0 to the three vertices by straight line segment. This concludes the proof of this case.

Case 2. $\delta(G) = 4$

The proof follows by a similar process by letting $N(v_0) = \{v_1, v_2, v_3, v_4\}$. Now, $G - v_0 + v_1v_3$ is a maximal planar graph and thus it has a straight line segment embedding. The proof follows by placing v_0 back to $G - v_0 + v_1v_3 - v_1v_3$. By considering the drawing of the embedding (Figure 33(a)), we are able to put v_0 back and connect v_0 to its neighbors in G by straight line segment.

Case 3. $\delta(G) = 5$

Again, we use the same technique and the drawing can be seen in Figure 33(b). ■

Figure 33. Location of v_0

- (•) The following Theorem consider pseudographs, i.e., loops and multiedges are allowed.

Theorem 51 (Euler-Poincaré Theorem)

Let G be a (p, q) -pseudograph which has a 2-cell embedding on S_n . Then, $p - q + f = 2 - 2n$ where f is the number of faces in the embedding.

Proof.

By induction on n and it's true when $n = 0$ (by Euler's planar graph formula). Assume that the assertion is true when $n = k \geq 0$ and G is a (p, q) -pseudograph which has a 2-cell embedding on S_{k+1} . Since $k + 1 \geq 1$, there exists a handle in the embedding, see Figure 34(a). It suffices to consider the embedding such that there exists at least one edge which passes through the handle (on the surface). Note that if we can pull back an edge without passing the handle, then pull it back, see Figure 34(b). Now, we apply the idea of "cut and past" to obtain a 2-cell embedding of $\tilde{G}$ on S_k .

By using a circle around the handle, we can cut the handle through the circle and obtain $\tilde{G}$, see Figure 34(c). As a consequence the graph $\tilde{G}$ is embedded in S_k . If there are t edges passing through the handle then $|\tilde{G}| = p + 2t$, $\|\tilde{G}\| = q + 3t$ and the embedding in S_k has $f + t + 2$ faces. Hence,

$$(p + 2t) - (q + 3t) + (f + t + 2) = 2 - 2k.$$

This implies that $p - q + f = 2 - 2(k + 1)$. ■

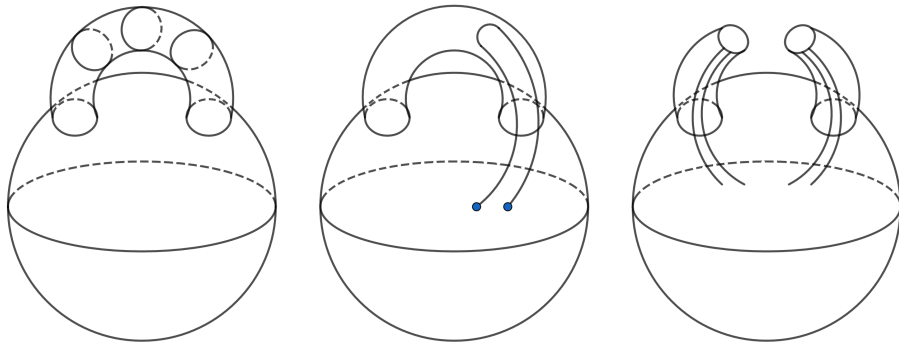
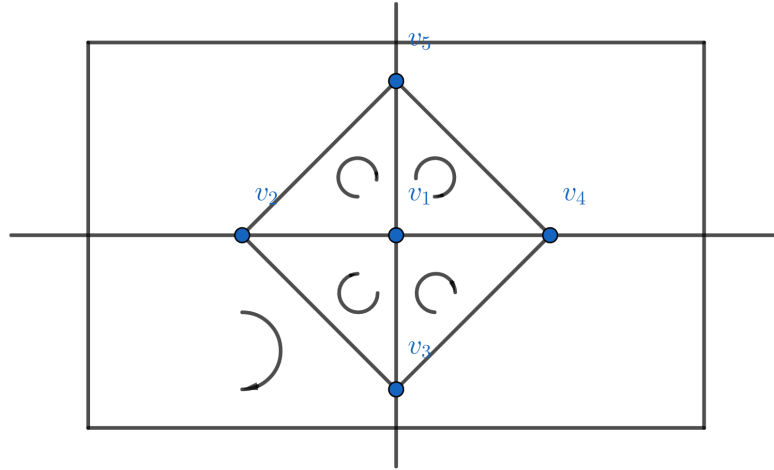


Figure 34(a) Figure 34(b), pull edges back Figure 34(c)

Exercise

Give a more precise proof.

(•) How to find a 2-cell embedding on S_k ?



The above figure provide an embedding of K_5 on S_1 , their regions are $((v_1, v_2, v_5))$, $((v_1, v_3, v_2))$, $((v_1, v_4, v_3))$, $((v_1, v_5, v_4))$, and $((v_2, v_3, v_5, v_2, v_4, v_5, v_3, v_4))$.

(•) Each are of D_5 occurs exactly once in a region (face).

(••) By considering each vertex, we observe that there are five permutations to determine this embedding.

$$\pi_1 = (2, 3, 4, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix} \text{ (Defined on } v_1 \text{.)}$$

$$\pi_2 = (3, 1, 5, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{pmatrix} \text{ (Defined on } v_2 \text{.)}$$

$$\pi_3 = (4, 1, 2, 5)$$

$$\pi_4 = (3, 2, 5, 1)$$

$$\pi_5 = (1, 4, 3, 2)$$

On the other hand, if we are given five permutations one from each vertex, then we have a 2-cell embedding. For example, $\pi_1 = (3, 2, 4, 5)$, $\pi_2 = (3, 1, 5, 4)$, $\pi_3 = (4, 1, 2, 5)$, $\pi_4 = (3, 2, 5, 1)$, and $\pi_5 = (1, 4, 3, 2)$.

Now, if we start from the arc (v_1, v_2) , then the permutation of π_2 will give the next vertex of the oriented 2-cell containing $(1, 2)$.

$$v_1 - v_2 - v_{\pi_2(1)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_{\pi_1(5)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_3 \cdots \Rightarrow \cdots \Rightarrow ((v_1, v_2, v_5, v_1, v_3, v_2, v_1, v_4, v_3)). \quad (v_{\pi_3(4)} = v_1 \text{ and } v_{\pi_1(3)} = v_2)$$

(*) Therefore, given a (p, q) -graph G , we can define p permutations for p vertices v_i such that each permutation is a cycle using the indices of vertices in $N_G(v_i)$. Then, a 2-cell embedding will be obtained.

Theorem 52 (Heffter's Rotational Embedding Scheme)

Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \dots, v_p\}$. For each 2-cell embedding of G on a surface, there exists a unique p -tuple $(\pi_1, \pi_2, \dots, \pi_p)$ where for $i = 1, 2, \dots, p$, π_i is a cyclic permutation of $V(i)$ that describes the subscripts of the vertices in $N_G(v_i)$ in counterclockwise order about v_i . Conversely, for each p -tuple $(\pi_1, \pi_2, \dots, \pi_p)$, there exists a 2-cell embedding of G on some surface such that for $i = 1, 2, \dots, p$, the subscripts of the vertices adjacent to v_i are in counterclockwise order about v_i given by π_i . Moreover, the set $\{\pi_1, \pi_2, \dots, \pi_p\}$ induces a mapping π such that $\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$ for each pair of adjacent vertices v_i and v_j , $1 \leq i \neq j \leq p$.

Proof.

The scheme was first observed and used by Dyck (1888) and Heffter (1891). A formalized version was obtained later in 1960 by Edmonds.

The main idea has been mentioned before the statement of Theorem 52. The genus of surface can be obtained after the number of face (regions) has been determined. ■

Theorem 53

$$\gamma(K_{2m,2n}) = (m-1)(n-1), \quad m \leq n.$$

Proof.

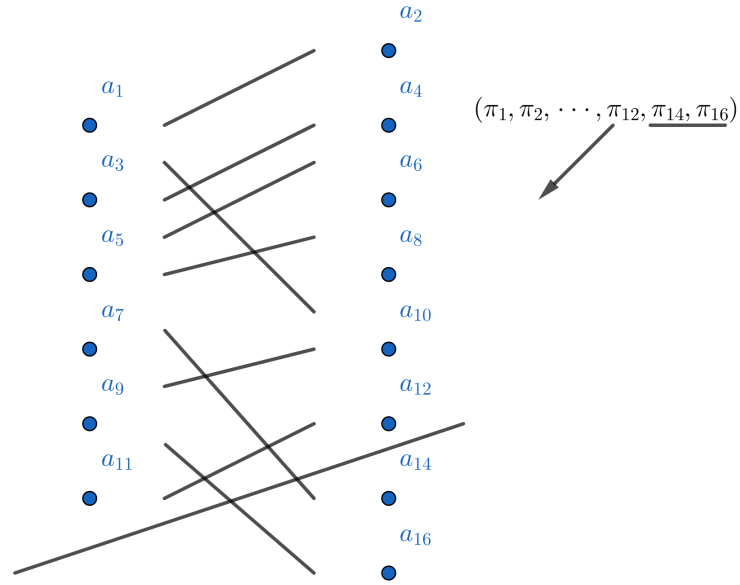
For convenience, let $K_{2m,2n} = (A_o, A_e)$ where $A_o = \{a_1, a_3, \dots, a_{4m-1}\}$ and $A_e = \{a_2, a_4, \dots, a_{4n}\}$. Note that m may not be equal to n . See Figure 35 for the case $2m = 6$ and $2n = 8$. Now, let

$$\left\{ \begin{array}{l} \pi_1 = \pi_5 = \dots = \pi_{4m-3} = (2 \ 4 \ 6 \ \dots \ 4n); \\ \pi_3 = \pi_7 = \dots = \pi_{4m-1} = (4n \ 4n-2 \ \dots \ 6 \ 4 \ 2); \\ \pi_2 = \pi_6 = \dots = \pi_{4n-2} = (1 \ 3 \ 5 \ \dots \ 4m-1); \text{ and} \\ \pi_4 = \pi_8 = \dots = \pi_{4n} = (4m-1 \ 4m-3 \ \dots \ 5 \ 3 \ 1). \end{array} \right.$$

We may check that this embedding does have $2mn$ regions each of them is bounded by a 4-cycle. Hence, by Theorem 51,

$$2\gamma(G) = 2 - p + q - r = 2 - (2m + 2n) + 4mn - 2mn = 2mn - 2m - 2n + 2.$$

Hence, $\gamma(G) = (m-1)(n-1)$. ■

Figure 35. $K_{6,8}$

- (••) They are $\prod_{i=1}^p (\deg_G(v_i) - 1)!$ p -tuples of $(\pi_1, \pi_2, \dots, \pi_p)$.
- (•••) The 2-cell embedding with the largest number of faces gives the genus of G .
- (*) The 2-cell embedding with the smallest number of faces gives the "maximum" genus of G , denoted by $\gamma_M(G)$.
- (**) Finding $\gamma(G)$ is a very difficult problem in general.
- (**) Finding $\gamma_M(G)$ is comparatively easier.

Theorem 54

Let $cr(G)$ denote the crossing number of G . Then,
 $cr(K_5) = cr(K_{3,3}) = 1$ and $cr(K_6) = 3$.

Proof.

Since $\gamma(K_5) = \gamma(K_{3,3}) = 1$, the proof follows by a drawing with "1" crossing number. Now, we consider K_6 . By Figure 35, $cr(K_6) \leq 3$. Let $cr(K_6) = k$. Then, we may convert the crossings into new vertices. Hence, we have a planar graph G (obtained from above): $|G| = 6 + k$ and $\|G\| = 15 + 2k$. By the fact $15 + 2k \leq 3(6 + k) - 6$, we have $k \geq 3$. ■

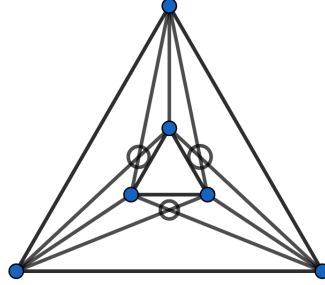


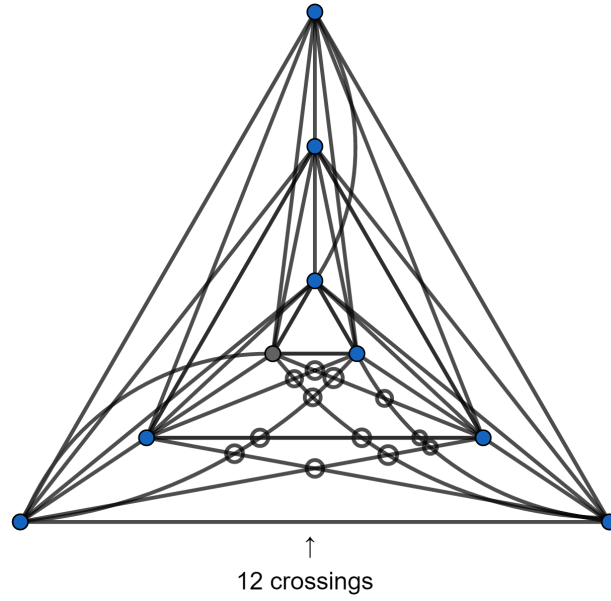
Figure 35.

Theorem 55

$$cr(K_9) = 36.$$

Proof.

For the upper bound, it suffices to give a drawing which has exactly 36 crossings. But, it is very technical to show the lower bound. Here, we provide a drawing for $cr(K_9) \leq 36$, see Figure 36.

Figure 36. $cr(K_9)$

Conjecture

$$cr(K_p) = \frac{1}{4} \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{p-3}{2} \right\rfloor. \quad (\text{True for } 1 \leq p \leq 10.)$$

Vertex-Coloring

- (•) k -coloring (proper): $\varphi : V(G) \rightarrow \{1, 2, 3, \dots, k\}$ s.t.
 $uv \in E(G) \Rightarrow \varphi(u) \neq \varphi(v).$
- (•) $\chi(G) = \min\{k \mid G \text{ has a } k\text{-coloring}\}$ (Chromatic number of G)
- (•) G is n -critical (chromatically) if $\chi(G - v) < \chi(G)$ for each
 $v \in V(G).$
- (••) Every graph G has an n -critical induced subgraph H .

Theorem 56

Every critically n -chromatic (n -critical) graph, $n \geq 2$, is $(n - 1)$ -edge-connected.

Proof.

First, if $n = 2$, then $G \cong K_2$ and thus G is 1-edge-connected. If $n = 3$, then $G \cong C_{2m+1}$, $m \geq 1$, (?) and G is 2-edge-connected. Let $n \geq 4$ and assume that G is not $(n - 1)$ -edge-connected. Hence, $V(G) = V_1 \cup V_2$ such that $|\langle V_1, V_2 \rangle| < n - 1$ ($\leq n - 2$). Let $G_1 = \langle V_1 \rangle_G$ and $G_2 = \langle V_2 \rangle_G$. Now, both of them are $n - 1$ colorable since G is $(n - 1)$ -critical. (See Figure 37.) Let the colorings be ϕ_1 and ϕ_2 resp.

Now, consider the vertices incident to edges in $\langle V_1, V_2 \rangle$. If for each edge $uv \in \langle V_1, V_2 \rangle$, $\phi_1(u) \neq \phi_2(v)$, then G has an $(n - 1)$ -coloring, a contradiction. Thus, assume that for some edges $uv \in \langle V_1, V_2 \rangle$, $\phi_1(u) = \phi_2(v)$. (We shall permute the colors of G_1 (ϕ'_1) in order that for each $uv \in \langle V_1, V_2 \rangle$, $\phi_1(u) \neq \phi_2(v)$).

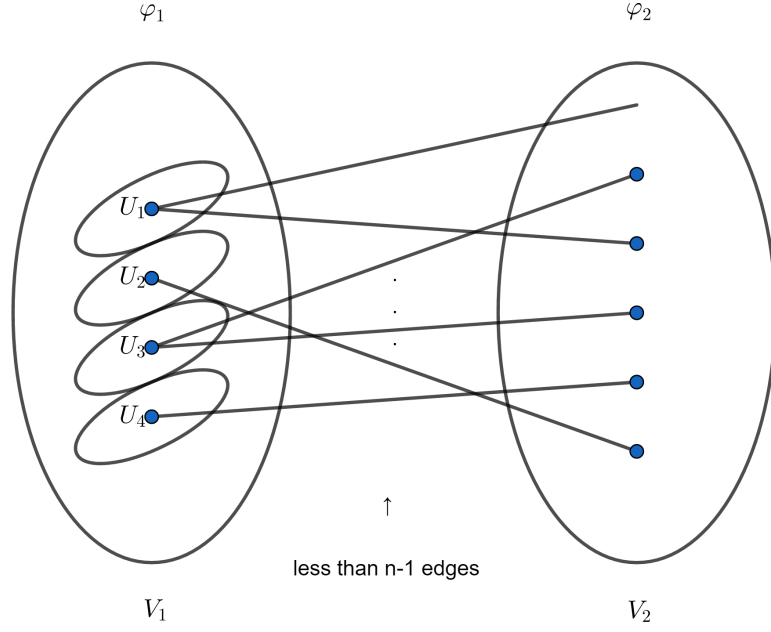


Figure 37.

Let $U_1, U_2, \dots, U_m$ be the subsets of V_1 such that $\varphi_1^{-1}(i) = U_i$, $i = 1, 2, \dots, m$ ($\leq n - 2$) and there is at least one edge joining U_i and $V(G_2)$ for each i . Furthermore, let n_i be the number of vertices in U_i which are incident to a vertices of $V(G_2)$. Hence, $\sum_{i=1}^m n_i \leq n - 2$.

Now, we start a process to recolor the vertices in V_1 . Starting U_1 . If $\forall x \in U_1$, $\varphi_1(x)$ has distinct colors with the colors of these vertices in V_2 which are incident to U_1 , then we go to consider U_2 . Otherwise, $\varphi_1(x) = \varphi_2(y)$ for some $x \in U_1$, and $xy \in \langle V_1, V_2 \rangle$. In this case, we permute the colors of $U_1, U_2, \dots, U_m$ (It was $1, 2, \dots, m$) such that the color used for the vertices in U_1 is distinct from the colors of vertices in V_2 which are incident to U_1 , there are n_1 of them. Since $\sum_{i=1}^m n_i \leq n - 2$, $n - 1 - n_1 > 0$ and thus there exists a color for U_1 .

Following this idea we consider U_2 . If there are vertices x in U_2 such that $\varphi_1(x) = \varphi_2(y)$ for some $xy \in \langle V_1, V_2 \rangle$ then permute the colors used

in $U_2, U_3, \dots, U_m$ where the color for U_1 is fixed. Again, since $n - 2 - n_2 \geq (n - 1) - n_1 - n_2 > 0$, a color for U_2 is available. Continuing this process, we end it up with an $(n - 1)$ -coloring of G , a contradiction to $\chi(G) = n$. ■

(•) If G is n -critical, then $\delta(G) \geq n - 1$.

Question

Is this graph $(n - 1)$ -connected?

Theorem 57

Let $k = \max_{H \leq G} \delta(H)$. Then, $\chi(G) \leq k + 1$.

Proof.

(1st)

Let $\chi(G) = n$ and H' be an n -critical induced subgraph of G . Then, $\delta(H') \geq n - 1$. Since $\max_{H \leq G} \delta(H) \geq \delta(H') \geq n - 1$, $k \geq n - 1$ and thus $\chi(G) = n \leq k + 1$. ■

(2nd) (G is a (p, q) -graph.)

Since $k = \max_{H \leq G} \delta(H)$, $\delta(G) \geq k$. Let $x_p \in V(G)$ and $\deg_G(x_p) = \delta(G) \geq k$. Moreover, let $G_{p-1} = G - x_p$. Again, G_{p-1} has a vertex of degree at most k . So, we obtain a sequence of induced subgraphs, $G = G_p \geq G_{p-1} \geq \dots \geq G_1$ such that $\delta(G_i) \leq k$ for $i = p, p - 1, \dots, 1$ such that $x_i \in V(G_i)$. Hence, we obtain a sequence $\langle x_1, x_2, \dots, x_p \rangle$ such that x_{j+1} is incident to at most k vertices in $\langle \{x_1, x_2, \dots, x_j\} \rangle_G$. This implies that we can use greedy algorithm to

color G starting from x_1 , and then $x_2, \dots, x_p$. All we need is at most $k + 1$ colors. Hence, $\chi(G) \leq k + 1$. ■

Theorem 58 (Brooks, 1941)

Let G be a connected graph which is neither a complete graph nor an odd cycle. Then, $\chi(G) \leq \Delta(G)$.

Proof.

By induction on $|G| = p$. We may assume that the graph we consider is 2-connected and Δ -regular where $\Delta \geq 3$. (?) (Note that a 2-regular connected graph G with $\chi = 3$ is an odd cycle.)

First, if G is 3-connected, let x_p be any vertex such that $\langle N_G(x_p) \rangle_G$ is not a complete subgraph of G . Such an x_p does exist since G is not a complete graph. Let x_1 and x_2 be two vertices in $N_G(x_p)$ such that $x_1 \not\sim_G x_2$. Now, we may construct a sequence corresponding to $V(G)$. Choose $x_{p-1} \in N_G(x_n) \setminus \{x_1, x_2\}$ (3-connected). Then, x_{p-2} is adjacent to either x_n or x_{n-1} . As a consequence, we have a sequence $\langle x_1, x_2, \dots, x_p \rangle$ such that " x_i is incident to at least one vertex in $\{x_{i+1}, x_{i+2}, \dots, x_p\}$ ". Now, we use the greedy algorithm to color all the vertices starting "from x_1 and x_2 with a common color".

Second, let G be 2-connected (but not 3-connected), i.e. $k(G) = 2$. Let S be a cut set with two vertices and $x_p \in S$. Hence, $G - x_p$ has a cut vertex, see Figure 38. Let x_1 and x_2 be two vertices in distinct blocks (2-connected maximal subgraph of G). Again, we use the idea mentioned above to construct a sequence $\langle x_1, x_2, \dots, x_n \rangle$ and the proof

follows by using the greedy algorithm for vertex coloring. The proof for $k(G) = 1$ is similar. ■

- (•) $\Delta(G) - \chi(G)$ can be arbitrarily large.
- (•) There are also graphs such that $\Delta(G) = \chi(G)$, for example even cycles, non-bipartite 3-regular graphs, say, Peterson graph.

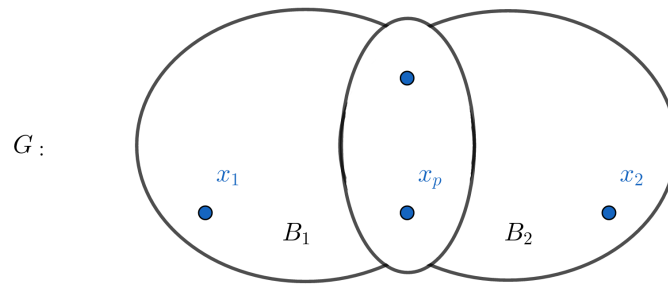
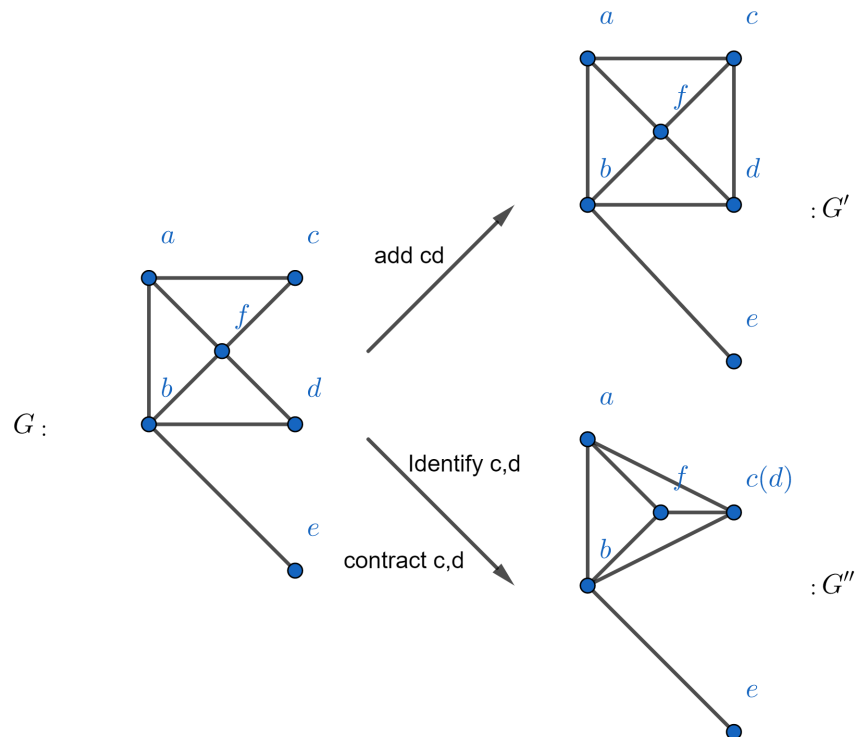


Figure 38. $k(G) = 2$

Another coloring algorithm



Observation

A coloring φ of G has two outcomes:

- (1) $\varphi(c) \neq \varphi(d)$ and (2) $\varphi(c) = \varphi(d)$.

Theorem 59

Let $p_H(k)$ be the number of distinct k -colorings of H . Then,
 $p_G(k) = p_{G'}(k) + p_{G''}(k)$ where G' and G'' are graphs obtained from G
 by adding xy and contracting x and y respectively for $x \not\sim_G y$. ($p_G(k)$ is
 known as the chromatic polynomial of G with k -colorings.)

Proof.

The proof follows by the fact that $\varphi(x) = \varphi(y)$ or $\varphi(x) \neq \varphi(y)$ but
 not both. ■

- (•) G is k -colorable if and only if $p_G(k) \geq 1$.
- (•) $\chi(G) = \min\{\chi(G'), \chi(G'')\}$.

Theorem 60

Let G be a (p, q) -graph with k components. Then,
 $p_G(x) = \sum_{i=0}^{p-k} (-1)^i \cdot a_i \cdot x^{p-i}$, where $a_0 = 1$, $a_1 = q$ and a_i is a positive
 integer for $0 \leq i \leq p - k$.

Proof.

By induction on $p + q$. Clearly, it's true for $p + q = 1$. Assume the
 assertion is true for the cases of smaller $p + q$ and let G be a
 (p, q) -graph with k components. First, if $m = 0$, then $p = k$, so

$p_G(x) = x^p$, thus $a_0 = 1$, $a_1 = q = 0$. Now, consider $m \geq 1$. Let uv be an edge of G and $G_0 = G - uv$. By induction,

$p_{G_0}(x) = x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i x^{p-i}$ where b_i is a non-negative integer for each i . (G_0 has at least k components.) Also,
 $p_{G_0''}(x) = x^{p-1} - \sum_{i=2}^{p-k} (-1)^i c_i x^{p-i}$ where c_i is a positive integer for each i .
 Note that $G'_0 \cong G$ (adding uv back).

This implies that

$$\begin{aligned}
 p_G(x) &= p_{G_0}(x) - p_{G_0''}(x) \\
 &= x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i x^{p-i} - x^{p-1} + \sum_{i=2}^{p-k} (-1)^i c_i x^{p-i} \\
 &= x^p - q \cdot x^{p-1} + \sum_{i=2}^{p-k} (-1)^i (b_i + c_i) x^{p-i} \\
 &= x^p - q \cdot x^{p-1} + \sum_{i=2}^{p-k} (-1)^i a_i x^{p-i}, \quad a_i > 0 \text{ for each } i.
 \end{aligned}$$

■

(•) If T is a tree of order p , then for each $k \geq 1$, there are

$$k \cdot (k-1)^{p-1} = k^p - \binom{p-1}{1} \cdot k^{p-1} + \dots = k^p - (p-1)k^{p-1} + \dots \text{ different } k\text{-colorings of } T, \text{ i.e., } p_T(k) = k \cdot (k-1)^{p-1}.$$

Theorem 61 (Nordhaus and Gaddum, 1956)

If G is a graph of order p , then

$$\begin{aligned}
 (1) \quad & 2\sqrt{p} \stackrel{\textcircled{1}}{\leq} \chi(G) + \chi(\bar{G}) \stackrel{\textcircled{1}}{\leq} p+1, \text{ and} \\
 (2) \quad & p \stackrel{\textcircled{2}}{\leq} \chi(G) \cdot \chi(\bar{G}) \stackrel{\textcircled{2}}{\leq} [(p+1)/2]^2.
 \end{aligned}$$

Proof.

First, we claim that $\chi(G) \cdot \chi(\bar{G}) \geq p$. For each vertex v of K_p let $\varphi(v) = (\varphi_1(v), \varphi_2(v))$ where φ_1 and φ_2 are chromatic colorings of G

and $\bar{G}$ respectively. Since two vertices of K_p are either adjacent in G or $\bar{G}$, all ordered pairs of $v \in V(K_p)$ are distinct. Hence,

$\chi(G) \cdot \chi(\bar{G}) \geq p$. (We need p colors for K_p .)

This implies that $\frac{\chi(G) + \chi(\bar{G})}{2} \geq \sqrt{\chi(G) \cdot \chi(\bar{G})} \geq \sqrt{p}$, ① holds.

Now, let $k = \max_{H \leq G} \delta(H)$. We claim that every induced subgraph H' of

$\bar{G}$ has minimum degree $p - k - 1$, i.e. $\max_{H' \leq \bar{G}} \delta(H') \leq p - k - 1$. Suppose

not. Let H'' be an induced subgraph of $\bar{G}$ such that $\delta(H'') = p - k$.

Since H'' is an induced subgraph of $\bar{G}$, $H'' \cong \bar{H}$ for some induced

subgraph H of G . Let $|H| = h$. Since $\delta(H'') = \delta(\bar{H}) = p - k$,

$\deg_H(v) \leq (h - 1) - (p - k)$ for each $v \in V(H)$. Therefore, in G ,

$\deg_G(v) \leq (h - 1) - (p - k) + (p - h) = k - 1$. On the other hand,

$k = \max_{H \leq G} \delta(H)$ and thus we have an induced subgraph $H''' \leq G$ such

that $\delta(H''') = k$. This implies that $V(H) \cap V(H''') = \emptyset$. By the fact

$|V(H''')| \geq k + 1$, $|H| = h \leq p - (k + 1)$ and thus $|\bar{H}| \leq p - (k + 1) = p - k - 1$.

$\delta(\bar{H}) = p - k$ is not possible. This concludes that

$\max_{H' \leq \bar{G}} \delta(H') \leq p - k - 1$ and thus $\chi(\bar{G}) \leq p - k - 1 + 1 = p - k$ (and $\chi(G) \leq 1 + k$), the proof of ①' follows.

Now, for ②', it follows by

$$\sqrt{\chi(G) \cdot \chi(\bar{G})} \leq \frac{\chi(G) + \chi(\bar{G})}{2} \leq \frac{p + 1}{2}. \quad \blacksquare$$

(•) A graph is said to be *self-complementary* if $G \cong \bar{G}$.

In this situation $\sqrt{p} \leq \chi(G) \leq \frac{p + 1}{2}$. $p = 5 \Rightarrow \chi(G) = 3 \rightarrow G \cong C_5$.

Problem.

Let $\omega(G)$ denote the order of a maximum clique, i.e., the order of

complete subgraphs with maximum order. Then, $\chi(G) \geq \omega(G) \rightarrow$
 Clique number of G . *When does the equality holds?*

(•) A subgraph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G .

(*) $\chi(H) - \omega(H)$ can be very large!

Theorem 62 (Mycielski's graph)

For every integer n , there exists a triangle-free graph G such that $\chi(G) = n$. ($\chi(G) - \omega(G) = n - 2$.)

Proof.

By induction on n and K_1, K_2, C_5 do have the property respectively for $n=1, 2$, and 3 . Now, assume that H is a triangle-free k -chromatic graph, i.e., $\chi(H) = k$. We construct a graph G based on H such that G is a triangle-free $(k+1)$ -chromatic graph.

Let $V(H) = \{v_1, v_2, \dots, v_p\}$ and $V(G) = V(H) \cup \{u_1, u_2, \dots, u_p, u_0\}$. Let $E(G) = E(H) \cup \{u_0 u_i \mid i = 1, 2, \dots, p\} \cup \{u_i v_j \mid v_j \in N_H(v_i)\}$. See Figure 38 for an example when $k=3$.

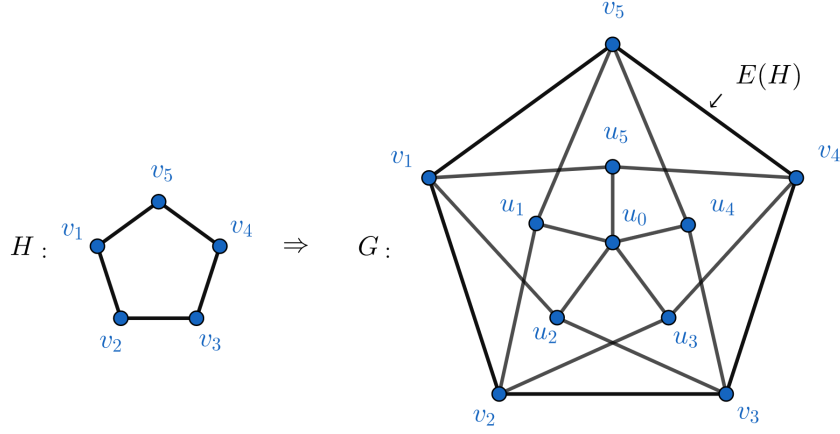


Figure 38. Grötzsch graph

Since $\langle \{u_1, u_2, \dots, u_p\} \rangle_G$ contains no edges, u_0 is not in any triangle. By assumption, $H \not\cong K_3$. So, the only possibility will be a triangle consists of u_i , v_j and v_k where $u_i v_j$ and $u_i v_k$ are edges of G . If they form a triangle, then $\langle \{v_i, v_j, v_k\} \rangle_H$ is a triangle in H . Hence, G is triangle-free.

Now, we claim $\chi(G) = k + 1$. Let φ be a k -coloring of H . Let $\tilde{\varphi} : V(G) \rightarrow \{1, 2, \dots, k + 1\}$ by letting $\tilde{\varphi}(u_i) = \varphi(v_i)$ and $\tilde{\varphi}(u_0) = k + 1$. Hence, we have a $(k + 1)$ -coloring of G , thus $\chi(G) \leq k + 1$. On the other hand, we show that $\chi(G) \geq k + 1$. Suppose not. Let φ' be a k -coloring of G and the colors used are $1, 2, \dots, k$. First, we assign u_0 the color k , i.e., $\varphi'(u_0) = k$. So, the colors used for $u_1, u_2, \dots, u_p$ must be in $\{1, 2, \dots, k - 1\}$. Since $\chi(H) = k$, k occurs somewhere in H , say v_i . (May have more vertices.) Now, we recolor v_i by using $\varphi'(u_i)$. Since u_i is adjacent to every vertex of $N_H(v_i)$, $\varphi'(u_i) \neq \varphi'(v)$ for each $v \in N_H(v_i)$ and thus we have a proper coloring of H using at most $k - 1$ colors. (?) $\rightarrow \leftarrow$ ■

(*) This theorem has been extended to a more general result obtained by Erdős and Lovász (1961): For any integers $m, n \geq 2$, there exists an n -chromatic graph whose girth exceeds m . (Theorem 62 is for $m = 3$.)

****Theorem 63 (Lovász, 1972) (Weakly Perfect Graph Theorem)**

A graph G is perfect if and only if $\bar{G}$ is perfect.

Note

The proof of this Theorem is not too long. But, the proof of next one is long.

****Theorem 63' (Maria Chudnovsky, Neil Rokertson, Paul Seymour and Robin Thomas, Annals of Mathematics, 164(2006), 15-229.)**

A graph G is perfect if and only if no induced subgraph of G or $\bar{G}$ is an odd cycle of length at least 5.

Proof of Theorem 63.

We prove a different version (stronger):

A graph G is perfect if and only if $|H| \leq \alpha(H) \cdot \omega(H) \longrightarrow (1)$

for all induced subgraphs H of G . ($\omega(H)$ is the clique number of H .)

(•) In $\bar{G}$, if A is an independent set, then in G , $\langle A \rangle_G$ is a clique (A is independent) $\langle A \rangle_G$ (a clique)

($\Rightarrow$) (Definition) If G is perfect, then for each induced subgraph H , $\chi(H) = \omega(H)$. Hence, the vertex set of H , $V(H)$, can be partitioned

into $\omega(H)$ subsets. Clearly, each subset has size at most $\alpha(H)$, hence $|H| \leq \alpha(H) \cdot \omega(H)$.

($\Leftarrow$) By induction on $|G|$. Assume that every induced subgraph H of G satisfying (1), but G is not perfect. (Every "proper" induced subgraph is perfect.)

Now, let $u \in V(G)$ and consider $G - u$. Let $\omega(G) = \omega$ and $\alpha(G) = \alpha$. By induction, $\chi(G - u) = \omega(G - u)$. If $\omega(G - u) < \omega(G)$, then by coloring u with a new color, we have $\chi(G) \leq \omega(G)$. This implies that G is perfect. (We can replace u with an independent set!) $\rightarrow \leftarrow$

Let K be the vertex set of a clique with ω vertices. Notice that if $u \notin K$, then K meets every color class (independent set) of $G - u$.

$\rightarrow$ (2) But, if $u \in K$, then K meets $\omega - 1$ color classes of $G - u$. $\rightarrow$ (3)

Now, we construct $\alpha\omega + 1$ independents in G by the followings. Let $A_0 = \{u_1, u_2, \dots, u_\alpha\}$ be an independent set of G with α vertices (independence number α).

Starting from $G - u_1$, and then $G - u_2, \dots, G - u_\alpha$, we have $\alpha\omega$ independent sets: $A_1, A_2, \dots, A_\omega, A_{\omega+1}, A_{\omega+2}, \dots, A_{2\omega}, \dots, A_{\alpha\omega}$. (Each of them contains ω independent sets.)

Observe that $K \cap A_i = \emptyset$ for all but one $i \in \{0, 1, 2, \dots, \alpha\omega\}$. (If $K \cap A_0 = \emptyset$, then $K \cap A_i \neq \emptyset$ for all $i \in \{1, 2, \dots, \alpha\omega\}$ (by (2))). On the other hand, if $K \cap A_0 \neq \emptyset$, then $|K \cap A_0| = 1$, say $K \cap A_0 = \{u_j\}$. (Except for u_j , all the other vertices of A_0 are not in K .) This implies that K meets $\omega - 1$ color classes of $G - u_j$ which implies that in $G - u_j$, there is an independent set A_i such that $K \cap A_i = \emptyset$. (By (3).)

$$M : \begin{array}{c} v_1 \\ A_0 \\ v_2 \\ A_1 \\ \vdots \\ v_3 \\ A_{\alpha\omega} \end{array} \left[\begin{array}{cc} & \\ M(i,j) = 1 & \\ \text{iff } v_j \in A_i & \end{array} \right] \quad (\alpha\omega + 1) \times p$$

Theorem 64

If G is a connected planar graph, then $\chi(G) \leq 5$.

Proof.

By induction on $|G|$. By Theorem 57, it suffices to consider an induced subgraph H whose minimum degree is 5.

Let $v \in V(H)$ such that $\deg_H(v) = 5$. By induction, $\chi(H - v) \leq 5$. Let φ be a 5-coloring of H and we consider the colors assigned on $N_H(v)$. Let them be $\varphi(v_1), \varphi(v_2), \dots, \varphi(v_5)$. Clearly, if any two of them are of the same color, then there is a color for v such that we have a proper 5-coloring of H . So, assume that $\varphi(v_i) = i$, $i = 1, 2, 3, 4, 5$ and the vertices are in clockwise order, see Figure 40.

Now, consider the induced subgraph $H_{1,3} = \langle \varphi^{-1}(1) \cup \varphi^{-1}(3) \rangle_H$. If v_1 and v_3 are in distinct components, then by changing the colors 1 and 3 in the $\varphi(v_1) = 3$ and $\varphi(v_3) = 3$. Hence, 3 is available for v .

On the other hand, there exists a path P connecting v_1 and v_3 . Hence, $v - v_1 - P - v_3 - v$ is a cycle such that v_2 and v_4 are in different regions. By a similar argument, we may change the color of v_2 to 4. Then, 2 is available for v . ■

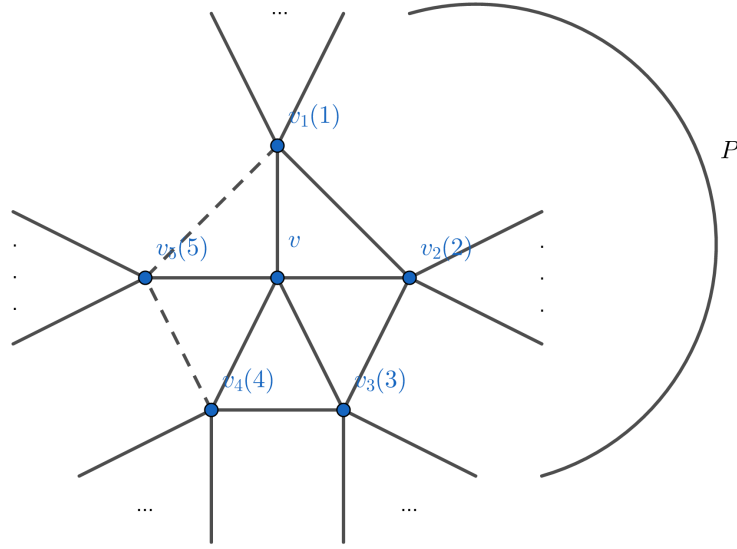


Figure 40.

(*) Theorem (4CT)

Every planar graph is 4-colorable.

The most recent proof was obtained by N. Robertson, D.P. Sanders, P.D. Seymour and R. Thomas (1996): A new proof of the 4CT, Electron. Res. Announc. A.M.S. 2, 17-25.

The first proof was obtained in 1976-1977, by K. Appel and W. Haken.

(•) The following theorem is not working for $n = 0$.

Theorem 65 (The Heawood Map Coloring Theorem)

For every positive integer n , $\chi(S_n) = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$. ($\chi(S_n)$: the maximum chromatic number among all graphs that can be embedded on S_n .)

Proof. (Outline)

The upper bound $\chi(S_n) \leq \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$ was obtained by Heffter in 1890. At that time, he claimed that it's an equality. But, unfortunately, the correct proof came out many years later by the effort of considering the embedding of K_p since for sure K_p needs p colors.

So, define $p = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$. It follows that $n \geq \frac{(p-3)(p-4)}{12}$ and thus $n \geq \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$. By the fact $r(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$, $\chi(S_n) \geq p = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$. ■

(•) There exists a procedure to construct all the graphs G with

$$\chi(G) \geq k.$$

(••) $\mathcal{G}_k$ is a collection of k -constructible graphs if $G \in \mathcal{G}_k$ can be constructed recursively by the following steps.

- (i) K_k is k -constructible.
- (ii) If G is k -constructible and $x, y \in V(G)$ are non-adjacent, then $(G + xy)/xy$ is k -constructible.

(iii) If G_1 and G_2 are k -constructible such that

$V(G_1) \cap V(G_2) = \{x\}$, $xy_1 \in E(G_1)$, $xy_2 \in E(G_2)$, then

$(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k -constructible, see Figure 41.

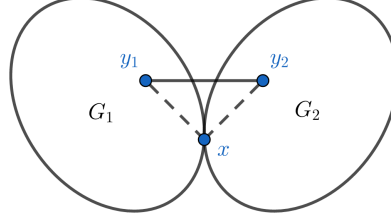


Figure 41. Hajós construction

(*) If G is k -constructible, then $\chi(G) \geq k$.

(i) is trivial.

(ii) If $(G + xy)/xy$ uses less than k colors, then by coloring x and y with the same color, we have $\chi(G) < k$.

(iii) If the graph obtained uses less than k colors, then either $\chi(G_1) < k$ or $\chi(G_2) < k$ depending on whether $\varphi(y_1) \neq \varphi(x)$ or $\varphi(y_2) \neq \varphi(x)$. Since y_1 and y_2 receive distinct colors, one of the above two conditions must hold.

Theorem 66 (Hajós, 1961)

Let G be a graph. Then, $\chi(G) \geq k$ if and only if G has a k -constructible subgraphs.

Proof.

($\Leftarrow$) It has been explained above.

($\Rightarrow$) Suppose not; then $k \geq 3$. Let G be a *maximal* counterexample, i.e. G is of maximum size such that G does not contain a

k -constructible subgraph. Now, G can not be a complete r -partite graph. For otherwise, $\chi(G) \geq k$ implies that $r \geq k$ and then G contains a k -constructible K_k . (Contract each partite set.) Hence there exist vertices x, y_1 and y_2 such that $xy_1 \notin E(G)$, $xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. By assumption of the maximality of G , both $G + xy_1$ and $G + xy_2$ contain k -constructible subgraphs, say H_1 and H_2 ; moreover, $xy_1 \in E(H_1)$ and $xy_2 \in E(H_2)$.

Let $H_2 - H_1$ denote the graph $\langle V(H_2) \setminus V(H_1) \rangle_{H_2}$ and H'_2 is an isomorphic copy of H_2 such that $V(H'_2) \cap V(G) = \{x\} \cup (V(H_2) \setminus V(H_1))$, see Figure 42. So, $V(H_1) \cap V(H'_2) = \{x\}$. Now, since $H'_2 \cong H_2$, let $\varphi : H_2 \rightarrow H'_2$ be an isomorphism. By (iii) $H_1 \cup H'_2 - xy_1 - x \cdot \varphi(y_2) + y_1 \cdot \varphi(y_2)$ is k -constructive, let this graph be H . Now, for each vertex v' in $V(H'_2) \setminus V(G)$, there exists a v such that $v' = \varphi(v)$. Furthermore vv' is not an edge of H . By (ii), we can identify v and v' and obtain a k -constructible subgraph of G after identifying all vertices in $V(H_2) \setminus V(H_1)$. ■

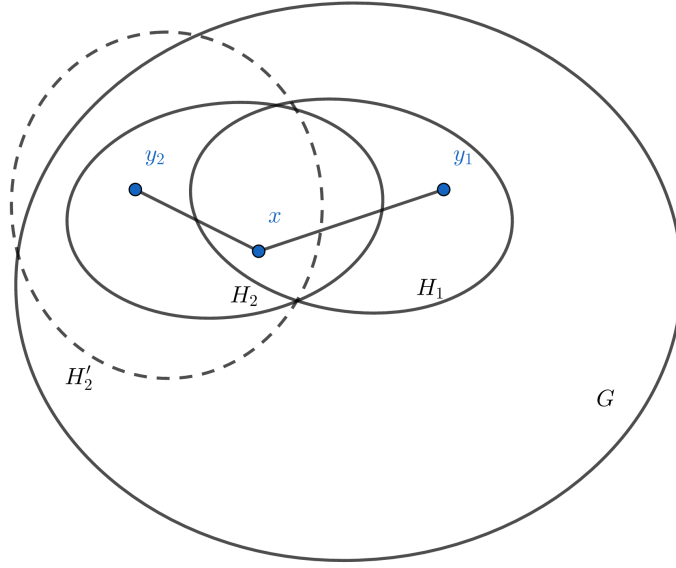


Figure 42.

- (•) A k -edge-coloring is a mapping $\pi : E(G) \rightarrow \{1, 2, \dots, k\}$ such that incident edges receive distinct images (colors).
- (•) $\chi'(G) = \min.\{k \mid G \text{ has a } k\text{-edge-coloring}\}$. (Chromatic index of G .) If $\chi'(G) = k$, then G is h -edge-colorable for each $h \geq k$.

Theorem 67 (Vizing, 1964)

If G is a simple graph, then $\Delta(G) \leq \chi' \leq \Delta(G) + 1$.

Proof.

The left hand inequality is easy to see, we prove the right hand inequality. By induction on $\|G\|$. We shall prove that G has a $(\Delta(G) + 1)$ -edge-coloring (coloring in short) for G and the assertion is true for smaller sizes, i.e., for each $e \in E(G)$, $G - e$ has a coloring π .

First, we observe that since each vertex v is of degree at most $\Delta(G)$, a color is missing around v . Second, if α and β be two colors

used in the coloring, then α and β induce a subgraph with components either paths or even cycles.

Finally, if " G has no colorings using $\Delta(G) + 1$ colors", then for each edge xy and any coloring of $G - xy$, there exists an $\alpha - \beta$ path from y ends in x provided α is missing at x and β is missing at y . See Figure 43 for missing colors.

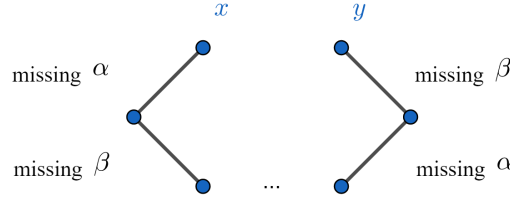


Figure 43.

(*) If $\alpha - \beta$ path does not connect x and y , then we may recolor one of the path (α, β) , to obtain a coloring of G using $\Delta(G) + 1$ colors.

Clearly, if x and y are missing the same color, then we can use that color to color xy and obtain a $\Delta(G) + 1$ coloring of G . $\rightarrow \leftarrow$

Claim: There is a way to recolor some edges in $G - xy$ such that x and y miss the same color.

Outline of proof.

Let $M(y)$ denote the colors missing at y , and $c_1 \in M(y)$. Now, consider $M(x)$. If $c_1 \in M(x)$, then color xy by c_1 results in a $\Delta(G) + 1$ coloring of G . (The claim holds.) Hence, $c_1 \notin M(x)$, let $c_0 \in M(x)$ and $\pi(xy_1) = c_1$. (See Figure 44) Then, consider $M(y_1)$ and let $c_2 \in M(y_1)$. Note that $c_2 \notin M(x)$. If $c_2 \in M(x)$, then we let $\pi(xy_1) = c_2$. Thus, c_1 becomes a missing color in $M(x)$, the coloring for xy is available,

$\pi(xy) = c_1$. This fact will continue: $c_2 \notin M(x) \Rightarrow \exists y_2, s.t. \pi(xy_2) = c_2$; and then $c_3 \in M(y_2)$, $\pi(xy_3) = c_3, \dots, c_{i+1} \in M(y_i)$, $\pi(xy_{i+1}) = c_{i+1}$. Since we only have $\Delta(G) + 1$ colors, there exists an l such that $\pi(xy_{l+1}) = c_{l+1} \in \{c_1, c_2, \dots, c_l\}$. W.L.O.G., let $c_{l+1} = c_k$, $k \in \{1, 2, \dots, l\}$.

Now, we have several cases to consider depending on whether $c_0 \in M(y_l)$ or $c_0 \notin M(y_l)$.

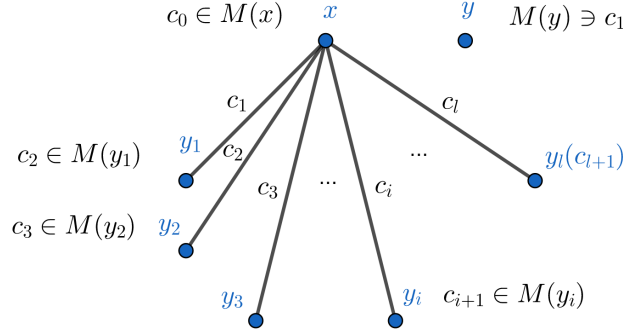


Figure 44.

(a) $c_0 \notin M(y_l)$.

Since $c_{l+1} = c_k$, $c_k \in M(y_l)$. Now, consider $c_k - c_0$ path starting from y_l .

- (i) It is a $y_l - y_k$ path. Since $\pi(xy_k) = c_k$, we may recolor them to a $c_0 - c_k$ path starting from y_k . (Note here that c_0 occurs in an edge incident to y_l . By the fact that the last color is c_k , both c_0 and c_k occur an even number of times.) Now, since $\pi(xy_k) = c_0$, the recoloring of $xy_1, xy_2, \dots, xy_{k-1}$ gives $c_1 \in M(x)$, we have the proof.
- (ii) It is a $y_l - y_{k-1}$ path. Since $c_k \in M(y_{k-1})$, this path is ended with color c_0 . That is to say c_0 is also available for xy_{k-1} (not only c_{k-1}). Hence, we color xy_{k-1} with c_0 instead of c_{k-1} , the proof follows by a similar recoloring process.

(iii) It is a $y_l - y_i$ path, $i \notin \{k-1, k\}$.

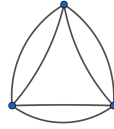
Then, either c_l or c_0 will be available for xy_i and the proof follows by recoloring process. The case (b) $c_0 \in M(y_l)$ can be done similarly. ■

(•) Based on the same proof technique, we also have a stronger result of Vizing's Theorem.

Theorem 67' (Vizing, 1964)

If G is a multigraph with multiplicity η , then $\chi'(G) \leq \Delta(G) + \eta$.

(•) The following graph has $\Delta(G) = 4$ and $\eta = 2$.



Definition (Class 1 and Class 2)

A graph (simple) is of Class 1 if $\chi'(G) = \Delta(G)$ and of Class 2 if $\chi'(G) = \Delta(G) + 1$.

Theorem 68 (König, 1916)

A bipartite graph is of Class 1.

Proof.

(1st)

By induction on $\|G\|$. Let $xy \in E(G)$ and $G - xy$ can be edge-colored with $\Delta(G)$ colors. Now, since $\deg_{G-xy}(x) < \Delta(G)$ and

$\deg_{G-xy}(y) < \Delta(G)$, a color is missing at x and also a color is missing at y . Let them be α and β respectively. Clearly, $\alpha \neq \beta$, and β occurs around x and α occurs around y . Now, we adapt the idea in proving Vizing's Theorem, let P be a longest $\alpha - \beta$ path from x :



First, if P is an $x - y$ path and the last edge has color α , then P is a path of even length. Hence, $P \cup \{xy\}$ is an odd cycle. G is bipartite. $\rightarrow \leftarrow$ Hence, x and y are in different components induced by the set of edges colored α and β . Now, we recolor all the edges of P by interchanging α and β . This gives a coloring in which β is missing at x and also at y . By coloring xy with β , we obtain a Δ -edge-coloring of G . ■

(2nd)

Lemma

Let G be a bipartite graph. Then, there exists a $\Delta(G)$ -regular bipartite graph $\tilde{G} \geq G$. (Exercise) By Lemma $\tilde{G}$ is a $\Delta(G)$ -regular bipartite graph and thus $\tilde{G}$ can be decomposed into $\Delta(G)$ perfect matchings by König's Theorem. This implies that $\chi'(\tilde{G}) = \Delta(G)$. Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) \leq \Delta(G)$. Hence, we conclude the proof.

(•) A graph G is said to be overfull if $\|G\| > \left\lfloor \frac{|G|}{2} \right\rfloor \cdot \Delta(G)$.

(••) If G is overfull, then G is of Class 2.

(•) If G is overfull, then $|G|$ is odd.

Theorem 69

Petersen graph is of Class 2.

Proof.

If G is the Petersen graph and $\chi'(G) = 3$, then G can be decomposed into 3 1-factors: F_1 , F_2 and F_3 (3 color classes). Now, consider the set of 5 link-edges e_1, e_2, e_3, e_4 and e_5 , see Figure 45.

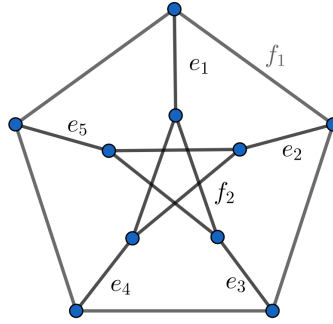


Figure 45. Petersen graph

At least one of F_1 , F_2 and F_3 will contain at least two link-edges by Pigeon-hole principle, let it be F_1 . Clearly, F_1 can not contain all the 5 link-edges. For otherwise, two C_5 's is the union of F_2 and F_3 which is impossible. So, there are three cases to consider.

$$(i) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 4$$

Let e_1 be the edge not in F_1 . But, now all the edges of $G - e_1$ not in $\{e_2, e_3, e_4, e_5\}$ are incident to an edge of $\{e_2, e_3, e_4, e_5\}$. So, no other edge can be chosen for F_1 .

$$(ii) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 3$$

Let e_1 and e_2 be the edges not in F_1 . Then, other than link-edges, we choose at most one more edge f_1 . The case e_1 and e_3 are not in F_1 has similar conclusion (only f_2 is available).

$$(iii) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 2$$

This case comes out that we can find two more edges which not link-edges. ■

(*) The proof of Theorem 69 implies that the Petersen graph contains no "Hamilton cycles".

Proof.

If G contains a Hamilton cycle C , then $\chi'(G) = 3$ by coloring the cycle with two colors and $G - C$ (1-factor) with another color. ■

Theorem 70

A 3-regular planar graph G is of Class 1.

Proof.

Let G be embedded in S_0 . Then, by 4-color Theorem, G is 4-face-colorable (or 4-map-colorable). Let the 4 colors used be obtained from the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$. Since each edge is in the boundary of two adjacent faces, let the edge be colored by $(a_1, b_1) \oplus (a_2, b_2)$ where (a_1, b_1) and (a_2, b_2) are the colors of these two adjacent faces. As a conclusion, we obtain a 3-edge-coloring of G , since $(0,0)$ will not be used. The coloring is proper since three adjacent faces will receive three different colors, see Figure 46. ■

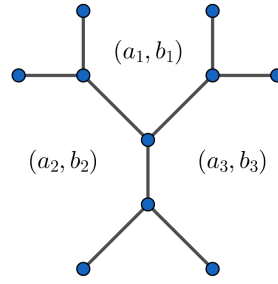


Figure 46.

Theorem 71 (Equitable edge-coloring)

If G has a k -edge-coloring f , then G has an equitable edge coloring, i.e., for any two $i, j \in \{1, 2, \dots, k\}$, $\|f^{-1}(i) - f^{-1}(j)\| \leq 1$.

Proof.

If there exist i and j such that $|f^{-1}(i) - f^{-1}(j)| \geq 2$, then we consider the graph H induced by the set of edges colored i and j . Then, H is a subgraph of G such that each component of H is either a path or an even cycle. Since i occurs more times than j , there exists an $i - j$ path:



whose end edges are colored i . Now, by switching the colors on this path, we obtain a new edge coloring of G such that i occurs one less time and j occurs one more. It turns out that we can obtain an k -edge-coloring s.t. $\|f^{-1}(i) - f^{-1}(j)\| \leq 1$. As a consequence, we are able to adjust all of them and obtain an equitable k -edge-coloring. ■

- (•) This Theorem is not difficult to prove, but very useful.
- (•) Without using 4CT, the proof of Theorem 70 is very difficult.

(*) It was conjectured that if G is planar and $\Delta(G)$ is large enough, then G is of Class 1.

Theorem 72

The complete graph K_n is of Class 2 if and only if K_n is overfull or equivalently n is odd.

Proof.

First, we claim that for each $m \geq 1$, K_{2m} is of Class 1. It suffices to give a $(2m - 1)$ -edge-coloring of K_{2m} . For convenience, let

$V(K_{2m}) = \mathbb{Z}_{2m} = \{0, 1, 2, \dots, 2m - 1\}$. For each color $i \in \{1, 2, \dots, 2m - 1\}$, let the set of edges colored i be

$$F_i = \{(0, i), (i + 1, i - 1), (i + 2, i - 2), \dots, (i + m - 1, i - m + 1)\}$$

(mod $2m - 1$). See Figure 46 for an example of $m = 5$ and $i = 3$.

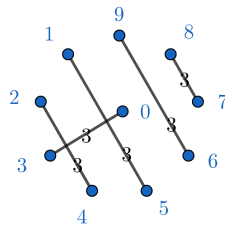


Figure 46. $\chi'(K_{10}) = 9$

Since $\Delta(K_{2m}) = 2m - 1$, $\chi'(K_{2m}) = 2m - 1$.

Now, by deleting 0 in K_{2m} , we obtain a $(2m - 1)$ -edge-coloring of K_{2m-1} . On the other hand, it is not difficult to check that K_{2m-1} is overfull for $m \geq 2$, this concludes that $\chi'(K_{2m-1}) > \Delta(K_{2m-1}) = 2m - 2$. ■

- (•) This Theorem is not difficult to prove, but it is very useful in the construction of "Combinatorial Designs".
- (••) Equivalently, K_{2m} can be decomposed into $2m - 1$ 1-factors, which is also known as a 1-factorization of K_{2m} .
- (•••) If G is an r -regular graph and $\chi'(G) = r$, then G has a 1-factorization.
- (* * *) It was conjectured that if G is r -regular and $r \geq \frac{|G|}{2}$, then G has a 1-factorization or equivalently $\chi'(G) = r$.

Theorem 71' (D. Hoffman et al.)

A complete multipartite graph G is of Class 2 if and only if G is overfull.

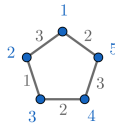
- (•) "Total coloring"

A k -total coloring of a graph G is a mapping

$\varphi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that

- (i) adjacent vertices receive distinct images,
- (ii) incident edges receive distinct images, and
- (iii) each vertex has a distinct image with the images of its incident edges.

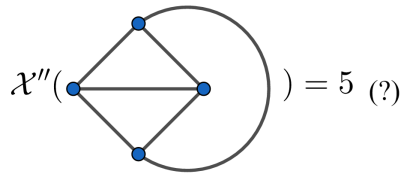
e.g.

A 4-total coloring of C_5

(•) $\chi''(G) = \min.\{k \mid G \text{ has a } k\text{-total coloring}\}$. (Total chromatic number)

Theorem 73

$$\chi''(K_{2n+1}) = \chi''(K_{2n}) = 2n + 1.$$



Proof.

$\chi''(K_{2n+1})$ can be obtained by using $\chi'(K_{2n+1}) = 2n + 1$.

TCC Conjecture $\chi''(G) \leq \Delta(G) + 2$

Note that $\chi''(G) \geq \Delta(G) + 1$. As to the total coloring of K_{2n} , we claim that $2n$ colors are not enough.

Observe that each color class has at most one vertex and $n - 1$ edges. So, $2n$ color classes will contain at most $2n$ vertices and $2n(n - 1)$ edges. Hence, there are $2n^2$ elements (vertices and edges) in total. But, K_{2n} has $2n + \frac{2n(2n - 1)}{2}$ elements to color, which is $2n^2 + n$. Clearly, $2n$ color is not enough. Since K_{2n+1} is $(2n + 1)$ -total colorable, K_{2n} is also $(2n + 1)$ -total colorable. The proof follows. ■

- (•) Based on TCC Conjecture, a graph G is called Type 1 if $\chi''(G) = \Delta(G) + 1$ and Type 2 otherwise.

Theorem 74

$K_{m,n}$ is of Type 1 if and only if $m \neq n$.

Proof.

($\Rightarrow$) If $m = n$, then there are $2n + n^2$ elements to color. Since each color class contains at most n elements, $\Delta(G) + 1 = n + 1$ colors are not enough, $n(n + 1) < 2n + n^2$. Hence, $\chi''(K_{n,n}) \geq n + 2$, and thus $K_{n,n}$ is not of Type 1.

($\Leftarrow$) On the other direction, let $m = n + k$. Now, $\Delta(K_{m,n}) = n + k$. By the edge-coloring of $K_{m,n}$, we have an $n \times (n + k)$ Latin rectangle based on $\{1, 2, \dots, n + k\}$, see Figure 47. Since $k \geq 1$, we may extend this rectangle to $(n + 1) \times (n + k)$ and the last row can be used to color the vertices of A . Finally, color all vertices of B by one extra color, we have $K_{m,n} \leq n + k + 1 = \Delta(K_{m,n}) + 1$. ■

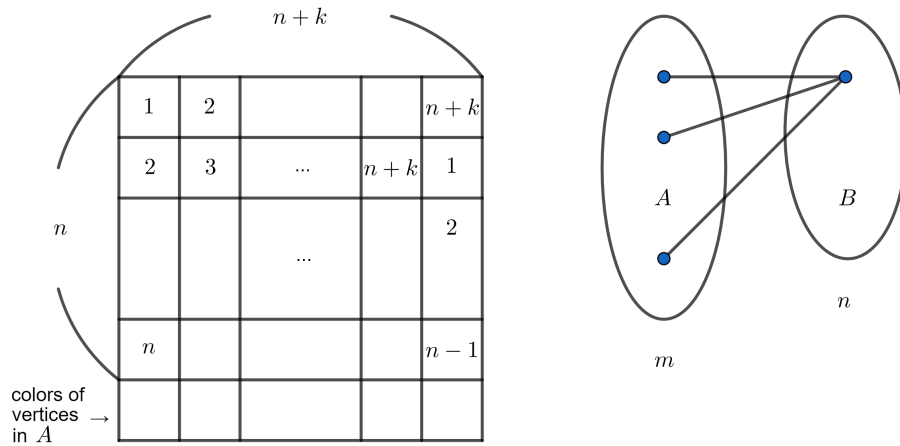
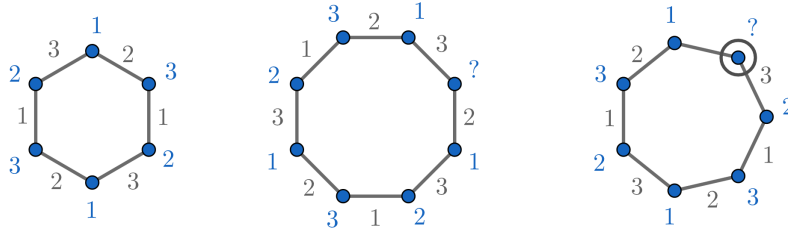


Figure 47. An edge-coloring of $K_{m,n}$

- (•) A cycle C_n is of Type 1 if and only if $n \equiv 0 \pmod{6}$.



If we use three colors, then starting from one vertex and one edge, all the colors of the others are forced!

- (•) The deficiency of a graph G , $def(G)$, is defined as

$$\sum_{v \in V(G)} (\Delta(G) - \deg_G(v)).$$

- (••) G is *conformable* if G has a vertex coloring

$$\varphi : V(G) \rightarrow \{1, 2, \dots, \Delta(G) + 1\} \text{ such that}$$

$$def(G) \geq |\{i \mid |G| - |\varphi^{-1}(i)| \equiv 1 \pmod{2}\}|.$$

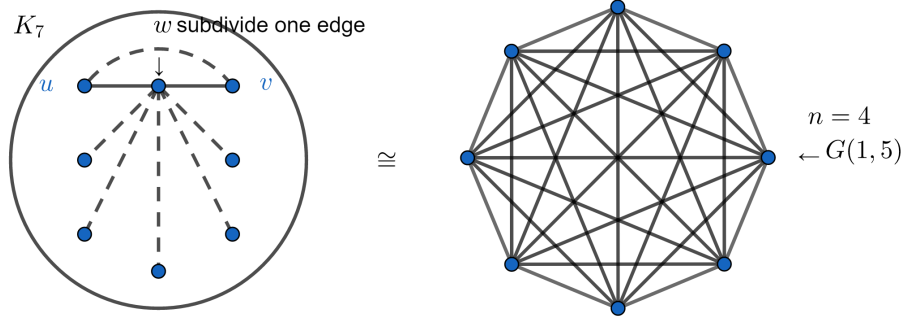
Theorem 75

Let S_i be a star with i edges. Then, $K_{2n} - S_1 - S_{2n-3} = G(1, 2n-3)$ is of Type 2.

Proof.

Assume that $G(1, 2n-3)$ is of Type 1, i.e., there exists a total coloring ψ of $G(1, 2n-4)$ using $\Delta(G) + 1 = 2n - 1$ colors. Let uv be the edge subdivided by w , see Figure 48. First, if $\psi(u) = \psi(v) = 0$, then let $\psi(w) = 1$, $\psi(uw) = 2$ and $\psi(wv) = 3$. Let r_j the number of vertices v in which j occurs in either v or an edge incident to v . Hence,

$\sum_{j=0}^{2n-2} r_j = |K_{2n}| + 2\|G(1, 2n-3)\|$. (?) Now, 0 occurs in at most $2n-2$ vertices. Hence, $\sum_{j=0}^{2n-2} r_j \leq (2n-2) + 3 \cdot (2n) + (2n-5) \cdot (2n-1) = 4n^2 - 4n + 3 < |K_{2n}| + 2\|G(1, 2n-3)\|$. $\rightarrow \leftarrow$

Figure 48. $G(1, 5)$

On the other hand, if $\psi(u) \neq \psi(v)$, a similar argument shows that $2n-1$ colors are not enough. Hence, $G(1, 2n-3)$ is of Type 2. ■

(*) If G is of Type 1, then G is conformable.

Proof.

If there exists a color i such that $|G| - |\psi_{V(G)}^{-1}(i)| \equiv 1 \pmod{2}$, then i occurs in at most $|G| - 1$ vertices. Since every color occurs around a major vertex (Δ -degree), $\Delta(G) + 1$ colors are not enough if $\text{def}(G) < |\{i \mid |G| - |\psi_{V(G)}^{-1}(i)| \equiv 1 \pmod{2}\}|$. So, if G is not conformable, then G is not of Type 1. ■

Conjecture (Chetwynd and Hilton, 1988)

Let G be a simple graph with $\Delta(G) \geq \left\lfloor \frac{|G|+1}{2} \right\rfloor$. Then, G is of Type 2 if and only if there exists a non-conformable subgraph H of G such that $\Delta(H) = \Delta(G)$ (and G is not the Chen and Fu graph). $\leftarrow$ 後來加入.

(•••) The (original) conjecture was disproved by using Theorem 75.

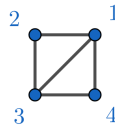
(•) The graph $G(1, 2n - 3)$ is known as "Chen and Fu graph".

(•) Prime Labeling

A prime labeling of a graph G is a mapping

$f : V(G) \xrightarrow{1-1} \{1, 2, \dots, n\}$ such that if $u, v \in E(G)$, then $f(u)$ and $f(v)$ are relatively prime, i.e., $\gcd(f(u), f(v)) = 1$.

Example



(•) $P(a, b)$: The of primes between integers a and b . $P(a, b]$, $P[a, b]$ and $P[a, b)$ can be defined accordingly.

(*) Bertrand's Postulate: (Bertrand-Chebyshev Theorem) For each $n \geq 2$, $P(n, 2n) \neq \emptyset$.

(•) $\alpha(G)$: Independence number of G

$\sigma(G)$: Vertex cover number of G

(S is a vertex cover of G if every edge e of $E(G)$ is incident at least one vertex of S .) $\Rightarrow V(G) \setminus S$ is an independent set.

Theorem 76

Let G be a graph of order n . Then the followings hold.

- (1) If $\alpha(G) < \left\lfloor \frac{n}{2} \right\rfloor$, then G has no prime labelings.
- (2) If S is a vertex cover of G and $|S| \leq |P(n, 2n)| + 1$, then G has a prime labeling.

Proof.

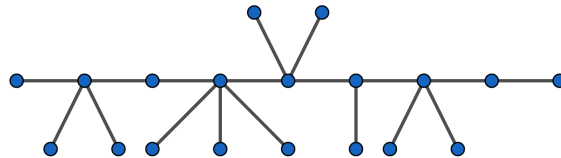
- (1) follows from the fact that in all prime labelings (if exist), the set of vertices with even labels induces an independent set. Hence, the graph must contain an independent set of size $\left\lfloor \frac{n}{2} \right\rfloor$.
- (2) Use the labels 1 and $x \in P(n, 2n)$ to label the vertices in S , we obtain a prime labeling of G . (The vertices in $G - S$ can be labeled arbitrarily.) ■

Use the facts above, we can easily find a prime labeling of a tree of order at most 10. The proof follows by letting $T = (A, B)$ and consider the cases $|A| = 1, 2, 3, 4, 5$.

Prime labeling Conjecture

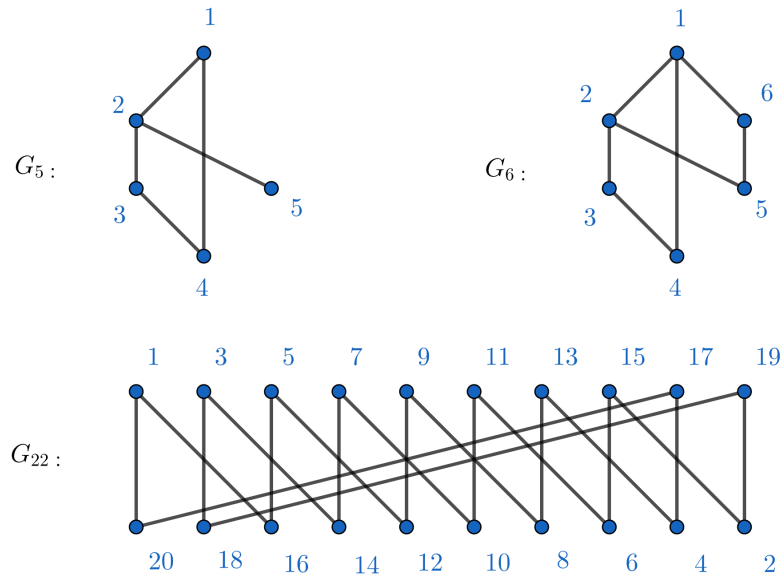
Every tree has a prime labeling.

- (•) The first goal should be the answer for *caterpillar*.



Prime sum graph

- (•) The prime sum graph of order n , G_n , is defined on $[n]$ and two vertices i and j in $[n]$ are adjacent if $i + j$ is a prime.



A part of edges, 23,19,41 are primes used for them.

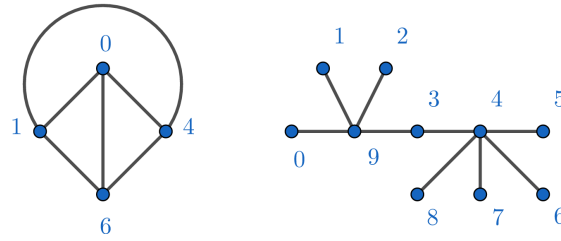
(1,22,19,4,15,8,11,12,7,16,3,20,21,2,17,6,13,10,9,14,5,18) ← Hamilton cycle.

- (••) So, we are interested in determining "for which n , G_n has a Hamilton cycle?". Clearly, n must be even. (?)

- (*) It is not difficult to check, for small $n \geq 6$, G_n is indeed hamiltonian.

(•) Graceful Labelings

A graceful labeling of a graph G (with $|G| \leq \|G\| + 1$) is a mapping $f : V(G) \xrightarrow{1-1} \{0, 1, 2, \dots, |G|\}$ such that $\{|f(u) - f(v)| \mid uv \in E(G)\} = \{1, 2, \dots, \|G\|\}$.

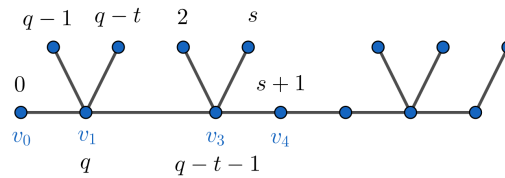


Theorem 77

Any caterpillar has a graceful labeling.

Proof.

Starting from the end vertex from one side, label the vertex v_0 with 0, then the vertex v_1 incident to v_0 is labeled with $\|G\| = q$. Now, the neighbors of v_1 are labeled by 2, 3, $\dots$ for pendant vertices and let the largest label be used in v_3 . Again, we shall start from the use of $q-1, q-2, \dots$ and let the smallest label for v_4 to use. The proof then follows. ■



(*) There are special trees which have a graceful labeling. But, to prove that all trees do have graceful labelings remains unsolved.

Graceful Tree Conjecture (Rigel-Kotzig)

All trees are graceful.

(*) We remark here, for forests, we can also find graceful labelings.

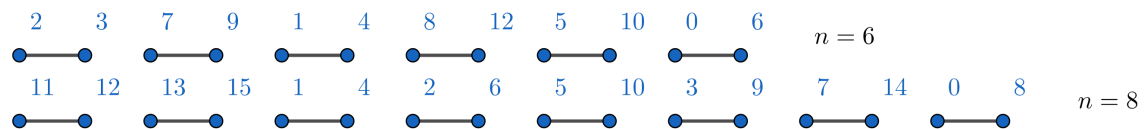
Theorem 77'

Each matching of size n , M_n , has a graceful labeling.

Proof.

This is a direct consequence of using Skolem sequences of order n .

(See a couple of examples below.) ■



(**) There are many problems in graph labelings, you may refer to the following reference for more informations.

A dynamic survey of graph labeling by J. A. Gallian.

(502 pages!)

Random Graphs

(*) The notation of random graphs is different from probabilistic method. Here are two theorems which uses probabilistic method.

(Find the lower bound of $R(s)$ is another example.)

Theorem 78

There exists a tournament T_n such that T_n has at least $n!/2^{n-1}$ directed Hamilton paths.

Proof.

In K_n , there are $n!$ Hamilton paths. (Starting from one of the vertices which have been selected.) Now, if each edge is assigned with an orientation, the probability for any one Hamilton path to be a directed Hamilton path is $\frac{1}{2^{n-1}}$. (each edge has $\frac{1}{2}$ chance to be in the right direction.) So, if χ is the random variable for the number of directed Hamilton paths, $E(\chi)$ (Expectation) $= n! \cdot \frac{1}{2^{n-1}}$ and therefore, there exists a tournament satisfying this value. ■

Theorem 79

The independence number of G , $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)}$.

Proof.

Let f be a random labeling of G by using $1, 2, \dots, |G|$. For convenience, let $V(G) = \{v_1, v_2, \dots, v_p\}$ and f is a 1-1 mapping from $V(G)$ onto $\{1, 2, \dots, p\}$. Now, for each v , there exists a unique $u \in N_G[v]$ such that $f(u) = \min \{f(x) \mid x \in N_G[v]\}$. Now, let S be the set of vertices v in $V(G)$ such that $f(v) = \min \{f(x) \mid x \in N_G[v]\}$. That's is v has the smallest label, then put v in S .

Now, clearly S is an independent set. (?) Moreover, the probability of being the smallest label among all its neighbors is $\frac{1}{\deg_G(v) + 1}$, we conclude the proof since $\alpha(G) \geq |S| = \sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)}$. ■

Random Graphs

There are different models, here we consider one of the most popular one.

Model A $G(n, p), 0 \leq p \leq 1$.

The probability of the existence of an edge (independently) is p and the graph induced by using existent edges is G_p .

- (•) We use G^n to denote the distribution of graphs of order n . Let q_n be the probability of the existence of "property" Q when the graphs considered are of order n .
- (••) If $\lim_{n \rightarrow \infty} q_n = 1$, then we say " Q " almost always holds. In this case, we say almost all graphs have property " Q ".

Theorem 80 (Gilbert, 1959)

Let p be a constant such that $0 < p \leq 1$. Then, almost all graphs are connected.

Proof.

Suppose not. Then, there are graphs G (of order n) which are not connected. Here, there exists a proper subset $S \subseteq V(G)$ such that $\langle S, V(G) \setminus S \rangle_G$ contains no edges. This implies that the probability q_n of the existence of disconnected graphs of order n satisfies

$$\begin{aligned}
0 &\leq q_n \\
&\leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (1-p)^{k(n-k)} \cdot p^c \quad (c \text{ is a constant}) \\
&\leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} n^k (1-p)^{k(n-k)} \\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (n \cdot (1-p)^{n-k})^k \\
&< \frac{y}{1-y} \quad (\text{where } y = (n \cdot (1-p)^{n-k})).
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} (n(1-p)^{n-k}) = 0$, $\lim_{n \rightarrow \infty} \frac{y}{1-y} = 0$. Hence, $\lim_{n \rightarrow \infty} q_n = 0$ and thus almost all graphs are connected. ■

(•) It is not difficult to see the connectedness should be quite strong, not only 1-connected. We can in fact claim that any cut set of size $k-1$ is not available for a fixed k .

Following the arguments from Theorem 80, we are able to show, if n is large enough, then a random graph obtained from constant p will give a graph with high connectedness.

Theorem 81

For every constant $p \in (0, 1)$ and $k \in \mathbb{N}$, almost all graphs are k -connected.

Proof.

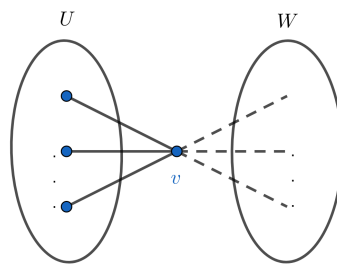
First, we claim G^p has property " $P_{i,j}$ ". ($P_{i,j}$ is the property that for any disjoint vertex sets U and W with $|U| \leq i$ and $|W| \leq j$, there exists a vertex $v \notin U \cup W$ such that $U = N_G(v)$ but $W \cap N_G(v) = \emptyset$. (See

Figure below.) Let $q = (1 - p)$. Then, the probability of such v is $p^{|U|} \cdot q^{|W|} \geq p^i \cdot q^j$. ($|U| \leq i$ and $|W| \leq j$)

Hence, the probability of no such v exists is

$$(1 - p^{|U|} \cdot q^{|W|})^{n-|U|-|W|} \leq (1 - p^i \cdot q^j)^{n-i-j} \text{ for } n \geq i + j.$$

Now, there are at most n^{i+j} $\langle U, W \rangle$ pairs, and thus the probability of " $P_{i,j}$ " (not $P_{i,j}$) is at most $n^{i+j} \cdot (1 - p^i q^j)^{n-i-j}$. By the fact i and j are constants, we have the probability "0" when $n \rightarrow \infty$.



To prove the theorem, let $i = 2$ and $j = k - 1$. Since almost all graphs G have property $P_{2,k-1}$. Let W be an arbitrary set of $k - 1$ vertices. Then, for any two vertices $x, y \in V(G) \setminus W$, then either x is adjacent to y or y have a common neighbor. Therefore, W is not a vertex cut of size $k - 1$. This implies that G is k -connected. ■

Not only the graph is with high connectivity, the graph does have very small diameter.

Theorem 82

Almost all graphs are of diameter 2.

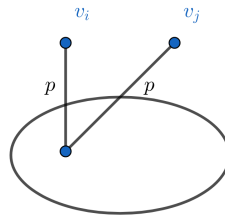
Proof.

Let $\chi_{i,j}$ be the indicator random variable such that

$$\chi_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ do not have a common neighbor; and} \\ 0 & \text{otherwise.} \end{cases}$$

So, the random variable χ of "no two vertices have a common neighbor is equal to $\sum_{j \neq i} \chi_{i,j}$. Hence,

$\mathbb{E}(\chi) = \sum_{j \neq i} \mathbb{E}(\chi_{i,j}) = \binom{n}{2} \cdot (1 - p^2)^{n-2} \rightarrow 0$ and we conclude that almost every pair of distinct vertices have a common neighbor. ■



(*) We may replace p by $p(n)$. Then, we have a more complicated situation to consider the random graphs.

(**) We may also consider the probability of edges based on the vertices they are incident to. That is, the next edge will come from somewhere near a vertex with larger degree.

Problem

Show that almost all graphs there is a unique vertex with maximum degree.

(•••) If this is true, then almost all graphs are of Class 1, since Vizing did prove that a Class 2 graph contains at least three major vertices (degree $\Delta(G)$).

For convenience in referring this result, we put it as a theorem.

Theorem 83

For $p \in (0, 1)$, almost all graphs obtained in Model A is of Class 1.

Proof.

Step 1.

Prove that almost all graphs have a unique major vertex.

Step 2.

By Vizing's result, every Class 2 graph has at least three major vertices, we conclude the proof. ■

Some ideas in Algebraic Graph Theorem.

- (•) The adjacency matrix $A(D)$ of a directed graph D is

$$A(D) = [x_{i,j}]_{n \times n}$$
where $V(D) = \{v_1, v_2, \dots, v_n\}$, $x_{i,j} = 1$ if and only if (v_i, v_j) is an arc of D .
- (•) If D is a graph (instead of digraph), then we view each edge of D as a pair of arcs in opposite directions, and thus $A(D)$ is a symmetric $(0,1)$ -matrix.
- (•) If G is a simple graph, then we can define the *Laplacian* of G , denoted by $L(G) = [l_{i,j}]_{n \times n}$ where $l_{i,i} = \deg_G(v_i)$ and $l_{i,i} = -1 = l_{j,i}$ if $\{v_i, v_j\} \in E(G)$.
- (••) We shall consider $A(G)$ where G is a simple graph in what follows.
- (•) The characteristic polynomial of $A(G)$ is defined as

$$\phi(A, x) =_{def} \phi(G, x) = \det(xI_n - A). \quad (A \approx A(G)).$$

- (•) The spectrum of A (matrix) is a list of its eigenvalues of A , the zeros of $\phi(A, x)$, together with their multiplicities.

Example

$$G \cong C_4.$$

$$\phi(C_4, x) = x^4 - 4x^2; \text{ zeros are } 2, 0, 0, -2 \text{ (0 is of multiplicity 2)}.$$

- (•) The *largest* eigenvalue of a graph G is called the index of G and the spectral radius of G is the maximum value of $\{|\lambda_i| \mid i = 1, 2, \dots, n, \text{ and } \lambda_i \text{ is an eigenvalue of } A\}$. In C_4 case, the index and spectral radius of G are equal.

$$(•) \text{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{pmatrix} \text{ and } \sum_{i=1}^t m_i = n.$$

Example

$$1. \text{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

$$2. \text{Spec}(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

Theorem 84

The diameter of a connected graph G is less than the number of distinct eigenvalues.

$$\text{diam}(K_n) = 1$$

$$\text{diam}(K_{m,n}) = 2$$

Proof.

Let r be the number of distinct eigenvalues, let them be $\lambda_1, \lambda_2, \dots, \lambda_r$. Then $\prod_{i=1}^r (x - \lambda_i)$ is the minimal polynomial of A . This implies that the linear combination of $I_n = A^0, A^1, \dots, A^r$ is the zero matrix. Now, consider $\text{diam}(G)$. Let $d(v_i, v_j) = \text{diam}(G) = k$. Then, $A^h(i, j) = 0$ for each $0 \leq h < \text{diam}(G)$. Hence, $A^0, A^1, \dots, A^k$ are linearly independent. (?) This implies that $k < r$, i.e., $\text{diam}(G)$ is less than the number of distinct eigenvalues. ■

Theorem 85

For every graph G , $\chi \leq 1 + \lambda_{\max}(G)$.

Proof.

For every graph G , let $\chi(G) = k$ and H be a vertex critical subgraph of G . That is, $\chi(H) = k$ and for each vertex $v \in V(H)$, $\chi(H - v) = k - 1$. By Theorem 56, $\delta(H) \geq k - 1$.

Now, consider $\lambda_{\max}$. If λ is an eigenvalue of $A(G)$, then there exists an eigenvector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Let

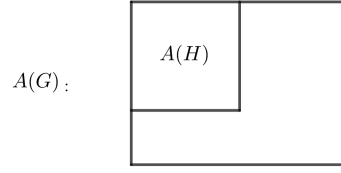
$x_j = \max_{i=1}^n \{x_i \mid \vec{x} = (x_1, x_2, \dots, x_n)\}$. Then

$$\lambda x_j = (A\vec{x})_j = \sum_{v_i \in N(v_j)} x_i \leq \deg_G(v_j) \cdot x_j \leq \Delta(G) \cdot x_j.$$

Hence, $\lambda_{\max} \leq \Delta(G)$ and Theorem 85 is an improvement of Brooks

Thm. On the other hand, $\lambda_{\max} \geq \delta(G)$. (?) This implies that

$$k \leq 1 + \delta(H) \leq 1 + \lambda_{\max}(H) \leq 1 + \lambda_{\max}(G). \quad \blacksquare$$



Theorem 86

For each bipartite graph G , there exists a graph $\tilde{G}$ such that $G \preceq \tilde{G}$ and $\tilde{G}$ is $\Delta(G)$ -regular.

Proof.

Let $G = (A, B)$ with $|A| \leq |B|$. Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. ($m \leq n$) First, we construct a graph $\bar{G} = (\bar{A}, \bar{B}) : (A \cup B', B \cup A')$ where $A' = \{a'_1, a'_2, \dots, a'_m\}$ and $B' = \{b'_1, b'_2, \dots, b'_n\}$ and $E(\bar{G}) = E(G) \cup \{\{b'_j, a'_i\} \mid a_i b_j \in E(G)\}$. (See Figure for example.) In fact, $\langle A \cup B \rangle_{\bar{G}} \cong \langle A' \cup B' \rangle_{\bar{G}}$. Hence,

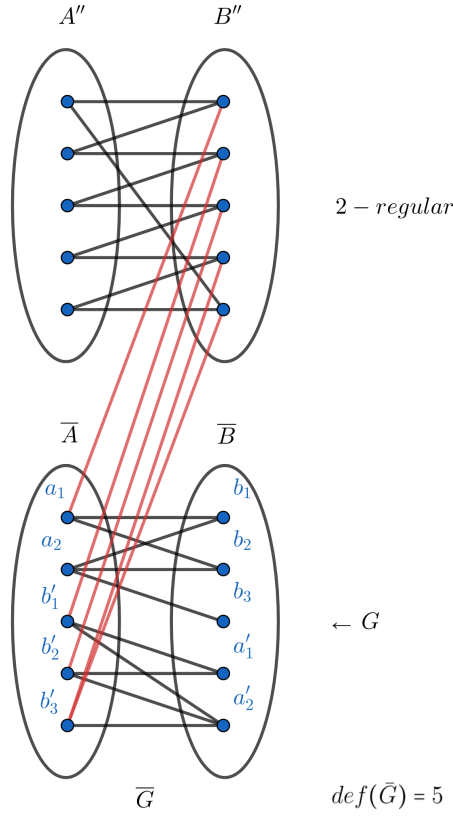
$$\sum_{v \in A \cup B'} \deg_{\bar{G}}(v) = \sum_{v \in B \cup A'} \deg_{\bar{G}}(v) = 2\|G\|, \quad |\bar{A}| = |\bar{B}|,$$

$$\sum_{v \in A \cup B'} (\Delta(G) - \deg_{\bar{G}}(v)) = \sum_{v \in B \cup A'} (\Delta(G) - \deg_{\bar{G}}(v)) = \text{def}(\bar{G}).$$

Now, based on $\text{def}(\bar{G})$, we have two cases to consider.

Case 1. $\text{def}(\bar{G}) \geq \Delta(G)$.

Construct a bipartite graph G'' such that $G'' = (A'', B'')$, $|A''| = |B''| = \text{def}(\bar{G})$ and G'' is $(\Delta(G) - 1)$ -regular. The proof then follows by connecting A'' to $\bar{B}$ and B'' to $\bar{A}$.



Case 2. $def(\bar{G}) < \Delta(G)$.

Let $|A''| = |B''| = \Delta(G)$, and $G'' \cong K_{\Delta(G), \Delta(G)} - M$, M is a matching of size $def(\bar{G})$. The proof follows by connecting the vertices in G'' (which are incident to M) and the vertices in $\bar{G}$ whose degrees are less than $\Delta(G)$.

So, the graph $\tilde{G}$ is defined on $(A'' \cup \bar{A}, B'' \cup \bar{B})$. Moreover, $\tilde{G}$ is $\Delta(G)$ -regular with partite set size $\max\{|A| + |B| + def(G), |A| + |B| + \Delta(G)\}$.

Theorem 87

Let G be a planar graph with $\Delta(G) \geq 10$, then G is of Class 1.

Proof.

We shall apply a lemma obtained by Vizing.

Vizing's adjacency lemma

First Form: If G is of Class 2, then every vertex of G is adjacent to at least two major vertices. In particular, G contains at least three major vertices.

Second Form: Let G be a connected graph of Class 2 that is minimal with respect to edge coloring. "If $uv \in E(G)$ and $\deg_G(u) = m$, then v is adjacent to at least $\Delta(G) - m + 1$ major vertices." $\rightarrow$ (*)

We shall apply the 2nd form to prove the theorem. Suppose not. Let G be a counterexample with minimum size. Thus, G is planar, $\Delta(G) = k \geq 10$ and $\chi'(G) = k + 1$. Clearly, G is minimal with respect to chromatic index. Since G is planar, G contains vertices of degree 5 or less, let S be the set of all such vertices. Define $H = G - S$. Again, H is planar, H contains a vertex w such that $\deg_H(w) \leq 5$. By the fact $\deg_G(w) > 5$, w is adjacent to some vertices of S . Let $vw \in E(G)$ where $v \in S$. In G , $\deg_G(v) \leq 5$. By (*), w is adjacent to at least $\Delta(G) - 5 + 1 (\geq 6)$ (major) vertices of degree $\Delta(G)$. This implies that w is adjacent to at least 6 vertices of H since all major vertices are in H . Hence, $\deg_H(w) \geq 6$. $\rightarrow \leftarrow$. This concludes the proof. ■

(•) $\sigma(G)$: Vertex cover number; $\alpha_1(G)$: matching number

Theorem 88 (König-Egeváry)

For bipartite graphs G , $\sigma(G) = \alpha_1(G)$.

Proof.

We shall apply max-flow min-cut theorem to prove the theorem. First, we define a network as in Figure 49. Let $A = \{a_1, a_2, \dots, a_m\}$, $B = \{b_1, b_2, \dots, b_n\}$ and $G = (A, B)$. Then, the network is defined by letting u and v be source and sink respectively, $\text{cap}(u, a_i) = 1$ for $i = 1, 2, \dots, m$, $\text{cap}(b_j, v) = 1$ for $j = 1, 2, \dots, n$, and $\text{cap}(a_i, b_j) = |G| + 1$ if $a_i b_j$ is in $E(G)$. (Note that all arcs are from A to B .)

Since $\sigma(G) \geq \alpha_1(G)$ as mentioned above, it suffices to show that $\alpha_1(G) \geq \sigma(G)$.

Now, let f be a maximum flow. It is easy to see that $\text{val } f = \alpha_1(G)$. This is due to the fact that all the arcs from u and into v are of capacity 1. (No two arcs can be out of two vertices in A and ended in a vertex of B .)

So, it is left to consider the minimum cut, let it be $K = (X, \bar{X})$ where $u \in X$, $v \in \bar{X}$, $A \cap \bar{X} = A'$ and $B \cap \bar{X} = B'$, see Figure 50. Hence, K contains arcs from u to A' , $A \setminus A'$ and $B \setminus B'$, see Figure 50. Hence, K contains arcs from u to A' , $A \setminus A'$ to $B \setminus B'$ and B' to v . Notice that $\text{cap } K \leq |G|$, for example, let $X = \{u\}$. This implies that the following $(\bullet)$ is true and $A' \cup B'$ is a vertex cover and $\text{cap } K = |A'| + |B'| = \text{val } f = \alpha_1(G)$. By the fact, $|A'| + |B'| \geq \sigma(G)$, we have $\alpha_1(G) \geq \sigma(G)$. ■

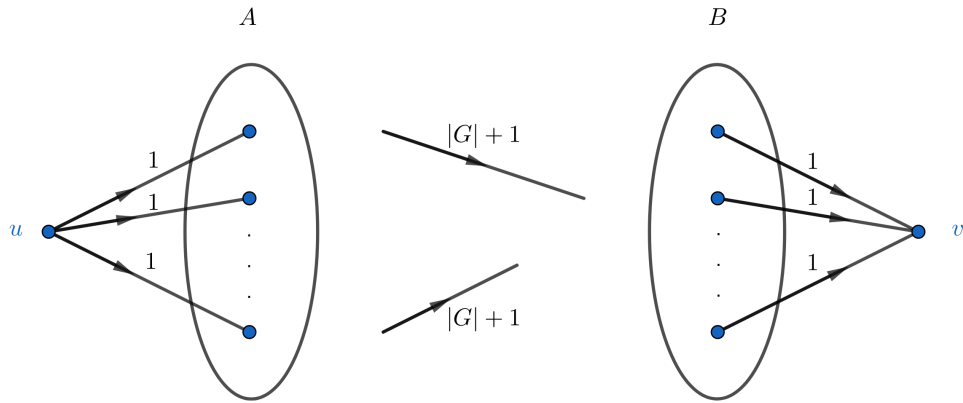


Figure 49. Net work

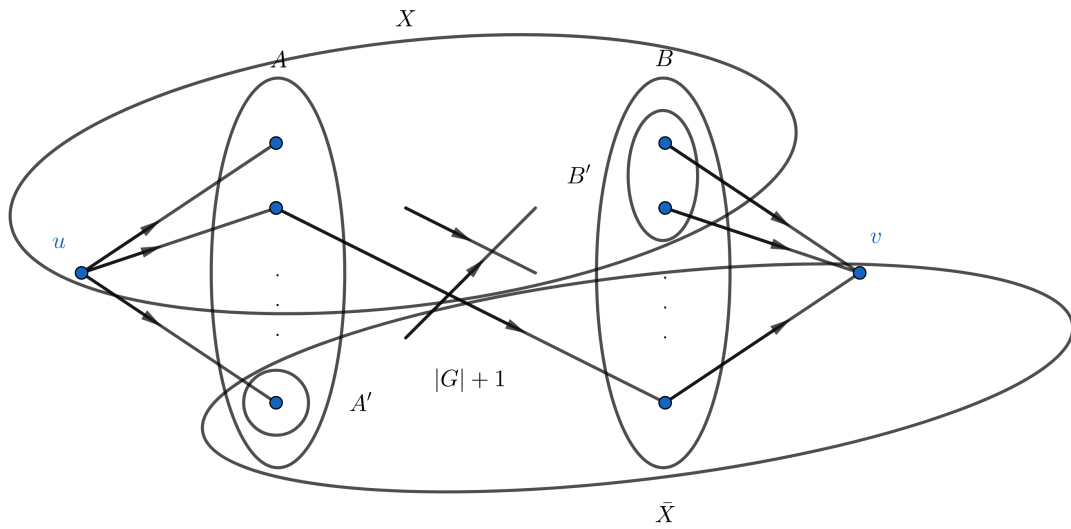


Figure 50. Cut

- (•) In $(X, \bar{X})$, there exist no edges from $A \setminus A'$ to $B \setminus B'$. For otherwise, it is not a minimum cut. This implies that all edges are incident vertices in $A' \cup B'$. $A' \cup B'$ is a vertex cover.

Theorem 89

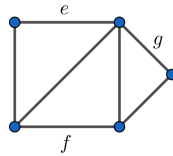
Let G be a graph of order p which has no isolated vertices. Then,
 $\alpha(G) + \sigma(G) = p$.

Proof.

Let S be a vertex cover with $\sigma(G)$ vertices of G . Then, $V(G) \setminus S$ is an independent set. Hence, $|V(G) \setminus S| \leq \alpha(G)$ and thus $p - |S| \leq \alpha(G)$. This implies that $p \leq \alpha(G) + |S| = \alpha(G) + \sigma(G)$. On the other hand, let T be an independent set of G such that $|T| = \alpha(G)$. Then, $G - T$ is vertex cover of G . By the fact $\sigma(G) \text{ (min.)} \leq |G - T| = p - \alpha(G)$, we have $p \geq \sigma(G) + \alpha(G)$. ■

Definition (Edge-cover)

An edge cover of a graph is a set of edges M such that all vertices of G are incident to M , i.e., for each $v \in V(G)$, v is incident to an edge in M . e.g.



$\{e, f, g\}$ is an edge cover.

The edge cover number of G , $\sigma_1(G) = \min.\{|M| \mid M \text{ is an edge cover}\}$.

(•) $\sigma_1(G) \geq \left\lceil \frac{|G|}{2} \right\rceil$. (Each edge can cover two vertices.)

Theorem 90

$\sigma_1(G) + \alpha_1(G) = p$. (G is a connected graph.)

Proof.

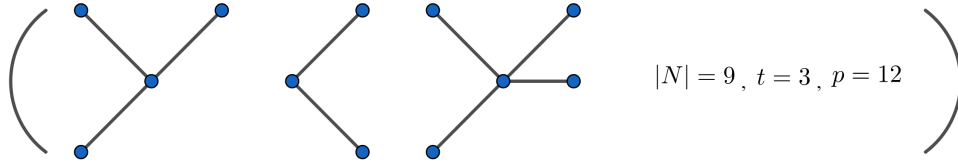
Let M be a matching in G with $\alpha_1(G)$ edges. Then, for each vertex

not in M , v is incident to a vertex in M if v is in an edge of G .

Assume that t vertices not in M , i.e., $p = 2|M| + t$. Now, by taking every edge in the matching M and the set of t edges not in M but incident to M , we have an edge cover with $|M| + t$ edges. This implies that $\sigma_1(G) \leq |M| + t$. As a consequence, we have $p = 2|M| + t = \alpha_1(G) + |M| + t \geq \alpha_1(G) + \sigma_1(G)$. On the other hand, let N be an edge cover of G with minimum number of edges, i.e., $|N| = \sigma_1(G)$. Notice that $\langle N \rangle_G$ is a disjoint union of stars. (You can not find



in $\langle N \rangle_G$.) Assume that there are t stars. Then, $p = |N| + t$. By the fact that in $\langle N \rangle_G$ we can find a matching of size t , $p \leq |N| + \alpha_1(G)$. ($\alpha_1(G)$ (maximum matching number) $\geq t$.) ■



Theorem 90'

Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_p\}$. Then, we can use the vertices of G as variables to obtain a generating function for all dominating sets of G .

Proof.

Let f be defined as follows:

$$f(v_1, v_2, \dots, v_p) = \prod_{i=1}^p (v_i + \sum_{u \in N_G(v_i)} u).$$

Then, each summand is a product $v_1^{c_1}v_2^{c_2}\cdots v_p^{c_p}$ where $0 \leq c_j \leq p$. Now, let $S = \{v_j \mid c_j > 0, j = 1, 2, \dots, p\}$. If $u \in V(G) \setminus S$, say $u = v_k$, then in the product $v_1^{c_1}v_2^{c_2}\cdots v_p^{c_p}$, $c_k = 0$. But, one of its neighbor has been selected. This implies that v_k is incident to a vertex of S . ■

(•) For small order graphs, this is a good way to find dominating sets.

In fact, the term with maximum of "0" in powers provide a dominating set with minimum size and thus the domination number is determined.

Theorem 91

Let G be a (p, q) -graph and $A(G) = A$. Then,

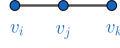
- (1) the number of triangles in G is $\frac{1}{6}tr(A^3)$;
- (2) the number of 4-cycles in G is $\frac{1}{8}[tr(A^4) - 2q - \sum_{i \neq j} a_{i,j}^{(2)}]$ where $a_{i,j}^{(2)}$ is the (i, j) -entry in A^2 ; and
- (3) the number of 5-cycles in G is $\frac{1}{10}[tr(A^5) - 5tr(A^3) - 5 \sum_{i=1}^p \sum_{j=1}^p (a_{i,j} - 2) \cdot a_{i,i}^{(3)}]$.

Proof.

It follows from the fact that the number of walks of length k from v_i to v_j is equal to $A^{(k)}(i, j)$. This can be proved by induction on k .

Hence, if triangles are concerned, then we consider $A^{(3)}(i, i)$, i.e., $tr(A^3)$. Since for each triangle (v_i, v_j, v_k) , there are 6 different ways of 3-walks: $\langle v_i, v_j, v_k, v_i \rangle$, $\langle v_i, v_k, v_j, v_i \rangle$, $\langle v_j, v_i, v_k, v_j \rangle$, $\langle v_k, v_i, v_j, v_k \rangle$ and $\langle v_k, v_j, v_i, v_k \rangle$, the result follows by using $\frac{1}{6}tr(A^3)$. For 4-cycles and

5-cycles, we have to take away those 4-walks (and 5-walks) which are not for cycles, for example



$\Rightarrow \langle v_i, v_j, v_k, v_i \rangle$. Check (2) and (3) yourself. ■

Theorem 92

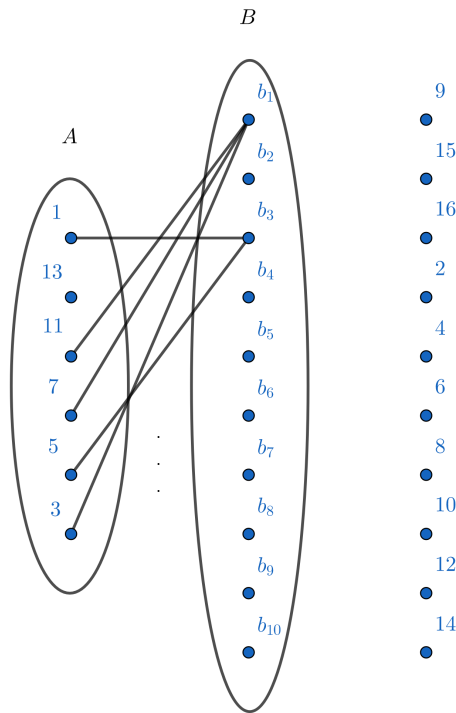
Let $G = (A, B)$ be a tree of order at most 16. Then, G has a prime labeling.

Proof.

(Sketched) We present a proof of the case $|A| = 6$ and $|B| = 10$. First, we label A by using $S = \{1, 13, 11, 7, 5, 3\}$. Then, the labeling of the vertices can be found from the set $\{1, 2, \dots, 16\} \setminus S$. Now, denote $B = \{b_1, b_2, \dots, b_{10}\}$. We claim that there exists a labeling of the vertices of B by using $[16] \setminus S$.

Let $B_i = \{x \in [16] \setminus S \mid \gcd(x, s) = 1 \text{ for each } x \in A \cap N_G(b_i)\}$. Now, consider $|S_k| = |\bigcup_{j=1}^k B_{i_j}|$. Clearly, if $k = 1$, then $|S_k| \geq 1$ since $\gcd(2, s) = 1$ for each $s \in S$. In fact, each B_i contains at least 4 elements, 2, 4, 8 and 16.

But the fact, "any two B_i 's have at most one common neighbor", we can verify that $|S_k| \geq k$ for $6 \leq k \leq 10$. Hence, by Hall's condition, the proof follows. ■



Theorem 93 (Alon, 1995)

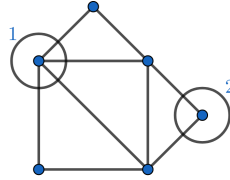
Let G be a graph of order p . Then, $\gamma(G)$ (the domination number) of G , $\gamma(G) \leq n[1 + \ln(\sigma(G) + 1)/(\sigma(G) + 1)]$.

Proof. (Probabilistic Method)

Let S be a subset of $V(G)$ with the probability of each vertex $p =_{\text{def}} \frac{\ln(\sigma(G) + 1)}{\sigma(G) + 1}$. Let $T = \{x \mid x \notin S, N_G(x) \cap S = \emptyset\}$. Since for each $y \notin S \cup T$, $N_G(y) \cap S \neq \emptyset$, $S \cup T$ is a dominating set of G . By the expectation of $\mathbb{E}(S \cup T) = \mathbb{E}(S) + \mathbb{E}(T) \leq np + n \cdot (1 - p)^{\sigma(G) + 1} \leq np + n \cdot e^{-p(\sigma(G) + 1)} = n \left(p + \frac{1}{\sigma(G) + 1} \right)$. This implies that there exists a dominating set of size at most $n \cdot [1 + \ln(\sigma(G) + 1)/(\sigma(G) + 1)]$. ■

(•) A greedy algorithm for finding the dominating set:

Choose the vertices of a dominating set one by one following the idea: A vertex that covers the "maximum number" of vertices which are not cover yet is pick.



" p is not a constant."

Theorem 94 (Omit the proof)

Let $p = p(n)$. Then, we have

- (1) $p(n) = n^{-2} \longrightarrow$ No edges.
- (2) $p(n) = n^{-\frac{3}{2}} \longrightarrow G$ has a nontrivial component which grows like a tree.
- (3) $p(n) = n^{-1} \longrightarrow$ Contain a cycle.
- (4) $p(n) = \frac{\ln n}{n} \longrightarrow$ Connected.
- (5) $p(n) = (1 + \varepsilon) \longrightarrow$ Contains a Hamilton cycle.
- (•) The growth rate is getting smaller.

Theorem 95 (More about eigenvalues of $A(G)$)

Let G be a connected graph of order p and A be its adjacency matrix. Then, we have the following basic properties.

- (1) For each eigenvalue λ , $|\lambda| \leq \Delta(G)$.

- (2) $\Delta(G)$ is an eigenvalue of G if and only if G is regular. Moreover, if $\Delta(G)$ is an eigenvalue of G , then the multiplicity of $\Delta(G)$ is 1.
- (3) If $-\Delta(G)$ is an eigenvalue of G , then G is regular and bipartite.
- (4) If G is bipartite and λ is an eigenvalue then $-\lambda$ is also an eigenvalue, moreover, they have the same multiplicity.
- (5) The maximal eigenvalue, $\lambda_{\max}(G)$ satisfies $\delta(G) \leq \lambda_{\max}(G) \leq \Delta(G)$.
- (6) If $H \leq G$, then $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$.

Proof.

- (1) Let $\vec{x}$ be an eigenvector with eigenvalue λ , i.e., $A\vec{x} = \lambda\vec{x}$. Let $\vec{x} = (x_1, x_2, \dots, x_p)$ and $|x_i| \leq 1$ (by re-scaling $\vec{x}$). Suppose that $|x_j| \geq |x_i|$ for each $i = 1, 2, \dots, p$. For convenience, let $x_j = 1$. ($|x_j|$ 最大, 則令 $\vec{x} \leftarrow \vec{x} / \pm |x_j|$.) Then,

$$\begin{aligned}
 |\lambda| &= |\lambda \cdot x_j| \\
 &= \left| \sum_{i=1}^p a_{j,i} \cdot x_i \right| \\
 &\leq \sum_{i=1}^p a_{j,i} \cdot |x_i| \\
 &\leq |x_j| \cdot \deg_G(v_j) \\
 &\leq \Delta(G).
 \end{aligned}$$

■

- (2) If $\Delta(G) = \Delta$ is an eigenvalue, then as in (1), let $|x_j| = 1$, and we have $\Delta = \Delta \cdot x_j = \sum_{i=1}^p a_{j,i} \cdot x_i$. Hence $x_i = x_j = 1$ and $\deg_G(v_j) = \Delta$, whenever $v_i \sim_G v_j$. Therefore, $\deg_G(x_i) = \Delta$. Now, by the same argument, if $v_k \sim v_i$, $\deg_G(v_k) = \Delta$, then G is Δ -regular by the fact

that G is connected. This also implies that the eigenvector is

$\vec{1} = (1, 1, \dots, 1)$. The reverse statement is easy to see.

- (3) If $-\Delta(G)$ is an eigenvalue, then by (2) we have $\deg_G(v_j) = \Delta$ and $x_i = -x_j = -1$ whenever $v_i \sim_G v_j$. Since two vertices are adjacent if they have distinct weights (x_i and x_j) 1 and -1, the vertex set of G can be partitioned into two subset V_1 and V_2 such that $v_i \sim_G v_j$ iff their corresponding weights are different (1 or -1). Hence, G is bipartite.

- (4) It follows by considering $\ker(A - \lambda I_p)$ and $\ker(A + \lambda I_p)$. Let $G = (V_1, V_2)$ and $\vec{b}(b_1, b_2, \dots, b_p)$ such that $b_i = 1$ if $v_i \in V_1$ and $b_i = -1$ if $v_i \in V_2$. Now, if $A\vec{x} = \lambda\vec{x}$ and $v_i \in V_1$, then

$$A \cdot (\vec{b} \otimes \vec{x})_i = \sum_{j=1}^p a_{ij} \cdot b_j \cdot x_j = \sum_{v_j \in V_1} a_{ij} x_j (= 0(v_i \in V_1)) - \sum_{v_j \in V_2} a_{i,j} x_j = -\sum_{v_j \in V_1} a_{ij} x_j - \sum_{v_j \in V_2} a_{i,j} x_j = -\sum_{j=1}^p a_{i,j} x_j = -\lambda x_i = -\lambda(\vec{b} \otimes \vec{x})_i.$$

$$\vec{b} \otimes \vec{x} =_{def} (b_1 x_1, b_2 x_2, \dots, b_p x_p)$$

This implies λ and $-\lambda$ occur the same number of times in solving $A\vec{x} = \lambda\vec{x}$, i.e., $m(\lambda)$ (multiplicity of λ) $= m(-\lambda)$.

- (5) By (1), we have $\lambda_{\max}(G) \leq \Delta(G)$. Now, we claim the other inequality. Let the numerical range of A be $V(A)$, i.e.,

$$V(A) = \{\langle A\vec{x}, \vec{x} \rangle = \vec{x}^t A \vec{x} \mid |\vec{x}| = 1\}. \text{ Hence, let } \vec{1} = (1, 1, \dots, 1) \text{ and we have } \frac{1}{p} \langle A\vec{1}, \vec{1} \rangle \in V(A).$$

$$\text{Now, } \lambda_{\max} = \max V(A) \geq \frac{1}{p} \langle A\vec{1}, \vec{1} \rangle = \frac{1}{p} \sum_{k=1}^p \deg_G(v_k) \geq \delta(G).$$

- (6) Let H be an induced subgraph of order $p-1$, i.e.,

$$H = \langle \{v_1, v_2, \dots, v_{p-1}\} \rangle_G. \text{ Then, } \lambda_{\max}(H) = \langle A'\vec{y}, \vec{y} \rangle \text{ where}$$

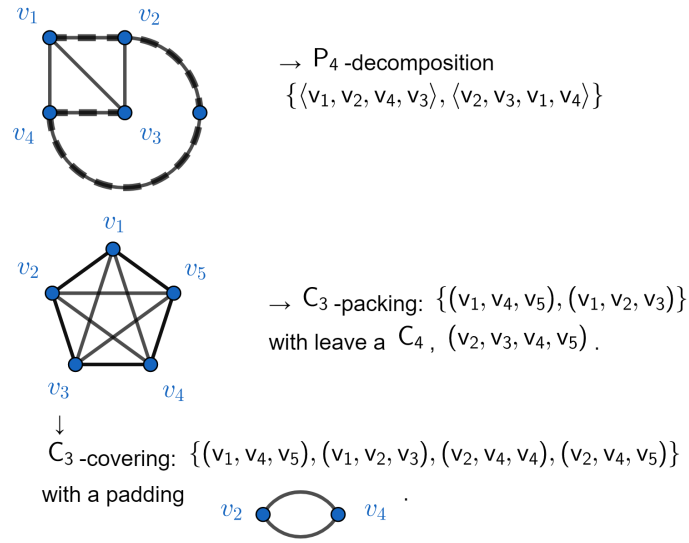
$$A' = A(H) \text{ and } \langle \vec{y}, \vec{y} \rangle = 1. \text{ Now, consider } \vec{x} = (y_1, y_2, \dots, y_{p-1}, 0)$$

where $\vec{y} = (y_1, y_2, \dots, y_p)$. Clearly, $\langle A'\vec{x}, \vec{x} \rangle = \langle A'\vec{y}, \vec{y} \rangle = \lambda_{\max}(H)$ and

$\langle \vec{x}, \vec{x} \rangle = 1$. Since $\langle A\vec{x}, \vec{x} \rangle \in V(A)$, i.e., $\lambda_{\max}(H) \in V(H)$. This implies that $\lambda_{\max}(G) \geq \lambda_{\max}(H)$. The other inequalities can be shown similarly. ■

- (•) A graph G has an $\mathcal{H}$ -decomposition if $E(G)$ can be partitioned into subsets $E_1, E_2, \dots, E_k$ such that for each $i = 1, 2, \dots, k$, $\langle E_i \rangle_G \in \mathcal{H}$.
- (•) If $\mathcal{H} = \{H\}$, then an $\mathcal{H}$ -decomposition of G can be referred to as an H -decomposition of G .
- (•) A graph G has an H -packing if $E(G)$ contains edge-disjoint subsets such that each of them induces a graph in $\mathcal{H}$. An H -packing of G can be defined accordingly.
- (•) A graph G has an H -covering if $E(G)$ is a subset of a edge-disjoint union of graphs in $\mathcal{H}$. An H -covering of G can be defined as well.
- (••) In packing, the edges not used induce a subgraph which is known as the "leave" of the packing. Similarly in covering, the extra edges used induce a padding of the covering.
- (•) If the graph G is the complete graph K_n , then the H -decomposition, H -packing and H -covering is also referred to as that of order n .

Examples



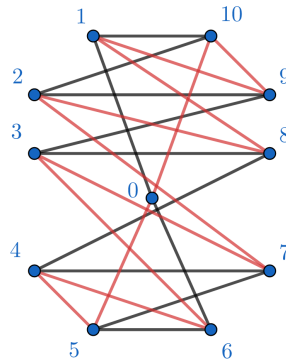
Theorem 96

For each odd integer $n \geq 3$, K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles. For each even integer n , K_n can be decomposed into $\frac{n}{2}$ Hamilton paths.

Proof.

The following construction is known as the Waleki's method. Let $V(K_n) = \mathbb{Z}_n$ and the cycles are:

$$\begin{aligned}
 &\left(0, 1, n-1, 2, n-2, \dots, \frac{n-1}{2}, \frac{n+1}{2}\right), \left(0, 2, 1, 3, n-1, \dots, \frac{n+1}{2}, \frac{n+3}{2}\right), \\
 &\left(0, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n+1}{2}, \frac{n-5}{2}, \dots, n-2, n-1\right). \quad \blacksquare
 \end{aligned}$$



By deleting a vertex in K_{n+1} (n even), we obtain the decomposition of K_n into $\frac{n}{2}$ Hamilton paths. ■

- (*) This theorem has been extended to cycles and paths with prescribed length. We list the theorem and omit their proofs.
(The cycle case is very complicate.)

Theorem 96' (Alspach et. al, 2001)

For each odd integer larger than 3 and an integer $3 \leq m \leq n$, the complete graph K_n (n is odd) and $K_n - I$ (n is even) can be decomposed into m -cycles provided $m \mid \binom{n}{2}$ (for odd n) and $m \mid \binom{n}{2} - \frac{n}{2}$ (for even n) respectively.

- (•) The case $m = 3$ was proved in 1847 by T. P. Kirkman, and the case $m = n$ mentioned above eas obtained long time ago.

- (*) An important tool for decomposition.

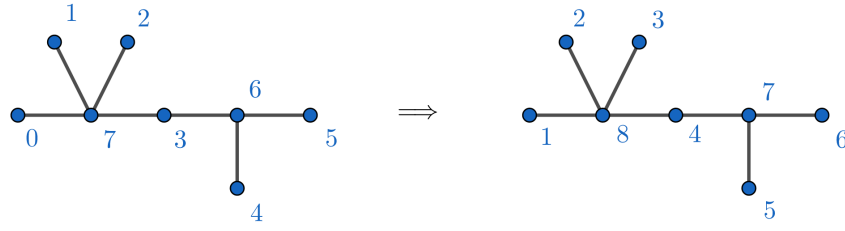
Definition (Graceful labeling, β -labeling)

A graceful labeling of a graph G is a 1-1 mapping $f : V(G) \rightarrow \{0, 1, 2, \dots, \max\{\|G\|, |G| - 1\}\}$ such that the weights of edges uv defined by $|f(u) - f(v)|$ are all distinct.

If G is connected, this *value* takes $\|G\|$.

Examples

(Shown easiler!)



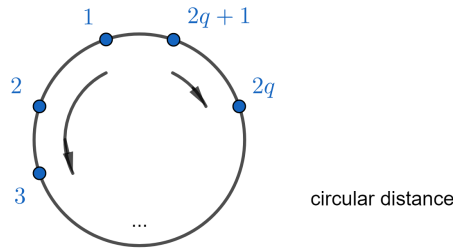
Theorem 97

Let G be a graph of size q and G has a graceful labeling. Then, K_{2q+1} can be decomposed into $2q + 1$ copies of G .

Proof.

Let $V(K_{2q+1}) = \mathbb{Z}_{2q+1}$. By arranging the vertices on a cycle, see Figure below, we notice that any two vertices have a circular distance at most q . More precisely, $\text{dist}(i, j) = \min\{j - i, (2q + 1) - (j - i)\}$ (for $j > i$).

Now, we can add the labels of G for each one of them (taking modulo $2q + 1$) and obtain the desired decomposition.

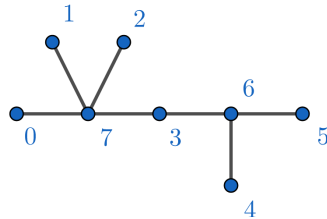


- (•) If G has a graceful labeling f and the labeling has an extra property such that $\exists c_f \in \mathbb{Z}_{2q+1}$ satisfying for each uv either $f(u) \geq c_f > f(v)$ or $f(v) \geq c_f > f(u)$, then G has an α -labeling.

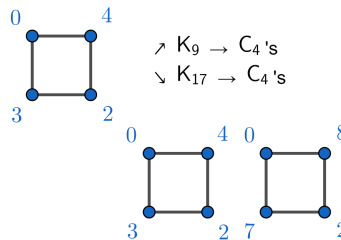
The idea of Theorem 97 is known as "difference method" and G with labeling is a "base graph".

Example

The labeling of the following graph



is an α -labeling since we can choose $c_f = 3$ or 4 or 3.5. The following labeling is also an α -labeling, $c_f = 2.5$.



Theorem 98

If G has an α -labeling and $\|G\| = q$, then $G \mid K_{2tq+1}$ where $t \in \mathbb{N}$.

($G \mid K_{2tq+1}$ denotes K_{2tq+1} has a G -decomposition.)

Proof.

By Theorem 97, $G \mid K_{2q+1}$ can be obtained by a graceful labeling of G . Now, if G has an α -labeling, we may change the labels to find t base graphs for the decomposition of K_{2tq+1} . As mentioned above on the case of C_4 's, for each label larger than c_f , we add $q, 2q, \dots, (t-1)q$ respectively. This gives a collection of t base graphs (with labels). By difference method, we have the proof. (All differences from 1 to tq have been used exactly once.) ■

- (*) If G has an α -labeling, then G must be a bipartite graph. The two partite sets of G are obtained by using the labels, larger than c_f and smaller than c_f respectively.
- (•) A graph G may have β -labeling but not α -labeling.
- (**) One of the most beautiful conjectures on labelings is "Graceful Tree Conjecture": Every tree has a graceful labeling. Of course, you may also conjecture that every tree has an α -labeling (but this is in general not true).
- (•) We have shown that for each graph G there exists a $\Delta(G)$ -regular graph H such that $G \leq H$. In fact, we can say more about this type of supergraph.

Theorem 99

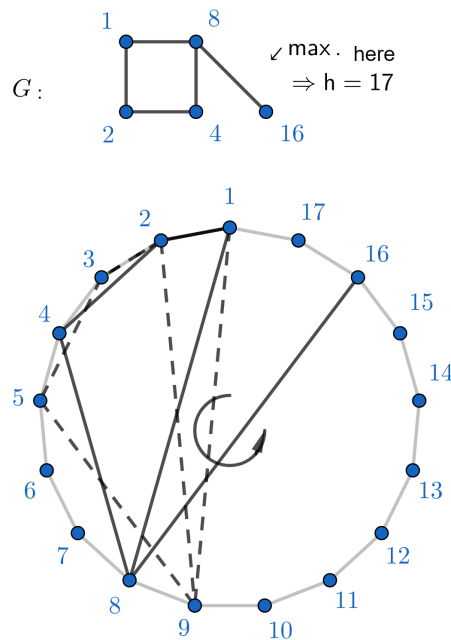
Let G be a graph of size q without isolated vertices. Then, there exists a regular graph H such that $G \mid H$. More precisely, H is a $2q$ -regular graph.

Proof.

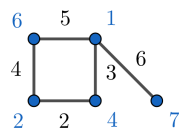
Let $G = \{v_1, v_2, \dots, v_p\}$ and f is a labeling of G such that $f(v_i) = 2^{i-1}$. Then, all edges will receive distinct weights $|f(u) - f(v)|$. We shall construct a $2q$ -regular graph of order h such that $h = 1 + 2 \max\{|f(v_i) - f(v_j)| \mid v_i v_j \in E(G)\}$.

Let $V(H) = \mathbb{Z}_h$ and G be the graph with its vertices the labels from f , see figure for an example. Then, by difference method, the base graph will generate a regular graph which use the weights in G exactly once. Since $\|G\| = q$, the graph obtained H will be a $2q$ -regular graph of order h . ■

(*) We can decrease the order of H by giving another labeling satisfying all $|f(u) - f(v)|$'s are different for $uv \in E(G)$.



Another example



H : 10-regular graph of order 13.

- (•) "Graph Decomposition" is one of the most important topic in Graph Theory, many results are also related to the study of Combinatorial Designs.

(*) Problem. For which graph G , $K_3 \mid G$?

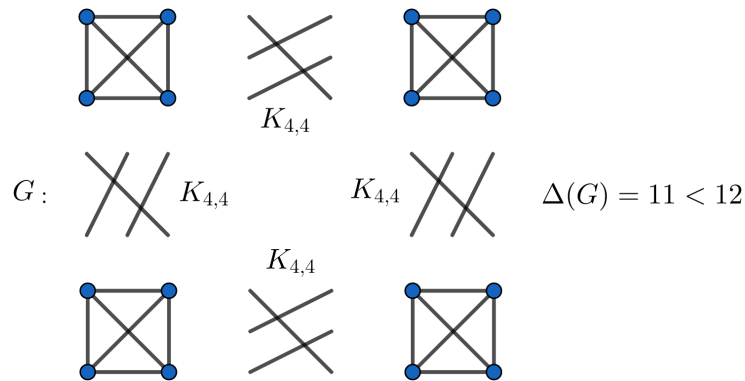
(*) Problem. For which graph G , $C_3 \mid G$?

Theorem 100

There exists a graph G with $\delta(G) < \frac{3}{4}|G|$ such that $K_3 \nmid G$.

Proof.

For general n , the construction is similar to the following graph of order 16.



Since there are four bipartite subgraphs $K_{n,n}$ in G , the K_3 -decomposition needs to use up all these edges by using one from K_n and two from $K_{n,n}$. ($K_{n,n}$ contains no odd cycles!) Hence, we need at least $\frac{1}{2}(4n^2)$ edges from four K_n 's. But, $4K_n$ has $2n(n-1)$ edges which are not enough! The K_3 -decomposition of G is not possible. ■

Nash-Williams Conjecture

For any graph G of order p , G has a K_3 -decomposition provided $\delta(G) \geq \frac{3}{4}p$ and $3 \mid \|G\|$.

How about C_4 -decomposition? Keep moving forward!