

Transposition Distance

(•) The term "transposition" comes from biology and refers to "transposons", which are sequences of DNA that can be displaced in a genome.

Note: Algebraic transposition is a 2-cycle in a permutation, for (ij) which represents $i \rightarrow j$ and $j \rightarrow i$.

Definition (Interval of permutation).

An interval (or segment) of a permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ is a set $\{\pi_a, \pi_{a+1}, \dots, \pi_b\}$ with $1 \leq a \leq b \leq n$. The absolute value is meaningful for signed permutation, and the elements π_a and π_b are the extremities of the interval.



Definition (Transposition $\tau(i, j, k)$)

For any permutation π in S_n , the transposition $\tau(i, j, k)$ with $1 \leq i < j < k \leq n+1$ applied to π exchanges the intervals

transforming π into $\pi \circ \tau(i, j, k)$. Therefore,

$$T(i, j, k) = \begin{pmatrix} 1 & 2 & \dots & i-1 & \boxed{i \dots j-2 \ j-1} & j & j+1 & \dots & k-1 & \boxed{k \dots n} \\ 1 & 2 & \dots & i-1 & \boxed{j \ j+1 \dots k-1} & i & \dots & j-2 \ j-1 & k & \dots & n \end{pmatrix}$$

(*) 把由 i 到 $j-1$ 的 interval 移到 $k-1$ 的後面, k 的前面。
(插入於 $k-1$ 与 k 之間)。

(*) We use $td(\pi)$ to denote the transposition distance of π ,
i.e., after $td(\pi)$ Transpositions, we end it up with i (identity).

e.g. $\pi = (5 \ 1 \ 3 \ 2 \ 4)$

$$\pi \circ \gamma(3, 4, 5) = (5 \mid \boxed{2} \boxed{3} 4)$$

2.9. $\pi = (\boxed{4} \mid \boxed{6} \mid \boxed{2, 5} \mid 7, 3)$, $\tau(3, 5, 6)$

$$\pi \circ \tau(3, 5, 6) = (4 \mid 5 \boxed{6} 7 \mid 3)$$

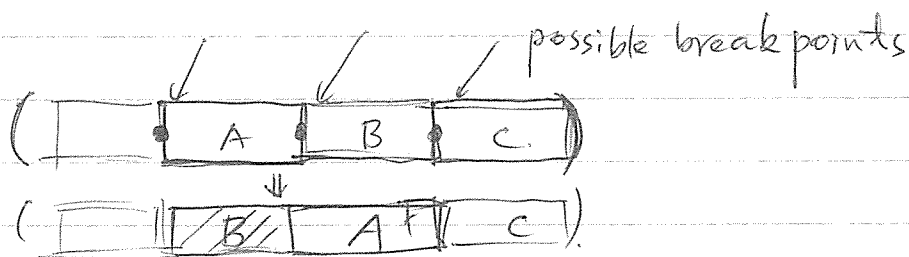
$\uparrow$
 $j-1 = k-1$

Proposition Let the number of breakpoints^{of π} be $\text{bp}(\pi)$. Then,

$$\text{Ad}(\pi) \geq \frac{\text{bp}(\pi)}{3}$$

Proof.

一次最多三个 b.p.!



Breakpoints $\begin{cases} \text{ascents} & \pi_{i+1} > \pi_i \quad (\text{But, not } \pi_{i+1} = \pi_i + 1) \\ \text{descents} & \pi_{i+1} < \pi_i \end{cases}$

(*) $\text{des}(\pi)$: The # of descents in π .

Proposition $\forall \pi \in S_n, \text{td}(\pi) \geq \max\left\{ \frac{\text{des}(\pi)}{2}, \frac{\text{des}(\pi^{-1})}{2} \right\}$.
It suffices to claim that the # of descents decrease only at most 2.

Proof. $\wedge$ Suppose not.

Let $\pi_{i-1} = a, \pi_i = b, \pi_{j-1} = c, \pi_j = d, \pi_{k-1} = e$ and $\pi_k = f$. Moreover,

$a > b, c > d$ and $e > f$. Following the transposition $\tau(i, j, k)$,

we have $a - d; e - b; c - f$; Also, $a < d, e < b$ and $c < f$.

But, this is not possible. If $a < d$ and $e < b$, then

$c > d > a > b > e > f \rightarrow \leftarrow$ If $a < d$ and $c < f$, then

$b < a < d < c < f < e \rightarrow \leftarrow$ Finally, if $e < b$ and $c < f$, then

$a > b > e > f > c > d \rightarrow \leftarrow$

Since the distance between π and i is the same as the distance between i and π^{-1} , and $\text{des}(\pi)$ may be different from $\text{des}(\pi^{-1})$, the assertion holds. ■

(*) A strip _{π} is a maximal interval of π containing no breakpoints of π .

(*) Reduced permutation

$$\underline{4\ 5\ 6} \cdot 3 \cdot 1 \cdot \underline{7\ 8} \cdot 2 \rightarrow (4\ 3\ 1\ 5\ 2)$$

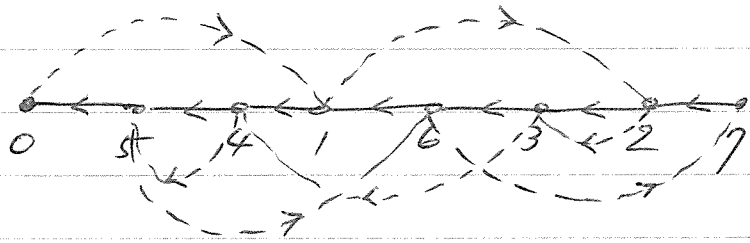
They have the same transposition distance.

Review "cycle graph of π ": $G(\pi)$

Let $c(G(\pi))$ denote the number of alternating cycles.

$$\pi = (5\ 4\ 1\ 6\ 3\ 2)$$

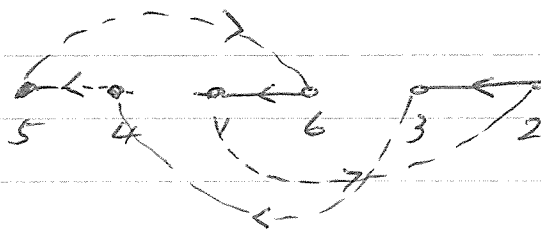
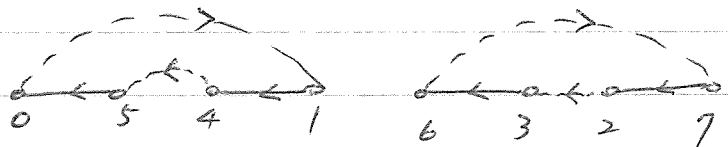
$G(\pi)$:

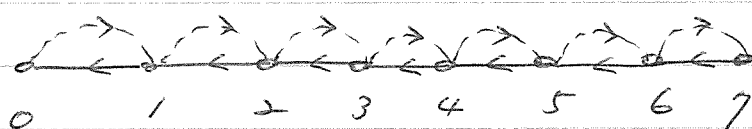


$c(G(\pi))$

||

3



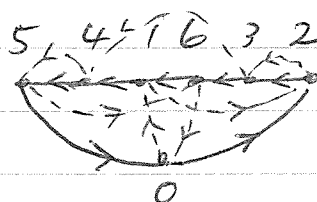


Identity
most alternating
cycles

Cycle graph of linear extension

(*) We can define the cycle graph of π by using circular extension instead of linear extension.

• Example,



"只加0, 不加7!"

虚线部份仍依顺序,
6接回0。

Proposition $\forall \pi \in S_n, \text{td}(\pi) \geq \frac{p(\pi) - c(G(\pi))}{2}$.

Proof. It suffices to show that a transposition may increase the number of alternating cycles by at most two.

Let $\tau(i, j, k) = \tau$ be the transposition, and $\Delta c(\tau) = c(G(\pi \circ \tau)) - c(G(\pi))$ be the number of alternating cycles which are changed.

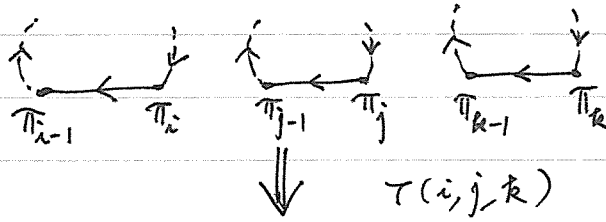
Now, we claim $\Delta c(\tau) = \{-2, 0, 2\}$.

By definition, the vertices of $G(\pi)$ which are involved in the transpositions are $\pi_{i-1}, \pi_i, \pi_{j-1}, \pi_j, \pi_{k-1}$ and π_k . Following the transposition $\tau(i, j, k)$, the black edges (π_i, π_{i-1}) , (π_j, π_{j-1}) and (directed)

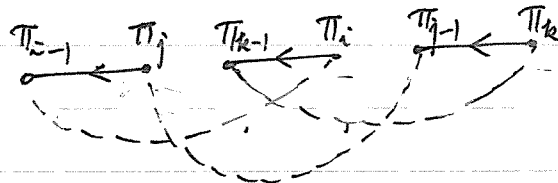
(π_k, π_{k-1}) are removed and adding new black edges $(\pi_j, \pi_{k-1}), (\pi_i, \pi_{k-1})$
(in new permutation)

and (π_k, π_{j-1}) .

Case 1. Three removed edges are in three alternating cycles
(of the decomposition of $G(\pi)$).

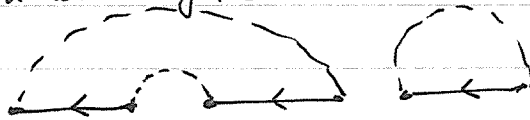


3 cycles

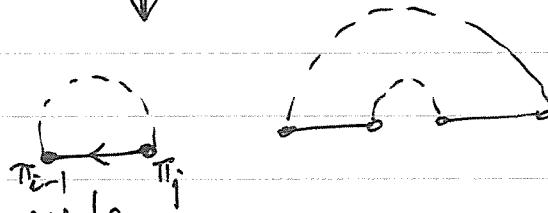


one cycle

Case 2 In two cycles



Two cycles

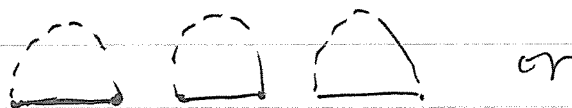


Two cycles

Case 3 In one cycle

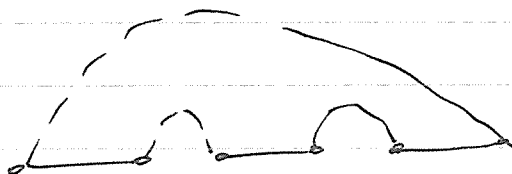


one cycle



Three cycles.

or



One cycle

Definition (Odd or even alternating cycles)

An alternating cycle is said to be odd if it contains an odd number of "black" edges (arcs), and even otherwise.

Example In the decomposition of $[0\ 5\ 4\ 1\ 6\ 3\ 2]$, there are two even cycles and one odd cycle.

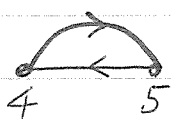
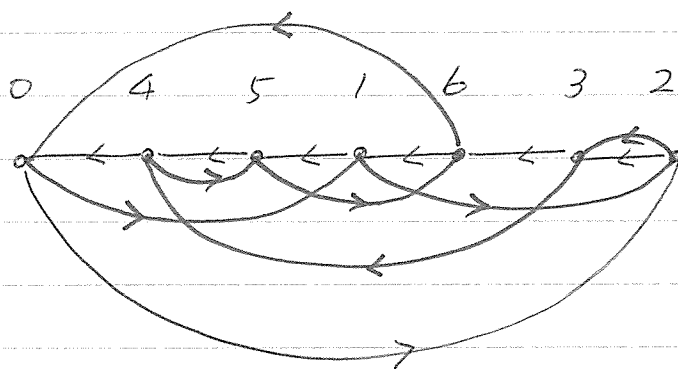
(*) The change of odd cycles in a transposition is ^{either or} $-2, 0, 2$.

Proposition $td(\pi) \geq \frac{pl(\pi) - c_o(G(\pi))}{2}$ where $c_o(G(\pi))$ denotes the number of odd cycles.

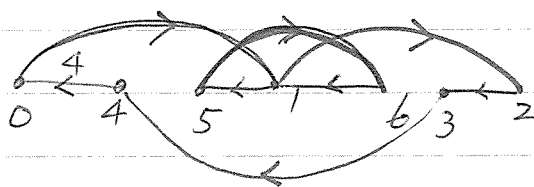
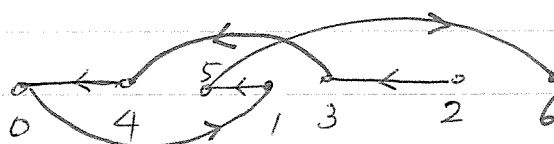
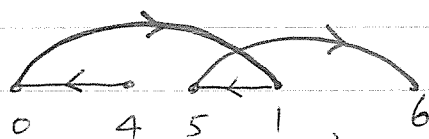
Reference

V. Bafna and P. A. Pevzner, Sorting by transpositions,

SIAM J. Discrete Math., Vol. 11, No 2, 224-240.



①



②



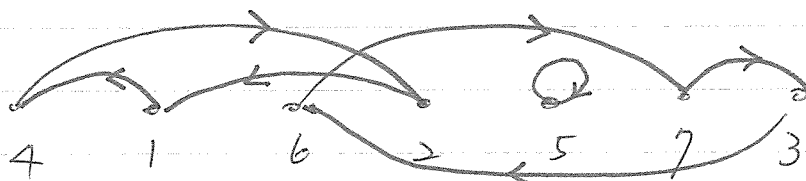
③

Definition (Γ -graph)

The Γ -graph of a permutation $\pi \in S_n$ is the digraph

$\Gamma(\pi)$ with ordered vertex-set $(\pi_1, \pi_2, \dots, \pi_n)$ and

$(\pi_i, \pi_j) \in A(\Gamma(\pi))$ if $\pi_i = j$.



$(4 \ 1 \ 6 \ 2 \ 5 \ 7 \ 3)$
 $\pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5 \ \pi_6 \ \pi_7$

3 odd directed cycles!

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 6 & 2 & 5 & 7 & 3 \end{pmatrix}$$

The cycles of $P(\pi)$ can be obtained by finding the cycle decomposition of π . Here, $\pi = \underline{(142)}\underline{(367)}\underline{(5)}$.

↑ ↗ odd cycles

So, the number of cycles (directed) in $P(\pi)$ is exactly the number of cycles in "cycle decomposition" (representation).

Proposition

For all $\pi \in S_n$, $td(\pi) \leq n - \checkmark \# \text{ of odd cycles. } c_o(P(\pi)).$

Proof. Try!!

(*) We can take the longest increasing subsequence of π to obtain an upper bound.

Proposition

Let $|LIS(\pi)|$ denote the length of a longest increasing sequence. Then, $td(\pi) \leq n - |LIS(\pi)|$.

$n - |\text{LIS}(\pi)|$ is also denoted by $\text{ulam}(\pi)$, called Ulam's distance.

Proof. We may use transposition to increase $\text{LIS}(\pi)$ one by one.
(at least one!)

Example

4 (1) 6 (2) (5) (7) 3

$\Downarrow$

4 (1) (2) (5) (6) (7) 3

4 1 2 4 5 6 7 3

3

1 2 3 4 5 6 7

(4) (5) (6) (7) [1 2 3]

From 4 to 7.

Another idea

Definition (Coding a permutation)

Given a permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$, the left code of the element π_i of π , denoted by $\text{lc}(\pi_i) \stackrel{\text{def}}{=} \downarrow$ next page

$|\{\pi_j \mid \pi_j > \pi_i \text{ and } 0 \leq j \leq i-1\}|$, for $1 \leq i \leq n$. ($\pi_0 = 0$, as an invention.)

Similarly, the right code of π_i , denoted by $rc(\pi_i) =_{\text{def}}$

$|\{\pi_j \mid \pi_j < \pi_i \text{ and } i+1 \leq j \leq n+1\}|$, for $1 \leq i \leq n$. ($\pi_{n+1} = n+1$).

(*) The left code of π is $(lc(\pi_1), lc(\pi_2), \dots, lc(\pi_n))$ and the right code of π is $(rc(\pi_1), rc(\pi_2), \dots, rc(\pi_n))$.

Example

$$\pi = (6 \ 3 \ 2 \ 1 \ 4 \ 5)$$

$\Downarrow$

$$lc(\pi) = (0, 1, 2, 3, 1, 1) \text{ and}$$

(: 左边项比该项大的个数!)

$$rc(\pi) = (5, 2, 1, 0, 0, 0).$$

(右边项比该项小的个数!)

Example $lc(i) = rc(i) = (0, 0, \dots, 0)$.

Definition (plateau (複數 plateaux))

sequence of

Let $\vec{s}$ be a $\wedge$ non-negative integers. A plateau of $\vec{s}$ is

any maximal length subsequence of contiguous elements in $\vec{s}$

that have the same nonzero value. The number of plateaux

in $\vec{s}$ is denoted by $\tilde{p}(\vec{s})$.

Example

$$\vec{s} = lc(\pi) \text{ where } \pi = (6 \ 3 \ 2 \ 1 \ 4 \ 5).$$

$$= (0, \underline{\underline{1}}, \underline{\underline{2}}, \underline{\underline{3}}, \underline{\underline{1}}, \underline{\underline{1}}). \quad \tilde{p}(\vec{s}) = 4.$$

$$\vec{\lambda} = \text{rc}(\pi) = (\underline{5}, \underline{2}, \underline{1}, 0, 0, 0)$$

$$\Downarrow$$

$$\tilde{p}(\vec{\lambda}) = 3$$

Lemma Given a permutation π , the leftmost (resp. rightmost) plateau of $\text{lc}(\pi)$ (resp. $\text{rc}(\pi)$) can be removed by a transposition to the left (resp. the right) without creating any new plateaux in the code.

Proof. We prove the left code case. Let the leftmost plateau be from position i to $j-1$. Then, all entries of $\text{lc}(\pi_i)$ before position i is "0". That is $\pi_1, \pi_2, \dots, \pi_{i-1}$ are in increasing order. So, if $\text{lc}(\pi_i) = a$, then $p(i, j, i-a)$ removes the first plateau from the left. $\square$

$$a=1$$

Example

$$\pi = (6 \textcircled{3} 2 \ 1 \ 4 \ 5)$$

$$\Downarrow$$

$$i=2, a=1 \quad \pi' = (3 \ 6 \ 2 \ 1 \ 4 \ 5)$$

$$\text{lc}(\pi') = (0, 0, 2, 3, 1, 1)$$

$$\pi'' = (2 \ 3 \ 6 \ 1 \ 4 \ 5)$$

$$\text{lc}(\pi'') = (0, 0, 0, 3, 1, 1)$$

$\vdots$

Proposition $ld(\pi) \leq \min \{ \tilde{p}(lc(\pi)), \tilde{p}(rc(\pi)) \}.$

$\tilde{p}(\pi)$

Proof. We may choose the one with less plateaux among

$lc(\pi)$ and $rc(\pi)$.