

(*) Topological Graph Theory studies the "drawing" of a graph on a surface.

(**) A proper drawing on a surface of a graph G with p vertices and q edges follows the rules:

(1). There are p points on the surface which corresponds to the set of vertices in G ; and

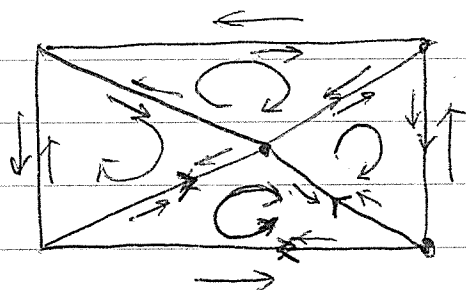
(2). There are q curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

2-manifold : A connected topological space in which every point has a neighborhood homeomorphic to the open unit disk defined on $\mathbb{R}^2$.

Bounded subspace : A subspace M of $\mathbb{R}^3$ is bounded if $\exists K \in \mathbb{R}^+$ such that $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 = K\}$.

Closed : M is closed if its boundary ∂M coincides with M .

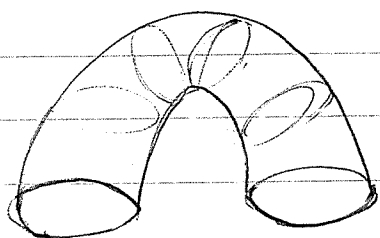
Orientable : M is orientable if for every simple closed curve C on M , a clockwise sense of rotation is preserved once around C . Otherwise, M is non-orientable.



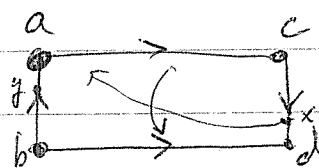
Orientable

Surface : A surface S_k is a compact "orientable" 2-manifold that may be thought of as a sphere on which has been placed (inserted) a number k of "handles" (holes),

Non-orientable : A surface obtained by adding to cross-caps Surface to a sphere (S_0) is a non-orientable surface N_k .



Handle

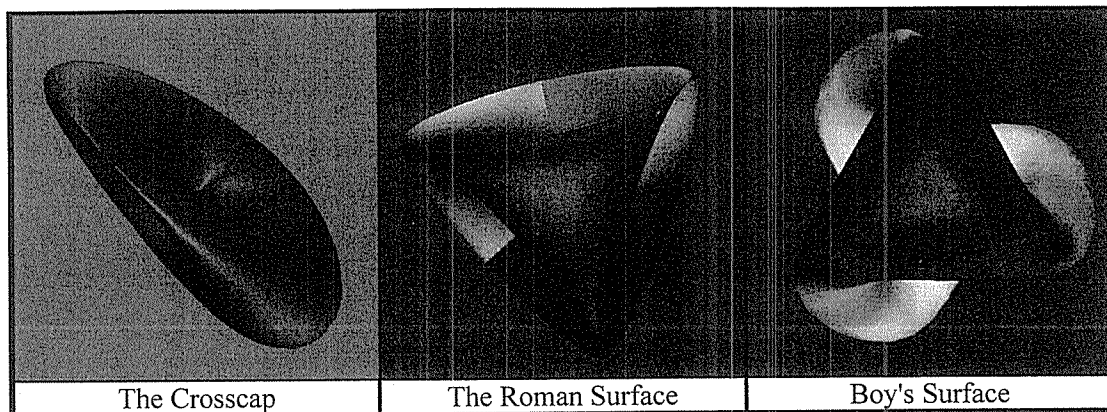


Möbius band

$a \rightarrow d, c \rightarrow b, x \rightarrow y$



(*) Adding a crosscap : Attach the boundary of a Möbius band to a cycle on S_0 .



Definition (Embeddable)

A (p, q) -graph G is said to be embeddable on a surface if it is possible to draw G properly on the surface.
(drawing without crossings)

Definition (Planar graph)

A graph is planar if it can be embedded in the plane, equivalently, embedded on the sphere.

Definition (2-cell embedding)

curve in that region can be continuously deformed or contracted in that region to a single point, equivalently, a 2-cell is topologically homeomorphic to $\mathbb{R}^2$. An embedding of G on a surface is a 2-cell embedding of G if all the regions so determined are 2-cells.

The following figure shows embeddings of $K_{3,3}$ on S_1 and S_2 respectively. S_k is a surface obtained by attaching k handles.

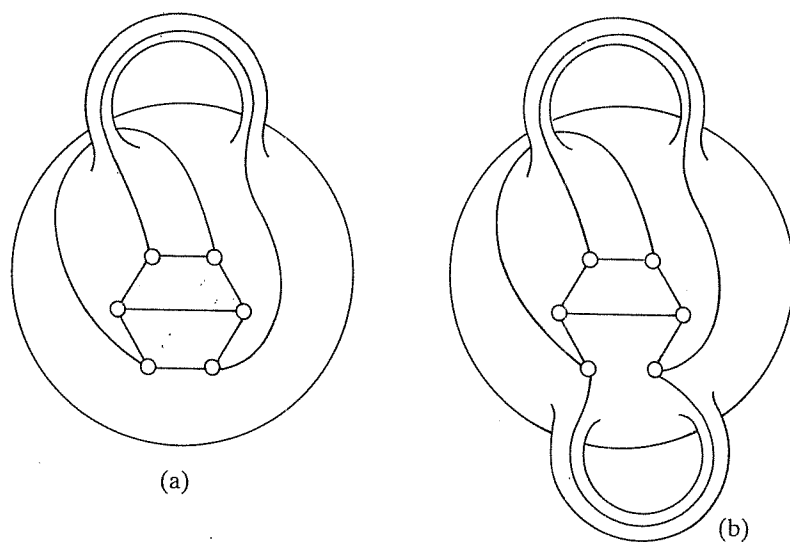


Figure 3.1. Embeddings of $K_{3,3}$ on surfaces S_1, S_2 .

(*) S_0 : Sphere

(*) $N_0 \simeq S_0$, N_h : Attach h crosscaps to $N_0(S_0)$.

Definition (Genus)

The number of handles (resp. crosscaps) ^(on a surface) is referred to as ^{orientable} genus of the _{orientable} surface (resp. non-orientable surface). We use $\gamma(G)$ (respectively $\tilde{\gamma}(G)$) to denote the smallest genus of all _{orientable} surfaces (resp. non-orientable surfaces) on which G can be embedded.

(*) If G is a planar graph, then $\gamma(G)$ (so is $\tilde{\gamma}(G)$) is equal to zero. But, G can be embedded on a surface with genus larger than "0".

(**) Given a graph G , determining $\gamma(G)$ is a difficult problem.

Theorem 46 (Euler)

Let G be a ^{connected} planar graph with p vertices, q edges and f faces (regions). Then $p - q + f = 2$.

Proof. By induction on q . Since G is connected, G has at least $p-1$ edges. (?) If G has $p-1$ edges and G is connected, then G is a tree which contains no cycles. This implies that $f=1$ and thus $p - (p-1) + 1 = 2$. The assertion is true for 'minimal' graphs. Let the hypothesis is true for $\|G\| \geq p-1$.

Now, consider G with $k+1$ edges. Clearly, G contains a cycle.

Let e be a cycle edge. Since G is a connected planar graph, $G-e$ is also a connected planar graph. Moreover,

$\|G-e\| = k$ and $G-e$ has k edges and $q-1$ faces. By

induction $p - k + q - 1 = 2$ and thus

$p - (k+1) + q = 2$. This concludes the proof. $\square$

Theorem 42 If G is a planar graph, then $\|G\| = 3|G| - 6$.

Proof. By observation, if G has maximum size, then each region of G is a triangle. Since each edge of G is in the boundary of exact two regions, $3 \cdot f = 2 \cdot q$ where f is the number of regions and q is the size of G , i.e., $q = \|G\|$. Now, by Euler's formula $p - q + f = 2$ equivalently

$$|G| - \|G\| + \frac{2}{3}\|G\| = 2$$

$$\Rightarrow 3|G| - 6 = \|G\|. \quad (G \text{ is a maximal planar graph!}) \quad \blacksquare$$

Corollary If G is a planar graph, Then $\|G\| \leq 3|G| - 6$.

Corollary In any ^{planar} graph, there exists at least one vertex of degree smaller than 6.

Corollary The degree sum of a planar graph is at most $6|G| - 12$.

This corollary is very useful.

(*) We can give a more accurate estimation of the above corollary

Theorem 48

Let G be a maximal planar graph (triangulated) of order p , and let p_i denote the number of vertices of degree i in G for $i=3, 4, \dots, \Delta(G)=d$. Then

$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \dots + (d-6)p_d + 12.$$

Proof. Since $p = \sum_{i=3}^d p_i$ and $2q = \sum_{i=3}^d i p_i$, we have

$$\sum_{i=3}^d i p_i = 2(3p - 6) = 6 \cdot \sum_{i=3}^d p_i - 12.$$

This implies the conclusion. ▀

Theorem 49

There are exactly five regular polyhedra.

Proof. Notice that a regular polyhedron is a polyhedron whose faces (regions) are bounded by congruent regular polygons and whose polyhedral angles are congruent.

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First, we convert a polyhedron into a regular planar graph.

(See Figure 32.) Let the number of vertices, edges and faces be

By Theorem 46, $p - q + f = 2$. Hence,

$$-8 = 4q - 4p - 4f = 2q + 2q - 4p - 4f$$

$$= \sum_{i \geq 3} i f_i + \sum_{i \geq 3} i p_i - 4 \sum_{i \geq 3} p_i - 4 \sum_{i \geq 3} f_i \quad (f_i; \# \text{ of } i\text{-face})$$

$$= \sum_{i \geq 3} (i-4) f_i + \sum_{i \geq 3} (i-4) p_i.$$

Since the polyhedron is regular, all degrees and face sizes are the same, let them be k and h respectively. Therefore,

$$-8 = (h-4) f_h + (k-4) p_k.$$

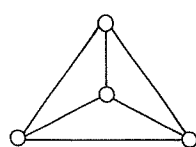
By the fact that every planar graph contains a vertex of degree less than six, we only have nine cases to consider:

$$3 \leq h \leq 5 \text{ and } 3 \leq k \leq 5.$$

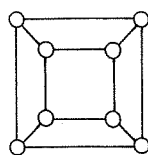
From direct checking, only 5 cases are possible, namely,

- (1) $f_3 = p_3 = 4$ (Tetrahedron) 四面体
- (2) $f_3 = 8$ and $p_4 = 6$ (Octahedron) 八面体
- (3) $f_3 = 20$ and $p_5 = 12$ (Icosahedron) 二十面体
- (4) $f_4 = 6$ and $p_3 = 8$ (Cube) 六面体
- (5) $f_5 = 12$ and $p_3 = 20$ (Dodecahedron) 十二面体。

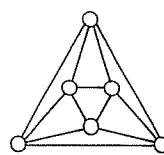
See Figure 3.2 for regular polyhedra.



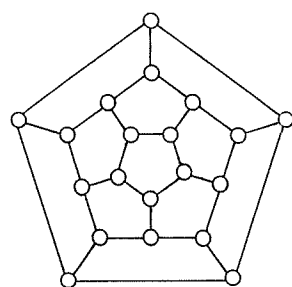
tetrahedron



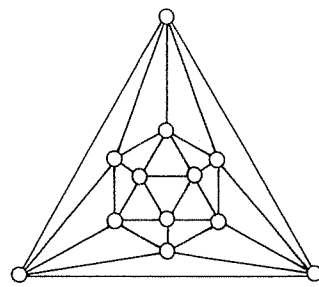
cube



octahedron



dodecahedron



icosahedron

Figure 32 The graphs of the regular polyhedra

Theorem 50 (Fáry, Wagner,
(1948) (1936))

A planar graph G can be embedded in the plane so that each edge is a straight line segment.

Proof. The proof is by induction on the order of G . It suffices to prove the case when G is a connected ^{maximal} planar graph.

Clearly, it is true for small orders. Assume the hypothesis is true for order k and let G be a connected maximal planar graph of order $k+1$.

Since G is maximal, $3 \leq \delta(G) \leq 5$.

Case 1. $\delta(G) = 3$

Let $v_0 \in V(G)$ such that $\deg_G(v_0) = 3$ and v_0 is adjacent to v_1, v_2 and v_3 . Since G is maximal, $\langle \{v_1, v_2, v_3\} \rangle_G \cong K_3$. This implies that $G - v_0$ is also a ^{maximal} planar graph. By induction $G - v_0$ has a straight line segment embedding. Now, put v_0 back to the graph $G - v_0$ such that v_0 is inside the region bounded by $\langle \{v_1, v_2, v_3\} \rangle$ and connect v_0 to the three vertices by straight line segment. This concludes the proof of this case.

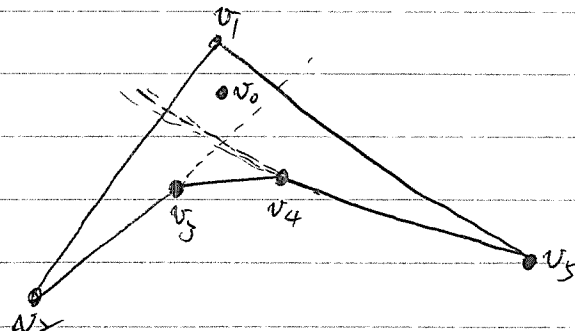
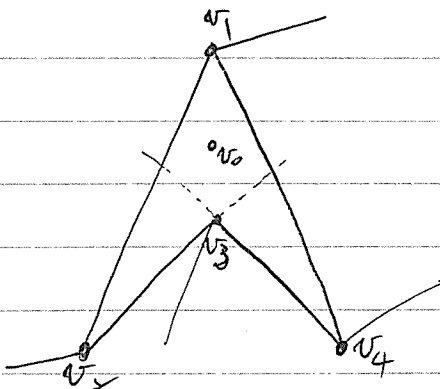
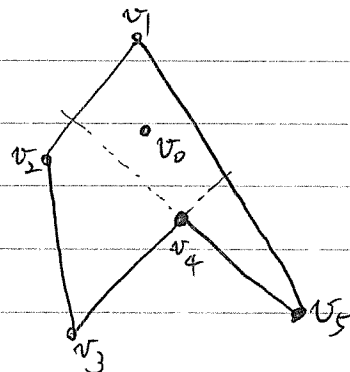
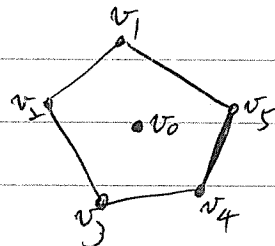
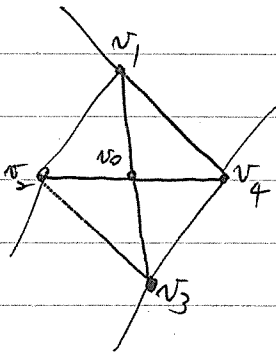
Case 2. $\delta(G) = 4$

The proof follows by a similar process by letting $N(v_0) = \{v_1, v_2, v_3, v_4\}$. Now, $G - v_0 + v_1v_3$ is a maximal planar graph and thus it has a straight line segment embedding. The proof follows by placing v_0 back to $G - v_0 + v_1v_3 - v_1v_3$. By considering the drawing of the embedding (Figure 33(a)), we are able to put

segment.

Case 3. $\delta(G) = 5$.

Again, we use the same technique and the drawing can be seen in Figure 33(b).



(a)

Figure 33 Location of v_0 .