

有些看起來很像?

↗

$C_i \sim C_j$ if and only if there exists a rotation π ($90^\circ, 180^\circ, 270^\circ$ or 360°) such that $\pi(C_i) = C_j$.

↓ An equivalence relation

$\{C_1, C_2, \dots, C_{16}\} / \sim \rightarrow$ Equivalence classes

Problem How many equivalence classes we have?

Theorem (Burnside's Lemma)

Let G be a permutation group of S . Then, the number of equivalence classes^{of S} induced by G is equal to

$$\sum_{s \in S} \eta(s) / |G| \text{ where } \eta(s) = |\{\pi \in G \mid \pi(s) = s\}|.$$

Proof. Let S_i be one of the equivalence classes of S induced

by G . Then, $\forall \pi \in G$, $\pi(s) \in S_i$ if $s \in S_i$. Clearly, some of

the permutations π satisfying $\pi(s) = s$ for example the identity.

Now, let $a, b \in S_i$ in case $a \neq b$. Let $\alpha_{a,b}$ be the number

permutations ^{π} in G such that $\pi(a) = b$. Claim: $\alpha_{a,a} = \eta(a) =$

$\alpha_{a,b}$.

Let $A_{a,b}$ be the set of permutations π mentioned above. Let
(resp. $A_{a,a}$)

$f \in A_{a,b}$. By the fact G is a group, $f^{-1} \in G$ and $f^{-1}(b) = a$. This

implies that $\forall g \in A_{a,a}$, $f \circ g \in A_{a,b}$; moreover, $f \circ g_1 = f \circ g_2$

implies that $g_1 = g_2$. Hence, $|A_{a,a}| \leq |A_{a,b}|$. On the other hand,

Let $f' \in A_{b,a}$. Then $f' \circ f(a) = a$, i.e., $f' \circ f \in A_{a,a}$. Again,

by the same argument $|A_{a,b}| \leq |A_{a,a}|$. We have the claim.

By the fact that all permutations in G either fix a or map a into b , $|G| = \sum_{x \in S_i} |A_{a,x}| = |S_i| \cdot \eta(a)$. Now, suppose that we

have t equivalence classes. $\sum_{a \in S} \eta(a) = \sum_{i=1}^t \left(\sum_{a \in S_i} \eta(a) \right) = t \cdot |G|$. ■

In general, $|S|$ is larger than $|G|$. Hence, we may use two-way counting to count $\psi(\pi) = |\{a \in S \mid \pi(a) = a\}|$, i.e., the number of elements in S which are fixed by π . For

example; in G , (page 1), we have $\pi_1(90^\circ)$, $\pi_2(180^\circ)$, $\pi_3(270^\circ)$, $\pi_4 = \text{id}$.

C_1, C_{16} are fixed by π_1 | C_1, C_{10}, C_{11} and C_{16} are fixed by π_2 ,

Also, C_1, C_{16} for π_3 , and C_1 to C_{16} by π_4 .

$$(o) \quad \sum_{a \in S} \eta(a) = \sum_{\pi \in G} \psi(\pi). \quad \left(\text{Counting pairs } (x, \pi) \text{ where } \pi(x) = x. \right)$$

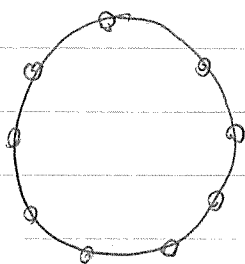
$$\# \text{ of equivalence classes} = \frac{2+4+2+16}{4} = 6$$

Of course, if we count $\eta(a)$, then we have $\eta(C_1) = \eta(C_{16}) = 4$,

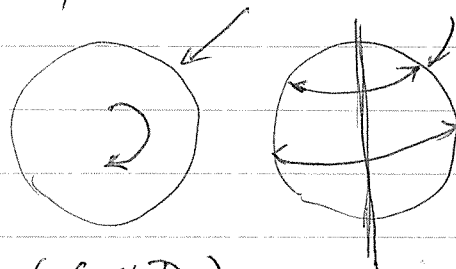
$\eta(C_{10}) = \eta(C_{11}) = 2$ (identity and π_2), and all the other C_i 's

$$\eta(C_i) = 1. \quad \sum_{a \in S} \eta(a) = 2 \cdot 4 + 2 \cdot 2 + 12 \cdot 1 = 24.$$

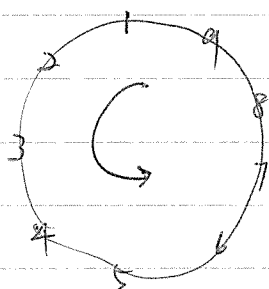
Example (參考離散講義 part 3).



用 9 顆寶石串成項鍊, 3 紅, 6 藍; 有多少
看起來不一樣的裝飾; 可以旋轉及鏡射。



(1) Permutation group G , $|G| = 18$, ($G \cong D_9$)
(9 rotations and 9 reflections)



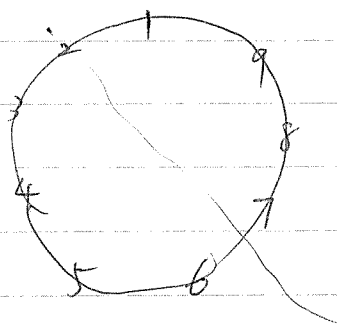
$$r_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{pmatrix} \quad G 40^\circ$$

$$r_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 \end{pmatrix} \quad G 80^\circ$$

$\vdots$

$$\begin{matrix} r_8 \\ r_9 \text{ (identity)} \end{matrix} \quad G 360^\circ$$

$$s_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$$



$$s_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 1 & 9 & 8 & 7 & 6 & 5 & 4 \end{pmatrix}$$

$\vdots$

$$(2) |S| = \binom{9}{3} = 84.$$

$$\sum_{\pi \in G} \psi(\pi) = \frac{0+0+3+0+0+3+0+0+84+36}{18} = 7.$$

4.9(?) δ_8
 δ_9

We may consider the example in 8-1 as functions

$$f: \begin{array}{|c|c|} \hline a & b \\ \hline d & c \\ \hline \end{array} \rightarrow \{0,1\}, \quad f: \{a,b,c,d\} \rightarrow \{0,1\}.$$

Hence, the functions can be described as

$$\begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (0,0,1,0) \\ (0,0,0,1) \\ (1,1,0,0) \\ (0,1,1,0) \\ (0,0,1,1) \\ (1,0,0,1) \\ (1,0,1,0) \\ (0,1,0,1) \\ (1,1,1,0) \\ (0,1,1,1) \\ (1,0,1,1) \\ (1,1,0,1) \\ (1,1,1,1) \end{array} \quad \left. \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array} \right\} (a,b,c,d)$$

(0) 把函数分成等价类! $G = \{\pi_1, \pi_2, \pi_3, \pi_4\}$.

eg. $\pi_1(a,b,c,d) = (b,c,d,a)$, $\pi_2(a,b,c,d) = (c,d,a,b)$.

$$f_1 \sim f_2 \Leftrightarrow \exists \pi \in G, \pi(f_1) = f_2$$

看成合成函数

• $F = R^D$ (In example, $R = \{0, 1\}$,
 $D = \{a, b, c, d\}$)
 $= \{f : f : D \rightarrow R\}$.

(..) $\forall r \in R$, let $w(r)$ denote the weight of r and

the store enumerator $\varphi_F = \sum_{r \in R} w(r)$. (Note the weight
of r may be only signs!)

(...) $\forall f \in F$, $w(f) = \prod_{d \in D} w(f(d))$.

e.g. let $w(0) = u$, $w(1) = v$, then the function $(0, 1, 0, 1)$
has weight $u^2 \cdot v^2$.

(*) We can see the number of 0's and 1's right away.

(**) Clearly, two functions are equivalent implies that
they have the same weight since we only apply
permutations. ($f_1 \sim f_2 \Rightarrow w(f_1) = w(f_2)$)

(***) Definition $\sum_{f \in F} w(f) = I_F$ (Inventory of F).

Example If we take F as the example in 8-1, then

$I_F = u^4 + 4u^3 \cdot v + 6u^2 v^2 + 4uv^3 + v^4$ provided $w(0) = u$ and $w(1) = v$.

Cycle Factorization

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 7 & 5 & 6 & 8 & 2 \end{pmatrix} = (143)(29)(576)(8)$$

Theorem A permutation has a unique cycle factorization (up to the order of cycles).

(*) Let $\#(f)$ denote the number of cycles of f in its cycle factorization. (The example above has $\#(f) = 4$.)

(*) Let $\pi(f) = (e_1, e_2, \dots, e_n)$ where e_i is the number of cycles of f with length i . Since f (above) has one cycle of length 1, one cycle of length 2 and two cycles of length 3, $\pi(f) = (1, 1, 2, 0, 0, 0, 0, 0, 0)$.

Clearly, $\boxed{\sum_{i=1}^n i \cdot e_i = n.}$

(**) For each f where $\pi(f) = (e_1, e_2, \dots, e_n)$, we associate f with a term in polynomials with n variables $z_1, z_2, \dots, z_n$,

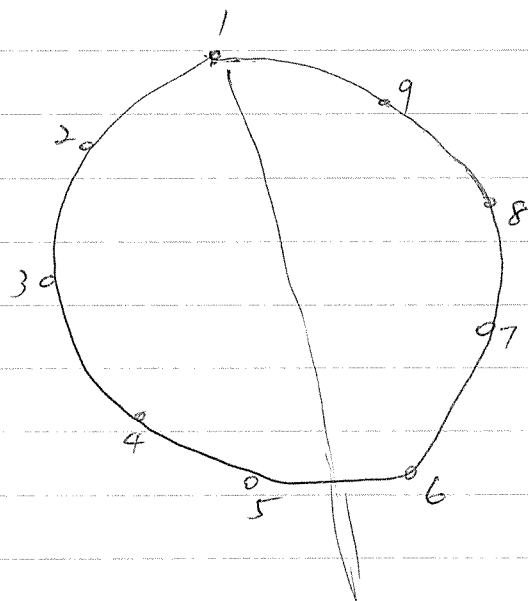
$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}.$$

(***) We have a G.F. for the permutation group G :

$$(*) \quad \sum_{f \in G} \text{mon}(f) = \sum_{\substack{f \in G \\ \pi(f) = (e_1, e_2, \dots, e_n)}} z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}.$$

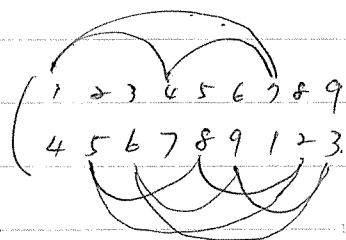
Example

$G =_{\text{def}} D_9$



(1) $|D_9| = 18$

(2) $D_9 = \{p_0, p_1, \dots, p_8; \delta_1, \delta_2, \dots, \delta_9\}$



$$t(p_0) = z_1^9, \quad t(p_1) = z_9, \quad t(p_2) = z_9, \quad t(p_3) = z_3^3$$

$$t(\delta_1) = z_1 z_2^4, \quad \dots, \quad t(\delta_9) = z_1 z_2^4$$

(3)
$$\sum_{f \in G} \text{mon}(f) = z_1^9 + 6z_9 + 2z_3^3 + 9z_1 z_2^4$$

Definition (Cycle Index)The cycle index of G , denoted by $P_G(z_1, z_2, \dots, z_n)$

$$= \frac{1}{|G|} \cdot \sum_{\substack{f \in G \\ t(f) = (e_1, e_2, \dots, e_n)}} z_1^{e_1} z_2^{e_2} \dots z_n^{e_n} \quad \text{where } |G| = n.$$

Surprising Theorem

(Omit the proof.)

→ Idea in next page

The number of equivalence classes of R^D (corresponding to G) is equal to $P_G(\underbrace{|R|, |R|, \dots, |R|}_{n\text{-tuple}})$, $|G| = n$.

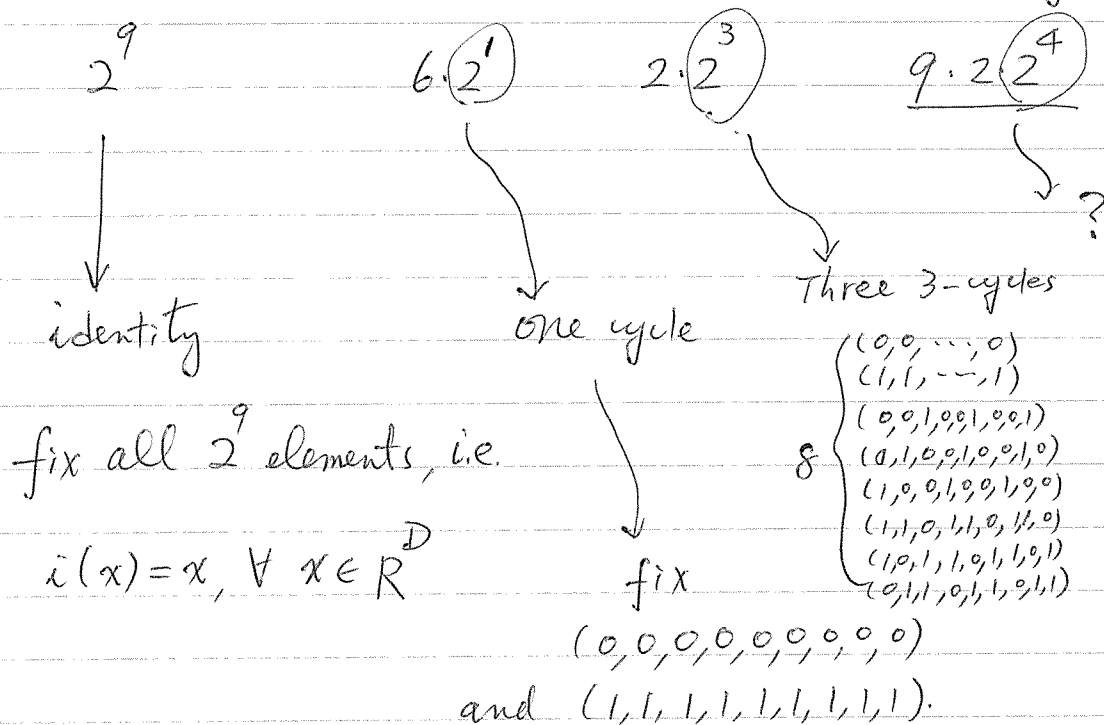
Example 九个宝石的例子, $|R| = 2$. $G =_{\text{def}} D_9$. (每个都可以红或蓝)

$$P_G(2, 2, \dots, 2) = (2^9 + 6 \cdot 2 + 2 \cdot 2^3 + 9 \cdot 2 \cdot 2^4) / 18 \quad (*)$$

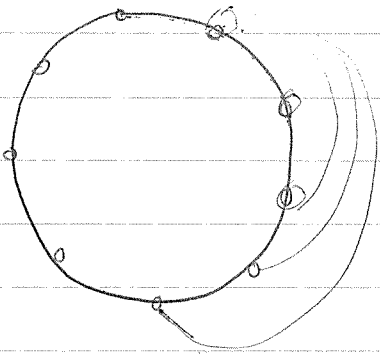
$$= (512 + 12 + 16 + 288) / 18 = 46.$$

If the number of colors are prescribed, say 3 reds and 6 blues, then we need to use another idea. (?)

(*) We can use Burnside's Lemma to explain the identity (*).



如果指定紅, 藍的個數, Burnside's Lemma 是最佳工具。 No. 8-9'



Check!

全紅寶石	①
全藍寶石	1
一個紅 -	①
一個藍 -	1
二個紅	④
二個藍	4
三個紅	②
三個藍	2
四個紅	⑩
四個藍	10
	46

$$\binom{9}{4} = \frac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} = 126$$

$$p_0 : 126$$

$$p_1, p_2, p_4, p_5, p_7, p_8 : 6 \times "0"$$

$$p_3, p_6 : 2 \times "0"$$

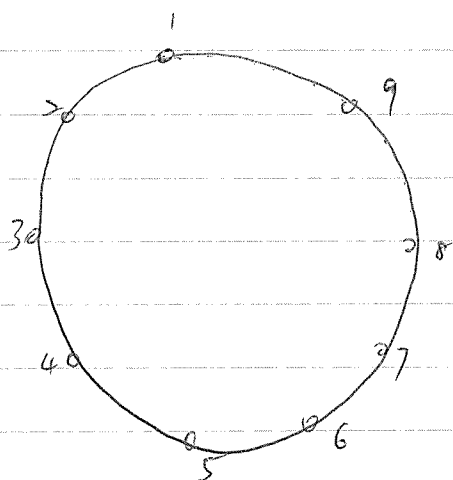
$$s_1 : 9 \times 6 = 54$$



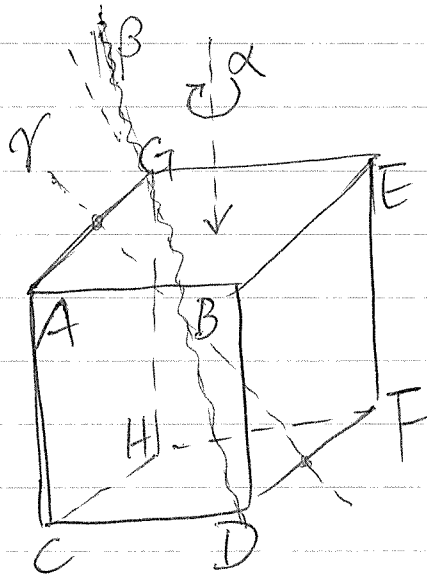
(*) If we have a cycle, say (abc) , then the elements fixed whenever $f(a) = f(b) = f(c) \in R$.

例如: a, b, c 同时是红色, 或蓝色, 甚至更多色!

结论: Cycles 的多少影响着可能被固定的元素。



如果只有一个 Cycle, (123456789) , 则九个位置必需都放同色宝石才可能固定。反之, 如果 cycle 是 $(1)(2) \dots (9)$, 则因为没取动, 怎么放都是被“它”所固定。



There are three types of rotations α , β , and γ

α : 以一面与对面的中心为轴旋转 $\square$.

β : 以最远距离的两角旋转 $\square$.

γ : 以对边的中心连线为轴旋转。
(最远)

α : 每两面有 4 个, 其中一个为 identity

3 个 { $(GEBA)(HFDC)$, $(GB)(AE)(HD)(FC)$, $(GABE)(HCDF)$ }

β : 每两角有 3 个, 其中一个为 identity

4 个 { $(G)(D)(\underline{BCF})(HEA)$, $(G)(D)(BFC)(HAE)$ }
 $\quad \quad \quad 120^\circ \quad \quad \quad 240^\circ$

γ : 对边, 180° , identity

$$6\gamma \left\{ \begin{array}{l} (GA)(DF)(BH)(CE) \\ \end{array} \right.$$

Cycle index

$$P_G(z_1, z_2, \dots, z_8) = 3 \cdot \overset{\alpha}{(z_4^2 + z_2^4 + z_4^2)} + 4 \cdot \overset{\beta}{(z_1^2 z_3^2 + z_1^2 z_3^2)} + 6 \cdot (z_2^4) + z_1^8$$

Problem Use n colors to color the eight corners, how many different patterns we shall have?

Sol. $|G| = 9 + 8 + 6 + 1 = 24$.

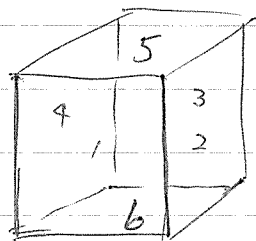
$$\begin{aligned} P_G(|z_1|, |z_2|, \dots, |z_8|) &= P_G(n, n, \dots, n) \\ &= 3(n^2 + n^4 + n^2) + 4(\underline{n^2 \cdot n^2 + n^2 \cdot n^2}) + 6n^4 + n^8 \\ &= n^8 + 17n^4 + 6n^2. \end{aligned}$$

Note that The # of equivalence classes is $(n^8 + 17n^4 + 6n^2)/24$.

$$(*) \quad 24 \mid n^8 + 17n^4 + 6n^2.$$

(用归纳法证明比较复杂!)

(*) 用 6 種顏色來塗六面體，每面都不同色，有多少 patterns?



Sol. $6! / 24 = 30.$

(沒有任一個旋轉可以 (Fix) 一個 pattern, 除了 identity 之外.)