

6. Enumerative principles

6.1 Combination, permutation and probability

Counting plays the most important role in our daily life.

Let S be a countable set. We use $|S|$ to denote the cardinality of S . Clearly, $|S|$ may be finite or infinite. If S is an infinite set, then $|S| = +\infty$. Otherwise, S is a finite set.

If $|S| = n$ for some positive integer n , then we refer S as an n -set. The most use n -set is $\{1, 2, \dots, n\}$, denoted by $[n]$ in short. The following observation is important in counting.

(*) Two sets S_1 and S_2 are of the same cardinality if and only if there exists a bijection from S_1 onto S_2 or S_2 onto S_1 .

With this observation, we can derive the following facts:

(F1) $|S_1| \leq |S_2|$ if and only if there exists a 1-1 mapping from S_1 into S_2 .

(F2) $|S_1| \geq |S_2|$ if and only if there exists an onto mapping from S_1 onto S_2 .

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In general, if S_1 is a proper subset of a finite set S_2 , then $|S_1| < |S_2|$. But, this is not true in the case that S_2 is an infinite set. Sometimes, this fact is utilized to define an infinite set which is countable.

Definition 6.1.1. A countable set S is infinite if there exists a proper subset of S , S' , such that $|S| = |S'|$.

For example, the set of ^{positive} even integers has the same cardinality as the set of all positive integers.

In this chapter, all sets considered are finite. If $|S| = 0$, then S is an empty set. A quick review from the idea of subsets, we notice that if S is an n -set, then there are 2^n distinct subsets of S . So, the first question arises: How many k -subsets do we have in an n -set where $0 \leq k \leq n$? To select a subset with k elements in an n -set is defined as

(F3) If $|S| = n$, there exists $n!$ distinct bijections from S onto itself. These bijections are also known as the permutation of S . For example, if $S = \{1, 2, 3\}$, then there are six permutations of S : $123, 132, 213, 231, 312, 321$ (ordered triples).

a combination of k objects from an n -set.

For example, if we have a set $S = \{1, 2, 3, 4, 5, 6\}$, then the combination of 2 objects from S are 12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, and 56. (We ignore the set notation here.) For

convenience, we use $\binom{n}{k}$ to denote the number all combinations
(read n -choose- k)

of k objects from an n -set.

Proposition 6.1.2. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ where k and n are non negative integers.

Proof. There are n choices of the first object, $n-1$ choices of the second object and $n-k+1$ choices for the k^{th} object. So,

in total, there are $n \cdot (n-1) \cdot \dots \cdot (n-k+1)$ choices. But each combination of k objects will occur $k!$ times (permutations), hence

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Example 1. How many ways are there in finding the "Combination" in Figure 6.1.

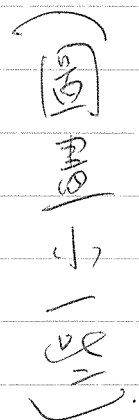


Figure 6.1. Binomial coefficients

By direct counting, we obtain 252 ways. Notice that this is the Pascal's triangle number. The reason can be seen from the observation of a, b, c in the figure. In order to find the number of ways ended at c , we can find the ways to a and b respectively, and then $c = a + b$. The above figure can be represented by combination numbers, see Figure 6.2.

(r, s) represents $\binom{r}{s}$ and r steps to there with s steps toward the left.

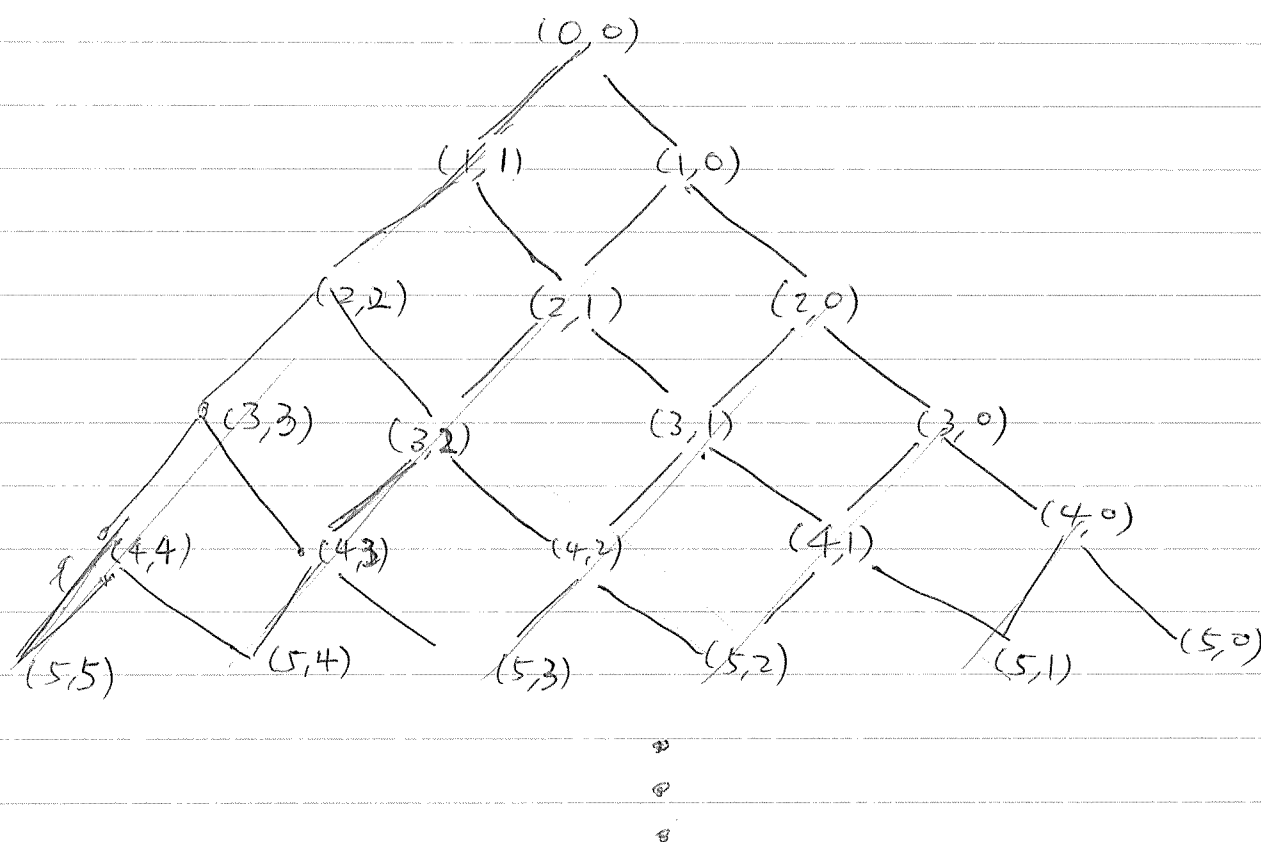


Figure 6.2. The Combination number

This notation can be applied to find the following equality. We can also use Proposition 6.1.2 to prove the fact.

Proposition 6.1.3.
$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Indeed, there are an enormous number of identities involving binomial coefficients. The reason we call the combination numbers "binomial coefficients" is very interesting. Of course, the coefficients are from the expansion of $(1+x)^n$ which is a product of n polynomials $1+x$. So, to find the coefficient of x^k , we have

to choose k out of n $(1+x)$ and choose x instead of 1 to make the product. There are exactly $\binom{n}{k}$ ways. Hence, we conclude

Proposition 6.1.4. $(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i, n \in \mathbb{N}.$

6' > From mathematical point of thought, the power n of $(1+x)$

can be generalized to a real number. First, we generalize the

combination number $\binom{n}{k}$ where $n \in \mathbb{R}$. For convenience, we use

$x^{\underline{k}}$ to denote $x \cdot (x-1) \cdot \dots \cdot (x-k+1)$. $x^{\underline{k}}$ is the k^{th} falling power

of x . Hence, $\binom{n}{k} = n^{\underline{k}} / k!$.

Definition 6.1.5.
$$\binom{x}{k} = \begin{cases} x^{\underline{k}} / k!, & \text{if } k \geq 0; \text{ and} \\ 0, & \text{if } k < 0 \end{cases} \text{ where } x \in \mathbb{R} \text{ and } k \in \mathbb{Z}.$$

Clearly, $\binom{x}{k}$ is a polynomial of x with degree k .

Example 2. $\binom{-1}{3} = (-1) \cdot (-2) \cdot (-3) / 3! = -1$ and $\binom{\frac{1}{2}}{3} = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) / 3! = \frac{1}{16}.$

Now, we can extend Proposition 6.1.3 to a more general version.

Proposition 6.1.6. Let $x \in \mathbb{R}$. Then $\binom{x}{k} + \binom{x}{k-1} = \binom{x+1}{k}$ where $k \in \mathbb{Z}$.

Proof. Let $f(x) = \binom{x+1}{k} - \binom{x}{k} - \binom{x}{k-1}$. Clearly, for $x \in \mathbb{N} \cup \{0\}$,

From this proposition, it is not difficult to see that

$$\sum_{i=0}^n \binom{n}{i} = 2^n \text{ (by plugging } x=1 \text{). By a similar trick, we see}$$

$$\text{that } \sum_{i=0}^n (-1)^i \binom{n}{i} = 0 \text{ (} x=-1 \text{). More identities can be discovered}$$

by way of the binomial polynomial $(1+x)^n$. We shall mention some of them in exercises.

$f(x)=0$. Since $f(x)$ is a polynomial of degree k , $f(x)=0$ can have at most k distinct solutions, unless $f(x)$ is a zero polynomial.

This concludes the proof. ■

With the extension of ^{the} combination number, we are able to define the binomial polynomial. First, since $\binom{n}{k} = 0$ if $k > n$, $(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i$. Now, if r is a real number, then $(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i$.

Note that even $i > r$, $\binom{r}{i}$ may not be zero, this is different from the case when n is a positive integer.

We remark finally that the extension of binomial polynomial is very useful in counting. For the rest of this section, we shall learn some basic notion about finite discrete probability.

Let the situation we consider has n possible outcomes, say $x_1, x_2, \dots, x_n$. Moreover, let them be mutually exclusive. By assigning a value to each outcome $p(x_i)$ a real number in $[0, 1]$ such that $\sum_{i=1}^n p(x_i) = 1$, we refer $p(x_i)$ as the probability that x_i occurs, i.e., the probability of x_i .

Let A be a set of events $x_1, x_2, \dots, x_m$ chosen from $\{x_1, x_2, \dots, x_n\}$.

Then, $p(A) = \sum_{j=1}^m p(x_j)$. If every event occurs with the same probability,

then, clearly $p(x_i) = \frac{1}{n}$ for each $i \in \{1, 2, \dots, n\}$, furthermore $p(E) = \frac{m}{n}$.

Example 3. In throwing at random two dices whose faces are numbers

1 to 6. The events are defined by the sum of two dices' numbers

facing upward. So, there are 11 possible outcomes $x_1, x_2, \dots, x_{11}$

in which x_i represents the event with sum $i+1$. Hence, $p(x_1) = \frac{1}{36}$,

$p(x_2) = \frac{1}{18}$, $p(x_3) = \frac{1}{12}$, $p(x_4) = \frac{1}{9}$, $p(x_5) = \frac{5}{36}$, $p(x_6) = \frac{1}{6}$, $p(x_7) = \frac{5}{36}$,

$p(x_8) = \frac{1}{9}$, $p(x_9) = \frac{1}{12}$, $p(x_{10}) = \frac{1}{18}$, $p(x_{11}) = \frac{1}{36}$.

It is easy to see that finding the probability of an event is
close related ^{to} counting the total objects (first) and then the

number of objects occurred in the event we are concerned.

← A more general approach of finding the
probability of certain events, ^{will be introduced later in Chapter 7.}

← For general form, the readers may refer to [1] for a
more detail introduction.

Exercise 1. Prove that $\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$.

Exercise 2. Let m be a positive integer. Find the number of different solutions of $x_1 + x_2 + \dots + x_n = m$ where x_i 's are non-negative integers.

Exercise 3. Let $|A| = m$ and $|B| = n$ such that $m \geq n$. Find the number of different surjective functions from A onto B .

Exercise 4. How many positive integers less than 10^{10} (in the decimal scale) have their digits in nondecreasing order?

6.2. Principle of Inclusion and Exclusion (PIE)

The counting principle introduced in this section is one of the most fundamental techniques. We start with an example.

Suppose we have a class of students who took two tests:

English and Mathematics. After the tests, we know 20 students

pass English, 15 students pass Mathematics and 10 students

both of pass. Then, So, how many students are there in the class?

who pass at least one of the two subjects

The answer can be depicted by using Figure 6.3.

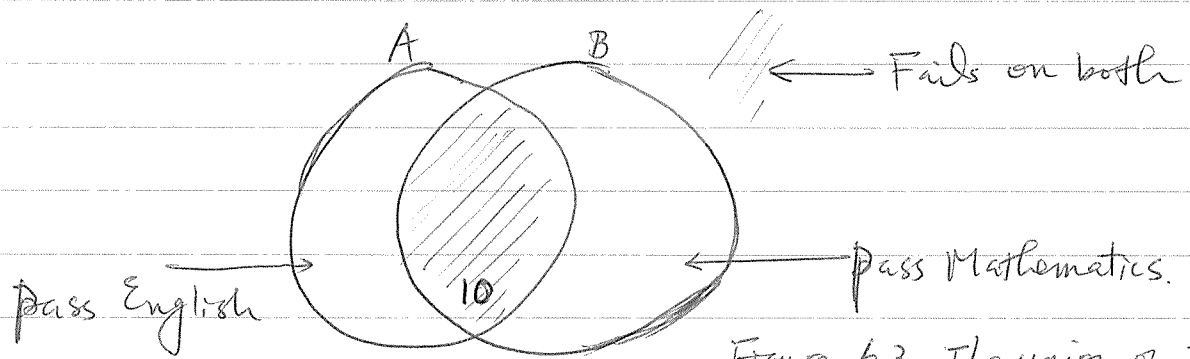


Figure 6.3. The union of two sets

Since the total number of students is the number of students

took the tests, it is the cardinality of the union of two sets.

Hence, the answer is $15 + 20 - 10 = 25$ ($|A \cup B| = |A| + |B| - |A \cap B|$).

We can extend this idea of counting to the union of more sets.

Let A be a set of cardinality N , $\alpha_1, \alpha_2, \dots, \alpha_r$ be the properties of objects in A , $N(\alpha_i)$ be the number of objects with property α_i , $i = 1, 2, \dots, r$.

Moreover, let $N(\alpha_i \alpha_j)$ denote the number of objects with properties α_i and α_j and let $N(\alpha_i')$ denote the number of objects without property α_i .

Now, it is easy to see that in the above example, we have two

properties: pass English (α_1) and pass Mathematics (α_2). So, if

the class has N students, then $N(\alpha_1' \alpha_2') = N - N(\alpha_1) - N(\alpha_2) + N(\alpha_1 \alpha_2)$
 $= N - (N(\alpha_1) + N(\alpha_2) - N(\alpha_1 \alpha_2)).$

The general form of PIE can be stated as follows.

Theorem 6.2.1. (PIE)

$$N(\alpha'_1 \alpha'_2 \dots \alpha'_r) = N - \sum_{i=1}^r N(\alpha_i) + \sum_{i \neq j} N(\alpha_i \alpha_j) - \sum_{\substack{i, j, k \\ \text{(different)}}} N(\alpha_i \alpha_j \alpha_k) + \dots \\ + (-1)^r N(\alpha_1 \alpha_2 \dots \alpha_r).$$

Proof. It suffices to show that any object without properties $\alpha_1, \alpha_2, \dots, \alpha_r$ will be counted exactly once in the right hand side of the equal sign, furthermore an object with some properties of $\alpha_1, \alpha_2, \dots, \alpha_r$ will not be counted.

The first part follows by fact that it is counted in the first term N and none of the others. Now, assume that an object does

have properties $\alpha_1, \alpha_2, \dots, \alpha_s, s \leq r$. Then, it will be counted

once in N , $\binom{s}{1}$ times in $\sum_{i=1}^r N(\alpha_i)$, $\binom{s}{2}$ times in $\sum_{i \neq j} N(\alpha_i \alpha_j)$, and

so on. Hence, the total number is $1 - \binom{s}{1} + \binom{s}{2} - \dots + (-1)^s \binom{s}{s}$

$+ \dots + (-1)^r \binom{s}{r} = \binom{s}{0} - \binom{s}{1} + \binom{s}{2} - \dots + (-1)^s \binom{s}{s} = (1 + (-1))^s = 0$. This

concludes the proof. 

Note that the formula described by PIE is also known as an inversion formula. It is very useful in counting.

Many applications can be obtained through PIE counting.

One of the most popular problems is known as the Hatcheck problem or Baseball Cap problem. A group of people with similar hats. After they throw up the hats, everyone gets a hat^{back}, which is not his own one. The problem is asking "what is the probability of this outcome?". Before answering the question, we need a definition.

Definition 6.2.2. (Derrangement)

Let N be a set of n objects. Let $N = [1, n]$. Then, a permutation f of N is a derrangement if for each $i \in N$, $f(i) \neq i$.

For example, $n=3$, $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Clearly, there are two such permutations for $n=3$.

Proposition 6.2.3. Let D_n denote the number of derrangements on an n -set. Then, $D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right]$.

Proof. Let α_i denote the property of a permutation ^{α} of $[1, n]$ such that $\alpha(i) = i$. Now, $N = n!$, $N(\alpha_i) = \binom{n}{1} \cdot (n-1)!$, $N(\alpha_i \alpha_j) = \binom{n}{2} \cdot (n-2)!$, and so on. Then, the proof follows by using PIE. ■

By the fact that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, $D_n/n! \approx e^{-1}$. Hence, the probability of getting a wrong hat back for each player is around $\frac{1}{e}$.

We can also use this principle to count the number of primes which are less than a fixed integer. The idea follows from the fact that if an integer x has a prime factor, then the prime factor is at most $\lfloor \sqrt{x} \rfloor$. Hence, in order to find the number of primes less than 10,000, all we need is to use the primes less than 100 and the PIE.

Exercise 5. Find the number of primes less than 1,000.

Exercise 6. Let $|A| = n$ and $|B| = m$ such that $n \geq m$. Find the number of distinct surjective functions (onto) from A onto B .

Exercise 7. Let $x_1 \leq 5$, $x_2 \leq 7$ and $x_3 \leq 9$. Find the number of different integer solutions for $x_1 + x_2 + x_3 = 15$.

Exercise 8. Let $\varphi(n)$ denote the number integers in $[1, n]$ which are relatively prime with n . Find $\varphi(105)$.

Exercise 9. Let n be a positive integer. Then, $\sum_{d|n} \varphi(d) = n$.

In counting, the following function plays an important role.

Definition 6.2.4. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ be a product of prime powers.

The function defined as follows is called the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n=1; \\ 0 & \text{if } e_i > 1 \text{ for some } 1 \leq i \leq s; \text{ and} \\ (-1)^s & \text{if } e_i = 1 \text{ for each } i=1, 2, \dots, s. \end{cases}$$

Example. $\mu(100) = \mu(2^2 \cdot 5^2) = 0$, $\mu(30) = \mu(2 \cdot 3 \cdot 5) = (-1)^3 = -1$.

Proposition 6.2.5. For $n > 1$, $\sum_{d|n} \mu(d) = 0$.

Proof. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$. Now, if $d|n$, then $d = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$

such that $0 \leq k_i \leq e_i$ for each $i=1, 2, \dots, s$. Clearly, if $k_i > 1$

for some i , then $\mu(d) = 0$ and the factor d contributes nothing to

the sum we are counting. So, consider the set D in which $d' \in D$

if $d' = p_1^{k'_1} p_2^{k'_2} \cdots p_s^{k'_s}$ with $k'_i = 1$, $i=1, 2, \dots, s$. Let D_j denote

the subset of D such that exactly j of k'_i 's are equal to 1.

$$\text{Now, we have } \sum_{d|n} \mu(d) = \sum_{d' \in D} \mu(d') = \sum_{j=1}^{s'} \sum_{d' \in D_j} \mu(d') = \sum_{j=1}^{s'} (-1)^j \binom{s'}{j}$$

$$= \binom{s'}{0} + (-1) \binom{s'}{1} + \cdots + (-1)^{s'} \binom{s'}{s'} = (1 + (-1))^{s'} = 0. \quad \blacksquare$$

6.3. Generating Functions

We start this section with an example on changing a one-dollar bill into coins: Penny (one cent), Nickel (5 cents) and dime (10 cents).

So, how many ways can we change the one-dollar bill into coins.

Let the number of pennys be x , nickels be y and dimes be z .

Then, we are looking at solving ^{integer solutions} (x, y, z) such that $x + 5y + 10z = 100$.

One idea in solving this problem is to consider the polynomial

$f(x) = (1 + x + x^2 + \dots + x^{100}) \cdot (1 + x^5 + x^{10} + \dots + x^{100}) \cdot (1 + x^{10} + x^{20} + \dots + x^{100})$ and the coefficient

of x^{100} provides the answer. For example, 10 pennys, 4 nickels and

7 dimes can be obtained by $x^{10} \cdot x^{20} \cdot x^{70}$ from the products respectively.

In fact, this function can answer a more general problem: In how

many ways we can combine these three types of coins to obtain

a value from 0 cents to n cents for all positive integers n . The

answer comes out from multiplying the three parts out and we have

$$f(x) = \sum_{i=0}^{+\infty} a_i x^i, \quad a_n \text{ is the answer.}$$

by extending the last term x^{100}

Definition 6.3.1.

Let $\langle a_0, a_1, a_2, \dots, a_n, \dots \rangle$ be a sequence. Then, the function

$f(x) = \sum_{i=0}^{+\infty} a_i x^i$ associate with the sequence is called the generating function for the sequence.

It is not difficult to realize that $f(x)$ is in fact a power series with one indeterminate x . If $f(x)$ has an explicit form, then solving a complicate solution may turn to a simpler one.

Example 1 Let $f(x) = (1+x+\dots) \cdot (1+x^5+x^{10}+\dots) \cdot (1+x^{10}+x^{20}+\dots)$.

Then, $f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}}$. Let $\frac{1}{1-x} = \sum_{i=1}^{+\infty} A_i x^i$, $\frac{1}{(1-x)(1-x^5)} =$

$\sum_{i=1}^{+\infty} B_i x^i$ and $\sum_{i=1}^{+\infty} C_i x^i = f(x)$. We are aiming at finding C_{100} .

Hence, $\sum_{i=1}^{\infty} A_i x^i = (1-x^5) \cdot \sum_{i=1}^{\infty} B_i x^i \Rightarrow A_n = B_n - B_{n-5}$, i.e.,

$$\begin{cases} B_n = A_n + B_{n-5}, & \text{Similarly} \\ C_n = B_n + C_{n-10}. & A_0 = B_0 = C_0 = 1. \end{cases}$$

Now, $A_n = 1$ for all n . $C_{100} = B_{100} + C_{90} = B_{100} + B_{90} + C_{80} = \dots$

$$= B_{100} + B_{90} + \dots + B_{10} + C_0 = A_{100} + B_{95} + B_{90} + \dots + B_{10} + C_0$$

$$= A_{100} + A_{95} + 2B_{90} + B_{80} + \dots = A_{100} + A_{95} + 2A_{90} + 2B_{85} + B_{80} + \dots = 1 > 1.$$

Example 2 Suppose that we have n collection of distinguished items. If we are allowed to choose the ^{items from the} same collection, then the number of different combination is denoted by H_r^n provided r items are selected from n items.

Now, we let $f(x) = (1+x+x^2+\dots)^n = \left(\frac{1}{1-x}\right)^n$ and H_r^n is equal to the coefficient of x^r in the expansion. For completeness, we provide the details in what follows.

$$f(x) = (1-x)^{-n} = (1+(-x))^{-n} = \sum_k \binom{-n}{k} (-x)^k = \sum_k \binom{-n}{k} (-1)^k x^k.$$

By the definition* of $\binom{-n}{k}$, we have

$$H_r^n = \binom{-n}{r} (-1)^r = \frac{(-n)(-n-1)\dots(-n-r+1)}{1 \cdot 2 \cdot \dots \cdot r} \cdot (-1)^r = \frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

Exercise 10. Find the explicit form of $f(x) = \sum_{i=0}^{\infty} (i+1)x^i$.

Exercise 11. Find the number of partitions of 101 into ^{positive} odd integers.

Exercise 12. Find the number of partitions of n into mutually distinct positive integers.

$$(*) \quad \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!} \quad \text{for each } x \in \mathbb{R}.$$

Instead of using the power series $\sum_{i=0}^{\infty} a_i x^i$ to represent the generating function for the sequence $\langle a_0, a_1, a_2, \dots \rangle$, we may use a more general form to describe the generating function.

Definition 6.3.2. Let $\mu_k(x)$ be a function satisfying

$$\sum_{k=0}^{\infty} a_k \mu_k(x) \neq \sum_{k=0}^{\infty} b_k \mu_k(x) \text{ provided } \langle a_0, a_1, \dots \rangle \text{ and } \langle b_0, b_1, \dots \rangle \text{ are}$$

two distinct sequences (non-equal sequences). Then, $F(x) = \sum_{k=0}^{\infty} a_k \mu_k(x)$

is a generating function of the sequence $\langle a_0, a_1, \dots \rangle$ and $\mu_k(x)$ is called the indicator function for a_k .

Note that in previous definition for generating function, we choose $\mu_k(x) = x^k$. In what follows, we consider $\mu_k(x) = x^k/k!$.

Definition 6.3.3. (Exponential Generating Functions)

$F(x) = \sum_{k=0}^{\infty} a_k x^k/k!$ is called the exponential generating function of the sequence $\langle a_0, a_1, \dots \rangle$.

The notion "exponential" comes from the fact that e^x is the exponential generating function of the sequence $\langle 1, 1, \dots \rangle$. Another important fact shows that we have permutation of items into our counting.

For example, we would like to count the number of different surjective functions f from an n -set onto an m -set. Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$. By the definition of a surjective function, for each $b_j \in B$, $f^{-1}(b_j) \subseteq A$ and $\{f^{-1}(b_j)\}_{j=1}^m$ is in fact a partition of A .

For the partition part, we may use $g(x) = (x + x^2 + x^3 + \dots)^m$ and find the coefficient of x^n . But, we also require the order of the partition,

Hence, an exponential generating function is useful. Now,

let $G(x) = (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)^m$ and find a_n in $\sum_{k=0}^{\infty} a_n \frac{x^n}{n!}$.

$$\begin{aligned} \text{Now, } G(x) &= (-1 + e^x)^m = \sum_{i=0}^m \binom{m}{i} (-1)^i \cdot (e^x)^{m-i} \\ &= \sum_{i=0}^m \binom{m}{i} (-1)^i \sum_{j=0}^{\infty} \frac{1}{j!} (m-i)^j x^j \\ &= \sum_{j=0}^{\infty} \left[\sum_{i=0}^m (-1)^i \binom{m}{i} (m-i)^j \right] \frac{x^j}{j!}. \end{aligned}$$

Hence, the coefficient of $\frac{x^n}{n!}$ is $\sum_{i=0}^m (-1)^i \binom{m}{i} (m-i)^n$. This

is the number of surjective functions we desire. Notice that if m

is larger than n , then the summation is equal to zero. This fact

can be observed by way of the definition on a surjective function

and the combinatorial property of a generating function. But, it is

also an interesting summation (itself) to figure out.

Exercise 13. Show that if $m > n$, then $\sum_{i=0}^m (-1)^i \binom{m}{i} (m-i)^n = 0$.

Exercise 14. Let $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ denote the number of ways to partition an n -set into m ^{non-empty} subsets. Prove that $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix} \right\} + m \left\{ \begin{smallmatrix} n-1 \\ m \end{smallmatrix} \right\}$ and find $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\}$.

6.4 Recurrence Relations

Recurrence relations (also known as recursion) occur naturally in counting. A well-known example is the Hanoi Tower problem.

In order to move the n -ordered rings from one peg to another peg with the order holds, we need at least three pegs to get the job done. For the three pegs case, if we let the number

of moves be a_n , then $a_n = 2a_{n-1} + 1$ by the observation that

a_{n-1} must be used to transfer $n-1$ top rings to one peg and

then transfer the largest one before put $n-1$ rings back on

top. Now, by a direct counting $a_n = 2a_{n-1} + 1 = 2(2a_{n-2} + 1) + 1$

$$= 2^2 a_{n-2} + 2 + 1 = 2^3 (a_{n-3}) + 2 + 2 + 1 = \dots = 2^{n-1} + 2^{n-2} + \dots + 1 = 2^n - 1$$

Since $a_1 = 1$.

So, a general form of recursion would be a function looks like

$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-r})$ where $r \leq n$. The following form is known as a ^{linear} homogeneous recurrence of r^{th} order :

$$a_{n+r} = c_1 a_{n+r-1} + c_2 a_{n+r-2} + \dots + c_r a_n, \quad n = 0, 1, 2, \dots \quad (*1)$$

It is well-known that we can associate this recurrence relation with an equation known as the characteristic polynomial equation (x2)

$$f(x) = x^r - c_1 x^{r-1} - c_2 x^{r-2} - \dots - c_r = 0. \quad \downarrow \text{By solving this equation}$$

together with r initial values $a_0, a_1, \dots, a_{r-1}$, we are able to obtain an explicit form of a_n . Here is the theorem.

Theorem 6.4.1.

Let the characteristic polynomial of the recurrence relation (*1)

be (*2) such that $f(x) = 0$ has q distinct zeros $\alpha_1, \alpha_2, \dots, \alpha_q$ and

the multiplicity of α_i is μ_i for $i = 1, 2, \dots, q$. That is, $\sum_{i=1}^q \mu_i \alpha_i = r$.

Then, the basic solutions of a_n are $\alpha_1^n, n\alpha_1^n, \dots, n^{\mu_1-1}\alpha_1^n; \alpha_2^n, n\alpha_2^n, \dots;$

$n^{\mu_2-1}\alpha_2^n; \dots; \alpha_q^n, n\alpha_q^n, \dots, n^{\mu_q-1}\alpha_q^n$. Moreover, the general solution for

a_n is a linear combination of the above r basic solutions, i.e.,

if we use $\beta_i, i=1, 2, \dots, r$ to denote the basic solution, then

$$a_n = \sum_{i=1}^r c_i \beta_i \text{ where } c_i\text{'s are constants.}$$

Here, we omit the proof. The readers may refer to [] for a proof. Instead, we give several examples to describe the idea of solving this type of recursion.

Example 1. The recurrence relation $a_n = a_{n-1} + a_{n-2}$, $a_0 = a_1 = 1$

is known as the Fibonacci sequence. By using $x^2 - x - 1 = 0$,

we obtain $\alpha = \frac{1 \pm \sqrt{5}}{2}$ and thus $a_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$. Since

$a_0 = a_1 = 1$, $c_1 + c_2 = 1$, $c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$, we conclude

that $c_1 = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2}$ and $c_2 = \frac{-1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2}$. Thus,

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right].$$

Example 2. Let $a_n = a_{n-1} - a_{n-2}$, $a_1 = 1$ and $a_2 = 0$. Then, by

using the characteristic polynomial $f(x) = x^2 - x + 1$, we obtain

$\alpha_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\alpha_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Therefore, $a_n = c_1 \alpha_1^n + c_2 \alpha_2^n$.

By converting α_1 and α_2 into $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ and $\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}$,

we are able to conclude that $a_n = \cos \frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3}$ following a careful calculation.

Example 3 $a_n = 9a_{n-1} - 15a_{n-2} + 7a_{n-3}$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

By solving $x^3 - 9x^2 + 15x - 7 = 0$, we have $\alpha_1 = 7$ and $\alpha_2 = 1$ (a multiple root). Hence, $a_n = c_1 7^n + c_2 1^n + c_3 \cdot n \cdot 1^n$. Solving c_1 , c_2 and c_3 , we obtain $a_n = n$. (This is a very surprising conclusion.)

We may also use generating function to find the explicit form of a_n especially the recursion is not homogeneous.

Example 4 The following recurrence relation comes from subdividing a plane into regions by using cycles in standard position (any two cycles intersect at two points and no three cycles have a common intersection point), see Figure 1 for example.

$$a_n = a_{n-1} + 2(n-1), \quad a_1 = 2. \quad (\text{Problem of ovals.})$$

$$(\text{Way 1}) \quad \sum_{i=1}^{n-1} (a_{i+1} - a_i) = \sum_{i=1}^{n-1} 2i.$$

$$a_n - a_1 = 2 \cdot \frac{n(n-1)}{2},$$

$$a_n = n^2 - n + 2.$$

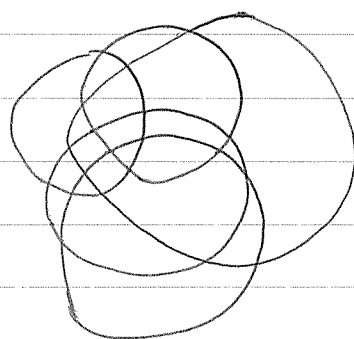


Figure 1. Intersecting cycles

(Way 2) Let $G(x)$ be the

generating function of $\langle a_n \rangle$, i.e., $G(x) = \sum_{k=1}^{\infty} a_k x^k$, with $a_1 = 2$.

$$\sum_{k=2}^{\infty} a_k x^k = \sum_{k=2}^{\infty} (a_{k-1} + 2(k-1)) x^k$$

$$= \sum_{k=2}^{\infty} a_{k-1} x^k + 2 \sum_{k=2}^{\infty} (k-1) x^k$$

$$G(x) - 2x = x G(x) + 2x^2 \left(\sum_{k=2}^{\infty} (k-1) x^{k-2} \right)$$

$$= x G(x) + \frac{2x^2}{(1-x)^2}$$

$$G(x) = \frac{2x}{1-x} + \frac{2x^2}{(1-x)^2}, \text{ This implies that } a_n = \frac{2n(n-1)}{2} + 2.$$

Example 5 Given a convex $(n+1)$ -gon with labeled n points.

Find the number of distinct triangulations (see Figure 2 for example).

Sol. By using one side as the base and consider the triangle using this base, we obtain a recurrence relation

$$h_n = \sum_{k=1}^{n-1} h_k h_{n-k} \text{ where } h_n \text{ denotes}$$

the number of ways in such

a triangulation. Now, by

letting $h_2 = 1$ and $h_3 = 2$, we

can use generating function to

find h_n .

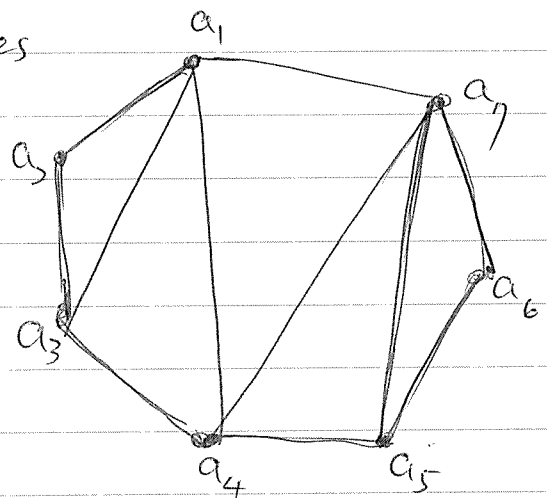


Figure 2. Triangulation of a 7-gon.

Let $H(x) = \sum_{k=1}^{\infty} h_k x^k$.

Then $(H(x))^2 = (x) - x$ by using the product of polynomials.

Hence, $H(x) = \frac{1 - \sqrt{1-4x}}{2}$ since $H(1) = 0$.

Now, by using Newton's Binomial formula, we conclude that

← $H(x) = \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} x^k$ following a careful

calculation. This implies that $h_n = \frac{1}{n} \binom{2n-2}{n-1}$.

Notice that $\frac{1}{n+1} \binom{2n}{n}$ is known as the Catalan numbers

for $n \in \mathbb{N}$. Moreover, there are quite a few enumeration problems

which are of answers of this type. The interesting readers may

refer to [] for a detail survey.

$\vec{a} =$

Exercise 15 Let $\langle a_1, a_2, \dots, a_{2n} \rangle$ be a $\{1, -1\}$ -sequence. $\vec{a}$ is

called an acceptable sequence provided for each $k \in \{1, 2, \dots, 2n\}$

$\sum_{i=1}^k a_i \geq 0$. Find the number of distinct acceptable sequences.

Exercise 16 Let $s(n, k)$ denote the number of k cycles in

the cycle representation of a permutation of n elements. Find $s(11, 7)$.

(*) Note that $s(n, k)$ is known as the Stirling number of the first kind and $S(n, k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the Stirling number of the second kind.

Exercise 17 Determine the number of ways to color the squares of an 8×8 chessboard by using three colors, red, blue and yellow such that an even number of squares are colored red and an odd number of squares are colored blue.

6.4. Pólya Counting

Let's look at an example. In Figure 3, we plan to color the four squares of a grid by using black and white colors. So, how many different patterns we can have. Clearly, two grids with different number of colors used should have different patterns. But, if we only have two black's and two white's then it is possible to have different patterns, see Figure 3.

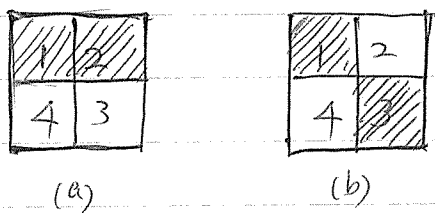
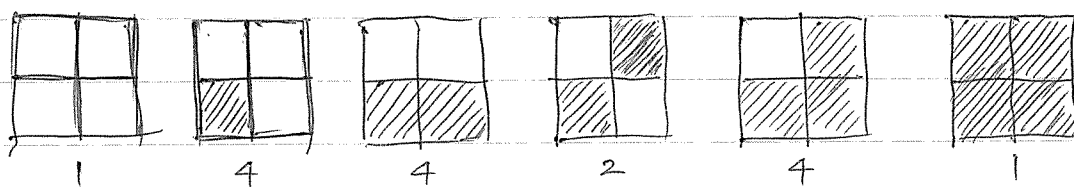


Figure 3. Colored grids.

Clearly, among all possible 2^4 colorings, four of them are looking like Figure 3 (a) and two of them are looking like Figure 3 (b).

For convenience, we use 4-tuples to denote the patterns (following the order shown on Figure 3). All 16 patterns are: 0000, 0001, 0010, 0100, 1000, 0011, 0110, 1100, 1001, 1010, 0101, 1110, 1101, 1011, 0111, 1111.

If the counting considers the rotation of grids, then 0011, 0110, 1100 and 1001 can be thought as the same configuration, so are 1010 and 0101. Following this idea, we only have 0000, 0001, 0011, 0101, 0111 and 1111 six different patterns:



The idea of taking "transformations" into enumeration is known as the Pólya counting. The transformations we use in the above example are $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$, rotations with (clockwise) degrees 90° , 180° and 270° . Together with identity, these four mappings are in fact a cyclic subgroup of S_4 (Symmetric group with 4 elements).

For easy understanding, we may view these six patterns as ¹¹generators¹¹ following the sense that two patterns ^{a and b}_a are the same provided we can obtain b from a by applying the group acting "a". For example, let G be the cyclic subgroup mentioned above, then $Ga = \{(0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0)\}$ when $a = (0,0,0,1)$. Here, $Ga = \{g \circ a \mid g \in G\}$ where $g \circ a$ is defined as a cyclic shift of the vector a .

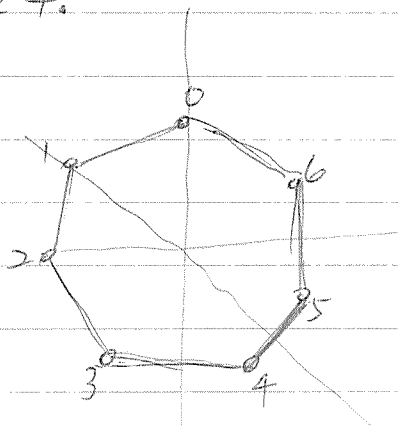
In a more precise description, two patterns a and b are called G -equivalent if there exists a permutation in G such that $g \circ a = g(a) = b$ for some $g \in G$. Hence, $(0,1,0,1)$ and $(1,0,1,0)$ are G -equivalent if $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \in G$ in the above example.

Now, we review some notions in Algebra before giving several counting formulas in Pólya counting. First, let S_n denote the symmetric group of all permutations of the set $\{1, 2, \dots, n\}$.

Then, it is well-known that the set of all cyclic permutations generated by the cyclic permutation $p = (1\ 2\ 3\ \dots\ n)$ is a cyclic group of order n , the group has elements $\{p_0, p_1, \dots, p_{n-1}\}$ where

p_0 is the identity element. The group is referred to as Z_n . Another beautiful subgroup of S_n is the dihedral group of order n , D_n in which we n more permutations to Z_n , known as symmetric

$w_0, w_1, \dots, w_{n-1}$,
reflections, see Figure 4.



$$w_0 = (16)(25)(34)$$

$$w_1 = (02)(36)(45)$$

$\vdots$

$$w_6 = (05)(14)(23)$$

Figure 4. $n=7$,

Let G be a permutation group.

Definition 6.4.1. Let a and b in S . Two configurations are said to be G -equivalent if configuration b can be obtained ^{from a} by applying a permutation of G .

Since G is a group, the G -equivalent relation induces an equivalence relation $\sim$ of S . So, our aim is to find the size of $S/\sim$, i.e., the number of equivalence classes. The example mentioned above does have 6 equivalence classes.

For convenience, a configuration b obtained ^{from a} by applying $g \in G$ will be called " b is induced from a by g ".

The following result plays an important role in counting. [4]

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Theorem 6.4.2. (Burnside's Lemma)

Let G be a permutation group which is acting on a set S of configurations. Then, the number of equivalence classes is given by $\frac{1}{|G|} \sum_{\pi \in G} \psi(\pi)$ where $\psi(\pi)$ is the number of elements in S that are invariant under the permutation π , i.e., for $x \in S$, $\pi(x) = x$.

Proof. Let $\eta(x)$ denote the number of permutations in G under which x is invariant, i.e., the number of permutations π in G such that $\pi(x) = x$.

Then, by two way counting of the total number of invariances under all permutations in G , we have

$$\sum_{\pi \in G} \psi(\pi) = \sum_{x \in S} \eta(x).$$

Now, let $\{a_1, a_2, \dots, a_k\}$ be an equivalence class. Notice that there are exactly $\eta(a)$ permutations in G that map a into b for any two elements a and b in the same equivalence class.

(If $\pi_1(a) = a$ and $\pi_2(a) = b$, then $(\pi_2 \circ \pi_1)(a) = b$.) This implies

that $\eta(a) = \eta(a_2) = \dots = \eta(a_k)$ and $\sum_{i=1}^k \eta(a_i) = |G|$. Hence, if

the number of equivalence classes $c(S)$, then

$$\sum_{s \in S} \eta(s) = c(S) \cdot |G|.$$

This implies that $c(S) = \frac{1}{|G|} \sum_{s \in S} \eta(s) = \frac{1}{|G|} \sum_{\pi \in G} \psi(\pi)$. ■

Since in general $|G| < |S|$, we have less calculation time in counting $\psi(\pi)$. For example, in the above example, $G = \{p_0, p_1, p_2, p_3\}$ and $\psi(p_0) = 16$, $\psi(p_1) = 2$, $\psi(p_2) = 4$ and $\psi(p_3) = 2$. Hence, the number of equivalence classes is $24/4 = 6$.

In applying Burnside's Lemma to the counting of the number of equivalence classes, one may find that the computation is quite involved, especially when both $|G|$ and $|S|$ are large. Moreover, if we wish to have further information about the properties of the equivalence classes, then extra effort is needed. The interesting readers may refer to  for more details.

Exercise 18. A circular ring is decorated with four red stones and 2 blue stones. How many ways are there if two decorations are the same if and only if they are D_6 -equivalent?

Exercise 19. If we use three colors to color the four corners of the grid mentioned in this section, then how many of them are different?

Exercise 20. If we use three types of stones, ^{two stones each,} to decorate a circular ring, then how many of them are different?

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