

作业问题

$$1. a_k = \int_0^\infty e^{-s} \cdot s^k ds, \quad a_0 = \int_0^\infty e^{-s} ds = \lim_{t \rightarrow \infty} \int_0^t e^{-s} ds$$

$$= \lim_{t \rightarrow \infty} \int_0^t e^{-s} s^k ds = \lim_{t \rightarrow \infty} -e^{-s} \Big|_0^t = \lim_{t \rightarrow \infty} -e^{-t} + 1 = 1.$$

$$a_k = k a_{k-1} \quad (\text{Not Homogeneous!})$$

$$\rightarrow a_k = k(k-1)a_{k-2} = k!$$

2. 化简成 Power Series (Details)

$$(1-4x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \cdot (-4x)^k$$

$$= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \cdot (-1)^k (4^k) x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot \dots \cdot (\frac{1}{2} - k + 1)}{k!} \cdot 2^k \cdot 2^k \cdot (-1)^k x^k$$

$$= 1 - \sum_{k=1}^{\infty} \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \dots \cdot \frac{(-3+2k)}{2}}{k!} \cdot 2^k \cdot 2^k x^k$$

$$= 1 - \sum_{k=1}^{\infty} \frac{1 \cdot 1 \cdot 3 \cdot \dots \cdot (2k-3)}{k!} \cdot 2^k \cdot x^k$$

$$= 1 - \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2k-3) \cdot 2 \cdot 4 \cdot \dots \cdot 2k}{k! \cdot k!} x^k$$

$$= 1 - \sum_{k=1}^{\infty} \frac{(2k)!}{(2k-1)k! \cdot k!} x^k \quad \leftarrow \text{上, 下各乘 } (2k-1)$$

$$= 1 - \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \binom{2k}{k} x^k$$

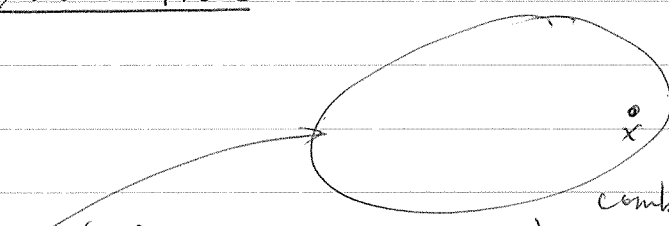
Lecture 7

Date / /

No. 7-1

Recurrence relations with two indices

Known examples

1. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. (Combination Number)  combinatorial proof. (Contain x or not.)

2. $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \cdot k$ (partition an n -set into k subsets) non-empty

3. $p_k^n = p_k^{n-1} + k \cdot p_k^{n-1}$ (k -permutations of n distinct elements)
↑ A special element does not appear
↑ the special element appears (Insert this element)

這些式子都可以直接用生成函數 $(\binom{n}{k}, \left\{ \begin{matrix} n \\ k \end{matrix} \right\}, p_k^n)$ 。以下我們用兩個“變數”的方式來找他們的生成函數。

Definition (General Form of recurrence relation with two indices).

$$f(n, r) = \sum_{j=0}^k \sum_{i=0}^h c_{j,i} \cdot a_{n-j, r-i}$$

$$= c_{0,0} \cdot a_{n,r} + c_{0,1} \cdot a_{n,r-1} + c_{0,2} \cdot a_{n,r-2} + \dots + c_{0,h} \cdot a_{n,r-h}$$

$$+ c_{1,0} \cdot a_{n-1,r} + c_{1,1} \cdot a_{n-1,r-1} + c_{1,2} \cdot a_{n-1,r-2} + \dots + c_{1,h} \cdot a_{n-1,r-h}$$

⋮

$$+ c_{k,0} \cdot a_{n-k,r} + c_{k,1} \cdot a_{n-k,r-1} + c_{k,2} \cdot a_{n-k,r-2} + \dots + c_{k,h} \cdot a_{n-k,r-h}$$

For convenience in solving the recursion, let

$$A_0(x) = a_{0,0} + a_{0,1}x + a_{0,2}x^2 + \cdots + a_{0,r}x^r + \cdots$$

$$A_1(x) = a_{1,0} + a_{1,1}x + a_{1,2}x^2 + \cdots + a_{1,r}x^r + \cdots$$

$\vdots$

$$A_n(x) = a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \cdots + a_{n,r}x^r + \cdots$$

$\vdots$

Then, we have a sequence of generating functions

$$\langle A_0(x), A_1(x), \dots, A_n(x), \dots \rangle.$$

$$\text{Now, Let } A(y, x) = \sum_{k=0}^{\infty} A_k(x) y^k$$

$$= \sum_{k=0}^{\infty} (a_{k,0} + a_{k,1}x + a_{k,2}x^2 + \cdots) y^k$$

$$= \sum_{k=0}^{\infty} (a_{k,0} y^k + a_{k,1} x y^k + a_{k,2} x^2 y^k + \cdots)$$

$a_{n,r}$ is the coefficient of $x^r y^n$.

(i) 一般而言, 双指标的递归关系很困难解; 特殊形式的问题是个例外。

(ii) 通常固定其中一个指标 (Index) 来找出生成函数。

Example 1 Find the generating function of $\binom{n}{k}$.

Sol. Fix "n".

$$\text{Let } G_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k, \quad \binom{n}{0} = 1.$$

$$\text{Then, } \sum_{k=1}^{\infty} \binom{n}{k} x^k = \sum_{k=1}^{\infty} \binom{n-1}{k-1} x^k + \sum_{k=1}^{\infty} \binom{n-1}{k} x^k.$$

$$G_n(x) - 1 = x G_{n-1}(x) + G_{n-1}(x) - 1$$

$$G_n(x) = (x+1) G_{n-1}(x) \Rightarrow G_n(x) = (1+x)^n.$$

Example 2 (可重複組合)

Find the number of r -combinations of n distinct items with unlimited repetitions. (Denote it by $\left\langle \begin{smallmatrix} n \\ r \end{smallmatrix} \right\rangle$.)

$$(0) \quad \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\rangle + \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle \quad (\text{選了一個之後, 還可以再重複, 所以 } n \text{ 不變。})$$

$$\text{Let } F_n(x) = \sum_{k=0}^{\infty} \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle x^k. \quad \left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle = 1, \left\langle \begin{smallmatrix} 0 \\ r \end{smallmatrix} \right\rangle = 0 \text{ for } r > 0 \quad (n \geq 0)$$

$$\sum_{k=1}^{\infty} \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle x^k = \sum_{k=1}^{\infty} \left\langle \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\rangle x^k + \sum_{k=1}^{\infty} \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle x^k$$

$$F_n(x) - 1 = x F_n(x) + F_{n-1}(x) - 1$$

$$F_n(x) = \left(\frac{1}{1-x} \right) F_{n-1}(x) = \left(\frac{1}{1-x} \right)^2 F_{n-2}(x) = \cdots = \left(\frac{1}{1-x} \right)^n.$$

$$\boxed{F_n(x) = (1-x)^{-n}} = \left(\frac{1}{1-x} \right)^n \leftarrow (1+x+x^2+\cdots)^n.$$

↑
G.F.
組合意義

Example 3 $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, Fixed k

Let $H_k(x) = \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^n$. $n < k \Rightarrow \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0$.

$$\sum_{n=1}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^n = \sum_{n=1}^{\infty} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} x^n + \sum_{n=1}^{\infty} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} x^n$$

$$H_k(x) - 0 = x H_{k-1}(x) + kx \cdot H_k(x)$$

$$H_k(x) = \frac{x}{1-kx} H_{k-1}(x)$$

$$H_k(x) = x^k \cdot \prod_{i=1}^k \left(\frac{1}{1-ix} \right)$$

$$= x^k (1+x+x^2+\dots) (1+2x+(2x)^2+\dots) (1+3x+(3x)^2+\dots) \dots (1+kx+(kx)^2+\dots)$$

(組合的意義?)

(•) $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is known as the Stirling number of the second kind.

(••) The Stirling number of the first kind is defined by

the following:

$$\sum_{k=0}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k = x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1).$$

e.g. $n=3$, $x(x-1)(x-2) = x^3 - 3x^2 + 2x$

(•••) The number of k cycles in a permutation of n elements.

$$\left(\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix} \right) = (1)(23), \quad \left(\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{matrix} \right) = (1234)$$

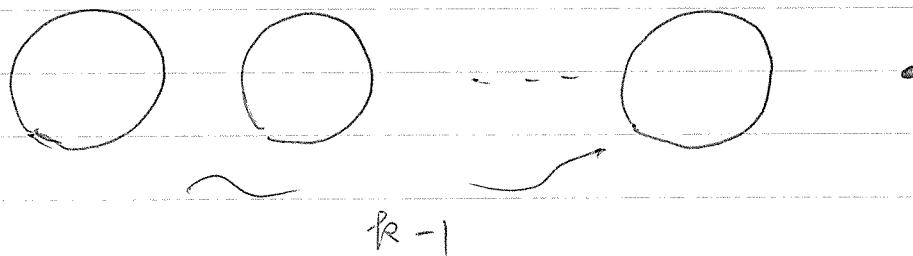
$k=2$ $k=1$

$$(*) \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = n!$$

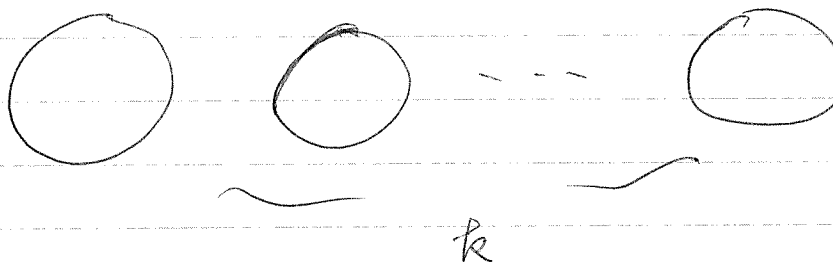
$$x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$$

$\left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\} = 6$ $\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} = 6$, $\left\{ \begin{matrix} 4 \\ 4 \end{matrix} \right\} = 1$
 $\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = 11$

$$\Delta(n, k) = \Delta(n-1, k-1) + (n-1)\Delta(n-1, k)$$



or



Since cycles have "orders", inserting new element
has $n-1$ choices.

Combinatorial arguments

$$x(x-1)(x-2)\cdots(x-n+1)$$



a cycle

One less cycle

x^n : n cycles

x^{n-1} : $n-1$ cycles, one cycle has "2" elements

x^{n-2} : $n-2$ cycles, ...

$$1+2+\cdots+(n-1) = \binom{n}{2}$$

Consider $\sum_{k=0}^{\infty} \boxed{(-1)^{n-k} \cdot \Delta(n, k)} x^k = F_n(x)$

$$\sum_{k=1}^{\infty} (-1)^{n-k} \Delta(n, k) x^k$$

$$= \sum_{k=1}^{\infty} (-1)^{n-k} \left[\Delta(n-1, k-1) + (n-1) \Delta(n-1, k) \right] x^k$$

$$= x \sum_{k=1}^{\infty} (-1)^{n-k} \Delta(n-1, k-1) x^{k-1} + \underbrace{(n-1)}_{(-1)} \cdot \sum_{k=1}^{\infty} \boxed{(-1)^{(n-1)-k}} \Delta(n-1, k) x^k$$

$$= x \left(\sum_{k=1}^{\infty} (-1)^{(n-1)-(k-1)} \Delta(n-1, k-1) x^{k-1} \right) + (n-1) \cdot F_{n-1}(x)$$

$$= x F_{n-1}(x) - (n-1) F_{n-1}(x)$$

$$= (x - n + 1) F_{n-1}(x)$$

$$F_n(x) = (x - n + 1) F_{n-1}(x)$$

$$= (x - n + 1) \cdot (x - n + 2) F_{n-2}(x)$$

$$= (x - n + 1)(x - n + 2) \cdots (x - 1)(x) = x^{\underline{n}}$$

$(-1)^{n-k} \cdot \Delta(n, k)$: The number of k cycles in a permutation
of n elements.

(*) Review cycle representation of a permutation!

Pigeon-hole principle (Dirichlet's drawer principle) No. 7-5

(•) The first formalization of the idea is believed to have been made by Peter Gustav Lejeune Dirichlet in 1834 under the name Schubfachprinzip ("drawer principle" or "shelf principle").

(••) It is different from Dirichlet's principle for harmonic functions.

(*) Quantified Version (Pigeonhole principle)

For natural numbers k and m , if $n = km + 1$ elements are distributed into m sets, then at least one of the m sets will contain at least $k+1$ elements.

(**) This principle can be thought as an average principle. If
(Expectation principle)

$n > m$ and $\lfloor \frac{n}{m} \rfloor < \lceil \frac{n}{m} \rceil$, then $n = mk + r$, $r > 0$.

$\parallel$ $\parallel$
 k $k+1$

(***) We can also consider an infinite set S . If S is partitioned into a finite number of subsets, then at least one of them must be an infinite subsets of S .

Proof. If all subsets are finite, then S is also finite. ■

For reference

1. Schubfach-prinzip (German)
2. Principe de tiroirs (French)
3. zasada szufladkowa (Polish)
4. skatulyaelv (Hungarian)
5. principio dei cassetti (Italian)
6. Ladenprincipe (Dutch)
7. Skuffeprincippet (Danish)
8. 抽屉原理 (Chinese)
9. 引き出し論法 (Japanese)
10. 鴿籠原理 (台灣)

Examples (Wiki)

1. Sock-picking
2. Hand-shaking
3. Hair-counting
4. The birthday problem
5. Softball team
6. Subset sum problem
 $S = \{1, 2, \dots, 9\}$, $T \subseteq S$, $|T| \geq 6$,
 T contains two elements of sum "10".
 $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$ (holes).

Good problems:-

1. Given n arbitrary ^{ordered} integers $(a_1, a_2, \dots, a_n)$, there exist a set of consecutive integers whose sum is a multiple of n .

Proof. Let $s_i = \sum_{k=1}^i a_k$, $i=1, 2, \dots, n$. If there exists a $1 \leq j \leq n$ such that $s_j \equiv 0 \pmod{n}$, then we are done. Otherwise, let

$s_j \equiv d_j \pmod{n}$ such that $d_j \neq 0$. Hence, $1 \leq d_j \leq n-1$, $1 \leq j \leq n$.

By Pigeonhole principle, let $1 \leq j_1 < j_2 \leq n$ such that $d_{j_1} = d_{j_2}$. Then,

$$s_{j_2} - s_{j_1} = \sum_{k=j_1+1}^{j_2} a_k \equiv 0 \pmod{n}. \quad \blacksquare$$

(Stronger Version)

- (*) 2. Let A be a set of n integers. Then, there exists a subset $B \neq \emptyset$ of A such that $\sum_{x \in B} x \equiv 0 \pmod{n}$.

3. Let $n \in \mathbb{N}$ and $n = 2t+1$. Let A be a set of n integers,

$A = \{a_1, a_2, \dots, a_n\}$ and α be a permutation of A . Then,

$\prod_{i=1}^n (a_i - \alpha(a_i))$ is an even number.

For convenience,

Proof. Let $A' = \{a_1, a_2, \dots, a_t, a_{t+1}\}$ be a $(t+1)$ -subset of A such that

all the integers are of the same parity. (同奇或同偶)

Now, consider $\alpha(A') = \{\alpha(a_1), \alpha(a_2), \dots, \alpha(a_{t+1})\}$. Clearly, $A' \cap \alpha(A') \neq \emptyset$,

since $A', \alpha(A')$ are subsets of A and $|A| = 2t+1$. Hence, let

$\alpha(a_j) = a_i$ where $i, j \in \{1, 2, \dots, t+1\}$. This implies that

$a_j - \alpha(a_j) = a_j - a_i$ is an even integer. The proof

follows. (乘式中存在一偶数!)

4. Place 10 points inside a regular triangle (Figure 1). Show that (including boundary)

the maximum shortest distance of these points is "1" unit.

Proof: The layout as in the figure

shows that the shortest distance is 1 and

thus $d_{10} \geq 1$. By Pigeonhole principle,

the placement of 10 points will result in

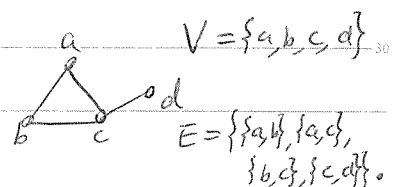
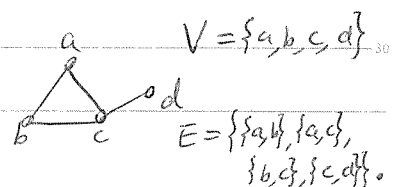
at least two points in a smaller triangle (9 smaller triangles).

Hence, $d_{10} \leq 1$.

5. A simple graph G is an ordered pair (V, E) where V is a non-empty

set and E is a collection of 2-subsets of V . V is called the

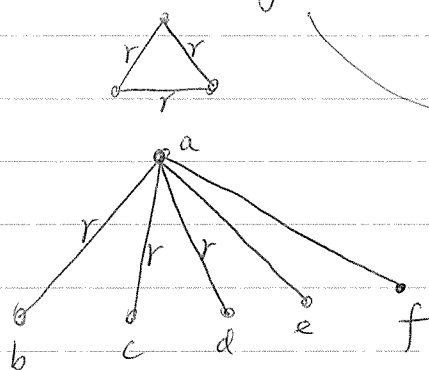
vertex set of G and E the edge set of G .



(*) If $|V|=n$, then $|E| \leq \binom{n}{2}$. The graph with $|E| = \binom{n}{2}$ is the complete graph of order n , denoted by K_n .

(**) 6. Color the edges of K_6 by using two colors red and blue, then either we obtain a red triangle or blue triangle.

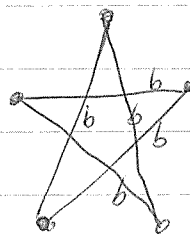
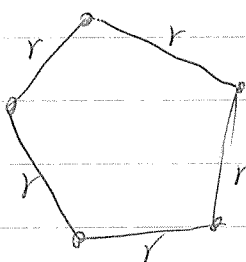
Proof. Consider



Mono-Chromatic triangle

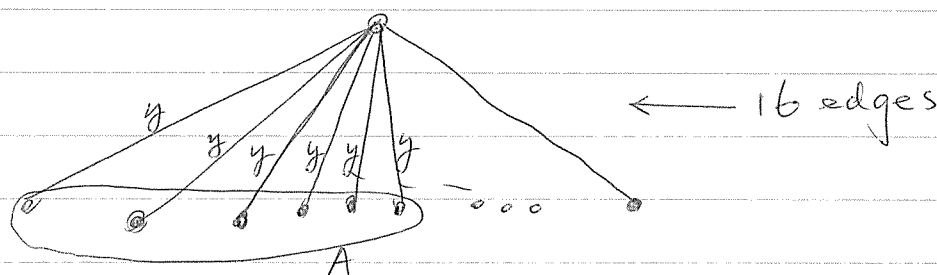
There are five colors on the above five edges. At least one of them occurs three times! (Pigeonhole principle) Say, red. Now, look at b,c, c,d and b,d. If one of them has color r, then we have a red triangle. On the other hand, all of them are colored blue, then we have a blue triangle. ■

Remark This fact is not true for K_5 .



7. Color K_7 with three colors, then there exists a mono-chromatic triangle.
 y, r, b

Proof:



By Pigeonhole principle, at least six of them are of the same color, say yellow. Then, consider the edges in A . If one of them is colored y , then we have a yellow triangle. On the other hand, no edges of A are colored y , then r and b for them. By 6., we have either a red triangle or a blue triangle. This concludes the proof. ■

Question What if four colors are used?

Remark 重要事項

期中考试 (Mid-Term)

範圍: Lecture 1~7, 離散演算 part 1, 2; 作業自行練習。

時間: 4月20日, 早上 10:10 ~ 12:00.